

Deformational Rigidity of Anosov diffeomorphisms near De la Llave's example

MARTIN LEGUIL

(joint work with Andrey Gogolev, Federico Rodriguez Hertz)

We consider the following *rigidity problem*: given two smooth diffeomorphisms f, g which are topologically conjugated by some homeomorphism h , namely,

$$(1) \quad h \circ f = g \circ h,$$

when is it possible to show that f and g are actually *smoothly* conjugated?

One basic obstruction for this to hold comes from periodic points, more precisely, from their multipliers. Indeed, assuming the conjugacy map h can be chosen C^1 , differentiating (1) at any periodic point $p = f^n(p)$, $n \geq 1$, yields

$$Dg^n(h(p)) = Dh(p)Df^n(p)Dh(p)^{-1},$$

hence $Df^n(p)$, $Dg^n(h(p))$ have the same multipliers.

For minimal systems, we may thus expect that a topological conjugacy can always be promoted to a smooth conjugacy. Yet, the example of circle diffeomorphisms shows that even “almost periodic” points can be problematic to keep the conjugacy as smooth as the systems themselves. Indeed, given two conjugated circle diffeomorphisms f and g whose orbits are “far from being periodic”, namely, if their (common) rotation number ρ satisfies a *Diophantine condition*, results of Arnold, Herman, Yoccoz, Khanin-Sinai etc. [1, 10, 15, 11] show that if f, g are C^ω , resp. C^∞ , then f and g are also C^ω , resp. C^∞ -conjugated, while if ρ is Liouville, then the conjugacy may not be very regular.

For hyperbolic systems, the picture is quite different. Indeed, assume that $f: \mathbb{T}^k \rightarrow \mathbb{T}^k$ is an *Anosov diffeomorphism* of the torus $\mathbb{T}^k := \mathbb{R}^k / \mathbb{Z}^k$, $k \geq 2$, i.e., there exists a Df -invariant splitting $T\mathbb{T}^k = E^s \oplus E^u$, where the stable bundle E^s , resp. unstable bundle E^u is uniformly contracted, resp. expanded by Df . On the one hand, by *structural stability*, any diffeomorphism g sufficiently C^1 -close to f is Anosov, and topologically conjugated to f . On the other hand, periodic points of f and g are dense in $\mathbb{T}^k$, and in fact, for any $p = f^n(p)$, it is easy to choose the perturbation g such that the multipliers of g at the continuation of p are distinct from those of f at p . In other words, for topologically conjugated Anosov diffeomorphisms, there are plenty of obstructions to lift for the conjugacy to be even differentiable. Yet, once it is differentiable, by some bootstrap phenomenon, it can sometimes be improved to be more regular. Moreover, in the spirit of Livshits Theorem [12], the hope is that periodic points carry all the obstructions for the smoothness of the conjugacy between two such systems. This was indeed shown for Anosov diffeomorphisms on $\mathbb{T}^2$, and also for 3-dimensional transitive Anosov flows, by De la Llave-Marco-Moriyón and Pollicott in a series of papers (see e.g. [3, 13]).

Let us now consider an Anosov flow $X^t: M \rightarrow M$ on some closed Riemannian manifold M , with generator $X := \frac{d}{dt}|_{t=0} X^t$. If Y^t is generated by Y that is a C^1 -small perturbation of X , then by Anosov's structural stability, these two flows are orbit-equivalent, that is, there exists a homeomorphism $\Phi: M \rightarrow M$

which sends X^t -orbits to Y^t -orbits. In general, such orbit equivalence usually cannot be improved to a conjugacy since the mismatch of periods of corresponding periodic orbits provide obstructions. It is a well-known corollary of the Livshits Theorem [12] that matching of all periods for a pair of transitive Anosov flows is a necessary and sufficient condition for the existence of a (time preserving) topological conjugacy.

Given two transitive Anosov flows X^t and Y^t on 3-manifolds which are topologically conjugated by a homeomorphism Φ ,

$$(2) \quad \Phi \circ X^t = Y^t \circ \Phi,$$

we can wonder whether Φ can (most of the time) be promoted to a smooth conjugacy. In other words, does the existence of a C^0 -conjugacy lift the obstructions coming from periodic multipliers? In a joint work with A. Gogolev and F.R. Hertz, we show that it is locally true in general, more precisely:

Theorem 1 (Gogolev-L.-Rodriguez Hertz [9]). *Let M be a 3-manifold such that the space $\mathcal{A}$ of C^∞ vector fields on M which generate transitive Anosov flows is non-empty. Then, there exists a C^1 -open and C^∞ -dense subset $\mathcal{U} \subset \mathcal{A}$ such that for any $X \in \mathcal{U}$, the Anosov flow X^t generated by X is locally rigid, i.e., if Y^t is an Anosov flow whose generator Y is sufficiently C^1 -close to X , we have:*

$$X^t \text{ and } Y^t \text{ are } C^0\text{-conjugate} \iff X^t \text{ and } Y^t \text{ are } C^\infty\text{-conjugate}.$$

This result follows from the following result:

Theorem 2 (Gogolev-L.-Rodriguez Hertz [9]). *Let X^t, Y^t be two 3-dimensional transitive C^∞ Anosov flows which are C^0 -conjugate by a homeomorphism Φ as in (2). Then, at least one of the following statements holds:*

- (1) Φ swaps SRB measures of the two flows, i.e., $\Phi_* m_X^+ = m_Y^-$ and $\Phi_* m_X^- = m_Y^+$, where $m_\dagger^\pm$ is the positive/negative SRB measure of the flow $\dagger = X, Y$;
- (2) at least one of the stable/unstable distributions $E_X^s, E_X^u, E_Y^s, E_Y^u$ is of class $C^{1+\alpha}$, $\alpha > 0$;
- (3) the flows X^t and Y^t are C^∞ -conjugate.

This result has a counterpart in finite regularity, although it requires a technical pinching condition. Moreover, the second case of the above list where one of the distributions is C^{1+} is highly non-generic. The previous statement generalizes a previous work by Gogolev-Rodriguez Hertz when the two flows are volume preserving, in which case, the conjugacy is smooth unless both flows are constant roof suspension flows.

The basic scheme of the proof is to show that periodic multipliers can be extracted from periodic expansions unless the stable/unstable distributions exhibit anomalous regularity. One key tool for that is the so-called *templates* introduced in the work of Tsujii-Zhang [14] on exponential mixing of 3-dimensional Anosov flows. Another important ingredient is the recent positive proportion Livshits Theorem of Dilsavor-Marshall Reber [4] which allows to globalize the argument, namely, it is sufficient to match multipliers for a positive set of periodic points to fall either in the first case or the third case of the above list.

Going back to Anosov diffeomorphisms, we can wonder whether a higher-dimensional version of the aforementioned rigidity results of De la Llave-Marco-Moriyón and Pollicott can be obtained. In other words, if f and g are two topologically conjugated Anosov diffeomorphisms with matchings multipliers, are they *smoothly* conjugated? For the 2-torus, the one-dimensionality of the stable and unstable foliations and the affine structures along their leaves plays an important role.

It turns out De la Llave [2] produced a family of counterexamples on the torus $\mathbb{T}^4$ as follows, see also [5, 6] for later expositions. Specifically, let A and B be automorphisms of $\mathbb{T}^2$ induced by hyperbolic matrices in $\mathrm{SL}(2, \mathbb{Z})$. We will assume that the smaller eigenvalues λ and μ of A and B , respectively, satisfy the following inequalities: $0 < \lambda < \mu < 1$. Define $\alpha = \log \mu / \log \lambda$ and notice that $\alpha \in (0, 1)$. Let $\phi_0: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a smooth function. A *De la Llave diffeomorphism* $L_{\phi_0}: \mathbb{T}^4 \rightarrow \mathbb{T}^4$ is defined as a skew-product

$$L_{\phi_0}: (x, y) \mapsto (Ax, By + \phi_0(x)), \quad (x, y) \in \mathbb{T}^2 \times \mathbb{T}^2.$$

Such diffeomorphism is Anosov and, if ϕ_0 is homotopic to a constant, is conjugate to the linear product automorphism L_0 . More generally, if ϕ_1 is homotopic to ϕ_0 then the corresponding De la Llave diffeomorphism L_{ϕ_1} is conjugate to L_{ϕ_0} with conjugacy h given by

$$h: (x, y) \mapsto (x, y + \psi(x)),$$

where $\psi: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is null-homotopic. Then we can explicitly solve for ψ in terms of $\phi_1 - \phi_0$ and check that $\psi \in C^\alpha(\mathbb{T}^2)$. However, for some simple choices of $\phi_1 - \phi_0$ (one Fourier mode) the function ψ is not in the $C^{\alpha+}$ regularity class. Consequently, the conjugacy h is merely C^α regular despite the fact that all multipliers at corresponding periodic points of L_{ϕ_0} and L_{ϕ_1} are the same.

In an ongoing project with A. Gogolev, we show that this phenomenon is not typical near such examples. Before stating the result, let us give some definitions. Consider a one-parameter family of Anosov diffeomorphisms $\{F_s\}_{s \in [0,1]}$, associated with a continuous family of conjugacies $\{h_s\}_{s \in [0,1]}$, such that $h_s \circ F_0 = F_s \circ h_s$ and $h_0 = \mathrm{id}$. Such family is called *isospectral* if for every periodic point $p = F_0^n(p)$ the linearized return maps $DF_s^n(h_s(p))$ all have the same collection of eigenvalues for all $s \in [0, 1]$. An Anosov diffeomorphism F_0 is called *deformation rigid* if for any isospectral one-parameter family based at F_0 the corresponding family of conjugacies h_s is, in fact, a family of C^1 diffeomorphisms. Our result is then:

Theorem 3 (Gogolev-L. [7]). *Let L_ϕ be a De la Llave diffeomorphism. Then there exists a C^1 -small neighborhood U of L_ϕ and a C^2 -open C^∞ -dense subset $V \subset U$ such that each Anosov diffeomorphism $F \in V$ is deformation rigid.*

Note that L_ϕ and its perturbations have 2-dimensional stable and unstable distributions. We consider perturbations F with a partially hyperbolic splitting, namely the (un)stable bundle splits into the sum of a strong (un)stable and a weak (un)stable bundles. One key step in our proof is to show that for any isospectral family $\{F_s\}_s$ based at a typical perturbation F , the conjugacies $\{h_s\}_s$ have to

preserve strong (un)stable foliations. For that, we consider leaves of these foliations at periodic points and use periodic expansions similar to those considered in [9].

REFERENCES

- [1] V.I. Arnol'd, *Small denominators. I. Mapping the circle onto itself*, Izv. Akad. Nauk SSSR Ser. Mat., **25** (1961), 21–86.
- [2] R. De la Llave, *Smooth conjugacy and S-R-B measures for uniformly and non-uniformly hyperbolic systems*, Comm. Math. Phys., **150**(2) (1992), 289–320.
- [3] R. De la Llave and R. Moriyón, *Invariants for smooth conjugacy of hyperbolic dynamical systems. IV.*, Comm. Math. Phys. **116**(2) (1988), 185–192.
- [4] C. Dilsavor and J.M. Reber, *A positive proportion Livshits theorem*, Proceedings of the American Mathematical Society **152**, no. 11 (2024), 4729–4744.
- [5] A. Gogolev, *Smooth conjugacy of Anosov diffeomorphisms on higher-dimensional tori*, J. Mod. Dyn., **2**(4) (2008), 645–700.
- [6] A. Gogolev and M. Guysinsky, *C^1 -differentiable conjugacy of Anosov diffeomorphisms on three dimensional torus*, Discrete Contin. Dyn. Syst., **22**(1-2) 2008, 183–200.
- [7] A. Gogolev and M. Leguil, *Deformational Rigidity of Anosov diffeomorphisms near De la Llave's example*, preprint.
- [8] A. Gogolev and F. Rodriguez Hertz, *Smooth rigidity for 3-dimensional volume preserving Anosov flows and weighted marked length spectrum rigidity*, preprint.
- [9] A. Gogolev, M. Leguil and F. Rodriguez Hertz, *Smooth rigidity for 3-dimensional dissipative Anosov flows*, preprint.
- [10] M.R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publications Mathématiques de l'IHÉS **49** (1979), 5–233.
- [11] K. M. Khanin and Y. G. Sinai, *A new proof of M. Herman's theorem*, Comm. Math. Phys., **112** (1987), 89–101.
- [12] A.N. Livsic, *Cohomology of dynamical systems*, Mathematics of the USSR-Izvestiya **36**, no. 6 (1972), 1296–1320.
- [13] J.M. Marco, R. Moriyón, *Invariants for smooth conjugacy of hyperbolic dynamical systems. I.*, Comm. Math. Phys. **109** (1987), 681–689.
- [14] M. Tsujii and Z. Zhang, *Smooth mixing Anosov flows in dimension three are exponentially mixing*, Annals of mathematics **197**, no. 1 (2023), 65–158.
- [15] J.-C. Yoccoz, *Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne*, Ann. Sci. Ecole Norm. Sup. (4), **17** (1984), 333–359.