EXPONENTIAL DECAY OF THE SIZE OF SPECTRAL GAPS FOR QUASIPERIODIC SCHRÖDINGER OPERATORS

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ABSTRACT. In the following we are interested in the spectral gaps of discrete quasiperiodic Schrödinger operators when the frequency α is Diophantine, the potential v is analytic, and in the subcritical regime. The gap-labelling theorem asserts in this context that each gap has constant rotation number, labeled by some integer. We prove that the size of these gaps decays exponentially fast with respect to their label. This refines a subexponential bound obtained previously by Sana Ben Hadj Amor in [A]. Contrary to her approach, which is based on KAM methods, the arguments in the present paper are non-perturbative, and use quantitative reducibility estimates obtained by Artur Avila and Svetlana Jitomirskaya in [AJ2] and [A1]. As a corollary of our result, we show that under the previous assumptions, the spectrum is 1/2-homogeneous.

CONTENTS

Introduction

In the following we consider one-dimensional discrete Schrödinger operators. We work in the quasiperiodic context, that is, given some function $v\colon \mathbb{R}/\mathbb{Z} \to \mathbb{Z}$, the potential at the point of index n is $\lambda v(\theta + n\alpha)$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is the frequency, $\lambda \in \mathbb{R}$ is the coupling constant and $\theta \in \mathbb{R}$ is the phase. The associate operator is denoted by $H = H_{\lambda v,\alpha,\theta}$. We assume that α satisfies some Diophantine condition: given $K, \tau > 0$, we denote $\alpha \in \mathrm{DC}(K, \tau)$ if for every $n \in \mathbb{Z}\setminus\{0\}$, $\|n\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq K/|n|^\tau$. Moreover, we restrict ourselves to the case of a small analytic potential, i.e., $v \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, and with the notations of [AJ2], $|\lambda| < \lambda_0(v)$. [1]

Since $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the spectrum $\Sigma = \Sigma_{\lambda,\alpha}$ does not depend on the phase θ. It is well-known that Σ is a compact subset of $\mathbb{R}$. Our study focuses on the spectral gaps of H, which by definition are the connected components of the complement of Σ in $\mathbb{R}$. To each energy $E \in \mathbb{R}$ we can associate a cocycle $(\alpha, S_{\lambda v,E})$ over the irrational translation $x \mapsto x + \alpha$, called the Schrödinger cocycle, and whose dynamics is closely related to the spectral properties of H. Indeed, any formal solution to the eigenvalue equation for H can be expressed in terms of this cocycle. A key fact is that the spectrum corresponds to energies E for which $(\alpha, S_{\lambda v,E})$ is *not* uniformly hyperbolic.

Let $E \in \Sigma$ be an energy in the spectrum. Under the above assumption that $|\lambda| < \lambda_0(v)$, Avila and Jitomirskaya [AJ2] have shown that the Lyapunov exponent [2] of $(\alpha, A) := (\alpha, S_{\lambda v,E})$ vanishes; in fact, they obtain a better control on the growth of the iterates of the cocycle. This corresponds to what is usually called the *subcritical regime*: for any $\delta > 0$, there exist $c_\delta > 0$, $C_\delta > 0$ such that for any $k \geq 0$,

$$\sup_{|\mathrm{Im}(x)|<c_\delta} \|A_k(x)\| \leq C_\delta e^{\delta k}.$$

Since $x \mapsto x+\alpha$ is uniquely ergodic and because Schrödinger cocycles are homotopic to the identity, it is possible to introduce a notion of fibered rotation number. This object allows us to characterize spectral gaps: indeed, an energy E belongs to a non-collapsed gap if the rotation number ρ is locally constant around E. Moreover each spectral gap can be assigned an integer label $m \in \mathbb{Z}$, such that the rotation number restricted to the spectral gap satisfies $2\rho \equiv m\alpha \mod \mathbb{Z}$. In the following, we denote by $G_m = (E_m^-, E_m^+)$ the spectral gap labeled by $m \in \mathbb{Z} \setminus \{0\}$.

Reducibility of Schrödinger cocycles is a fundamental question. It asks whether $(\alpha, S_{\lambda v,E})$ can be conjugated to a constant cocycle; if so the dynamics becomes much easier to understand since the iterates of the cocycle are conjugated to the powers of a constant matrix. Reducibility gives a lot of information, for instance on the growth of the cocycle, on the existence of absolutely continuous spectrum *etc.* (see for instance [AJ2] or [A1] for some examples). Avila and Krikorian weakened this notion and introduced the concept of almost reducibility: in such a case, it is not necessarily possible to conjugate the cocycle to a constant one, but we can get as close to it as desired.

One important approach to the problem of reducibility is given by KAM theory. In this case, we start with the perturbation of a constant cocycle and wonder if the resulting cocycle is still reducible. The method is based on an iteration process, where at each step the cocycle is conjugated to a new one which is closer to be constant. Each conjugacy is obtained by solving some cohomological equation, and the problem of small divisors is classically one limitation of these techniques.

Using such a perturbative approach, Ben Hadj Amor was able to get a subexponential bound on the size of spectral gaps. In our case, her results give

1. In the case of an Almost Mathieu operator, i.e. when $v = 2\cos(2\pi\cdot)$, we have $\lambda_0(v) = 1$.
2. See Subsection 1.2 for a definition, and Theorem 2.4 below for the corresponding statement.

Theorem (Theorem 2, [A])**.** *Assume that $\alpha \in \mathrm{DC}(K,\tau)$. Then there exists a constant $C_0 > 0$ that only depends on K and τ such that if $\left\|\lambda v^{(n)}\right\|_c \leq C_0 c^{600(21\tau+13)}$ for $c > 0$ and $n = 0, 1, 2$, then for any $m \in \mathbb{Z} \setminus \{0\}$,*

$$E_m^+ - E_m^- \leq Ce^{-\gamma|m|^\kappa}$$

for some constants $C, \kappa > 0$, and where $\gamma > 0$ only depends on K, τ and c.

The main part of her proof consists in showing an almost reducibility result thanks to a KAM scheme of faster convergence. She uses it together with some transversality properties to get a lower bound on the variation of the rotation number on some intervals. But on a non-collapsed spectral gap, the rotation number is constant, and this thus gives an upper bound on the length of the gap. One advantage of her methods is that they also work in higher dimensional situations, which is not the case for the approach we develop in the following.

Another method to study the reducibility properties of the Schrödinger cocycle is given by Aubry duality. It relates the localization properties of a dual Schrödinger operator $\hat{H}$ (the definitions are given in Subsection 1.5) to the reducibility of Schrödinger cocycles. More precisely, starting with a generalized solution $\hat{u}$ to the eigenvalue equation for $\hat{H}$, it is possible to construct a vector U, called a Bloch wave, which behaves nicely under the dynamics of $S_{\lambda v,E}$. Using this vector, we can build a matrix which will conjugate our cocycle to a constant one.

This relation was studied in more detail in [AJ2], where Avila and Jitomirskaya show that a similar link exists between the almost localization properties of $\hat{H}$ and almost reducibility of Schrödinger cocycles. Under our assumption that $|\lambda| < \lambda_0(v)$, they prove that dual Schrödinger operators are almost localized. In their paper, the analysis is made in terms of the resonances of a certain phase θ, and uses truncated Bloch waves; in particular, they show that almost localization implies quantitative almost reducibility results, where both the constant matrix and the conjugacy can be controlled by the resonances of θ. These estimates were later improved by Avila in [A1].

Thanks to these non-perturbative arguments, we obtain exponential decay of the length of spectral gaps with respect to their label. Recall that for $m \in \mathbb{Z} \setminus \{0\}$, we denote by $G_m = (E_m^-, E_m^+)$ the associated spectral gap. Our main result is then:

Theorem A. *Assume that the frequency α is Diophantine, i.e. $\alpha \in \mathrm{DC}$, that the potential v is analytic, and that $|\lambda| < \lambda_0(v)$. Then there exist constants $C, \gamma > 0$ such that for any $m \in \mathbb{Z} \setminus \{0\}$,*

$$E_m^+ - E_m^- \leq Ce^{-\gamma|m|}.$$

We give two proofs of this fact. The first one proceeds as follows. Given some integer m, we start with the energy $E = E_m^+$ located on the right boundary of the corresponding spectral gap. Following [AJ2], we introduce a phase $\theta = \theta(E)$ for which we may find a bounded solution to the eigenvalue equation for the dual Schrödinger operator. We will see that $2\theta = n\alpha \mod \mathbb{Z}$ for some $n \in \mathbb{Z}$; in particular, θ is non-resonant (see Subsection 1.5 for the definition), and this implies that the corresponding Schrödinger cocycle is reducible. More precisely, it follows from the estimates of [A1] that the cocycle $(\alpha, S_{\lambda v,E})$ can be conjugated to a constant parabolic cocycle, with off-diagonal coefficient decaying exponentially fast with respect to $|n|$, and where the conjugacy is subexponentially big. By an idea present in [AJ2] and [A1], we can relate these estimates to the label m of the spectral gap we consider. We then use this conjugacy to study the perturbed cocycle $(\alpha, S_{\lambda v,E-\varepsilon})$ for some small $\varepsilon > 0$. Thanks to an averaging argument similar those which can be found for example in [MP] or [P1], we then show that we can take ε exponentially small with respect to m such that the rotation number $\rho(E - \varepsilon)$ is different from

$\rho(E)$; this shows that $E-\varepsilon$ does not lie in the spectral gap anymore, which gives an exponential bound on the size of the latter.

The second proof we give is based on the monotonicity of Schrödinger cocycles with respect to the energy; indeed, recall that the derivative of the fibered rotation number with respect to E is always nonpositive. The proof also uses the fact that on the boundary of a spectral gap, we can conjugate the Schrödinger cocycle to some constant parabolic cocycle with good estimates both on the conjugacy and the off-diagonal coefficient. Monotonicity for the Schrödinger cocycle translates through the conjugacy into monotonicity for the conjugate cocycle. Then we see that after an exponentially small perturbation of the energy, we can change the rotation number, which concludes.

Let us say a few words on the continuous case, that is, for an operator $-\Delta + v$ defined on the real line, where v is a quasiperiodic analytic potential. In [MP], Moser and Pöschel introduce some not too rapidly increasing approximation function Ω. Assume that the frequency α satisfies $\|n\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \Omega(|n|)^{-1}$ for any $n \in \mathbb{Z}\setminus\{0\}$. Set

$$\mathcal{R}(\Omega) := \{m\alpha \mid \text{if } k \neq m \in \mathbb{Z}, \text{ then } \|(m-k)\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq 2\Omega(|k|)^{-1}\}.$$

Theorem (Theorem 1.3, [MP])**.** *Under the previous assumptions, there exist $C,\gamma > 0$ such that for any $m \in \mathbb{Z}\setminus\{0\}$, if $m\alpha \in \mathcal{R}(\Omega)$, then the spectral gap $\widetilde{G}_m = (\widetilde{E}_m^-, \widetilde{E}_m^+)$ with rotation number $m\alpha/2$ satisfies*

$$\widetilde{E}_m^+ - \widetilde{E}_m^- \leq Ce^{-\gamma|m|}.$$

More recently, but still in the continuous case, Damanik and Goldstein have obtained the following result, where the dependence of the constants in terms of the potential is made very explicit:

Theorem (Theorem B, [DG])**.** *Assume that $\alpha \in \mathrm{DC}$, that the potential v is analytic, and that its Fourier coefficients $(\hat{v}_k)_{k\in\mathbb{Z}}$ satisfy $|\hat{v}_k| \leq \varepsilon e^{-\gamma_0|k|}$ for some $\gamma_0 > 0$ and some small $\varepsilon > 0$. Then there exists $\varepsilon^{(0)} > 0$ such that if $\varepsilon \leq \varepsilon^{(0)}$, then for any $m \in \mathbb{Z}\setminus\{0\}$,*

$$\widetilde{E}_m^+ - \widetilde{E}_m^- \leq 2\varepsilon e^{-\frac{\gamma_0}{2}|m|}.$$

In fact, the main target of their paper is to show that conversely, there exists $\varepsilon^{(1)} > 0$ such that if for some $\varepsilon < \varepsilon^{(1)}$ and $\gamma > 4\gamma_0$, we have $\widetilde{E}_m^+ - \widetilde{E}_m^- \leq \varepsilon e^{-\gamma|m|}$, $m \in \mathbb{Z}\setminus\{0\}$, then for every $k \in \mathbb{Z}$, $|\hat{v}_k| \leq \varepsilon^{1/2} e^{-\frac{\gamma}{2}|k|}$.

The main goal of this theorem is its application to the problems of isospectral potentials (see [DGL2]) and the existence of a global solution to the KdV equation $\partial_t u + \partial_x^3 u + u\partial_x u = 0$ with small quasiperiodic initial data. As mentioned to us by professor Goldstein, it would be also very interesting to look for a statement analogous to the second part of the previous result, but whose proof would be based on reducibility arguments similar to those developed in the present paper.

As a consequence of Theorem A, we show homogeneity of the spectrum of discrete quasiperiodic Schrödinger operators which satisfy the previous hypotheses. We say that a closed set $\mathcal{S} \subset \mathbb{R}$ is homogeneous if there exist $\chi > 0$ and $\sigma_0 > 0$ such that for any $E \in \mathcal{S}$ and any $0 < \sigma \leq \sigma_0$, we have $\mathrm{Leb}((E-\sigma, E+\sigma)\cap\Sigma) > \chi\sigma$.

For discrete Schrödinger operators, Damanik, Goldstein, Schlag and Voda [DGSV] have obtained the following result. In their work, they assume that for some constants $K,\tau > 0$, the frequency α satisfies the condition $\|n\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{K}{n(\log n)^\tau}$, $n > 1$.

Theorem (Theorem H, [DGSV])**.** *Assume that there exists $L_0 > 0$ such that for any $E \in \mathbb{R}$, the Lyapunov exponent $L(E)$ of $(\alpha, S_{\lambda v,E})$ satisfies $L(E) \geq L_0$. Then $\Sigma_{\lambda,\alpha}$ is χ-homogeneous with some $\chi = \chi(\lambda v, K, \tau, L_0) > 0$.*

In the continuous setting, homogeneity of the spectrum was studied by Damanik, Goldstein and Lukic [DGL1]. In this paper the authors build on the results of Damanik and Goldstein [DG] recalled above, and prove that under the same assumptions, the spectrum is 1/2-homogeneous. Mimicking the proof given in [DGL1], we show:

Theorem B. *Let* $\alpha \in \mathrm{DC}$ *be a Diophantine frequency,* v *an analytic potential and assume that* $0 < |\lambda| < \lambda_0(v)$*. Then the spectrum* $\Sigma_{\lambda,\alpha}$ *of the associated Schrödinger operator* H *is* $1/2$*-homogeneous, i.e., there exists* $\sigma_0 > 0$ *such that for any* $E \in \Sigma$ *and any* $0 < \sigma \leq \sigma_0$*,*

$$\mathrm{Leb}((E-\sigma, E+\sigma) \cap \Sigma) > \frac{\sigma}{2}.$$

Note that under our assumptions, and contrary to the case considered in [DGSV], there is no uniform lower bound on the Lyapunov exponent; indeed, as we have recalled above, the Lyapunov exponent vanishes identically on the spectrum.

Structure of the paper

In the first part, we recall general facts on quasiperiodic Schrödinger operators. The second part is dedicated to the proof of a quantitative reducibility result for Schrödinger cocycles associated with energies located on the boundary of a spectral gap. In the third part, we explain how this fact together with an averaging argument imply exponential decay of the size of the gaps. In the fourth part, we give another proof, based on the monotonicity of Schrödinger cocycles with respect to the energy. As an application of our main result, we show in the last part that this implies homogeneity of the spectrum for the Schrödinger operators considered here.

Acknoledgements

I am very grateful to Artur Avila and Raphaël Krikorian for their guidance on this work, and to Julie Déserti for her constant support. I also thank Svetlana Jitomirskaya for a helpful discussion, Qi Zhou for introducing me to this problem and many comments on a first version of this paper, and Christophe Fiszka for several explanations on KAM theory. I am grateful to *Réseau Franco-Brésilien de Mathématiques* and IMPA for their financial help during a visit in Rio de Janeiro. I also thank Adriana da Luz for having invited me to talk about this work in the student seminar of Institut de Mathématiques de Bourgogne.

1. Preliminaries

For a bounded analytic function f defined on a strip $\{|\mathrm{Im}(x)| < c\}$, $c > 0$, we denote $\|f\|_c := \sup_{\{|\mathrm{Im}(x)|<c\}} |f(x)|$. If f is a bounded continuous function on $\mathbb{R}$, we also denote $\|f\|_0 := \sup_{x\in\mathbb{R}} |f(x)|$.

1.1. Schrödinger operators and Schrödinger cocycles.

The *(discrete) quasiperiodic Schrödinger* operators $H = H_{\lambda v,\alpha,\theta} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ are defined by

$$(Hu)_n := u_{n+1} + u_{n-1} + \lambda v(\theta + n\alpha)u_n,$$

where $v\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is called the *potential*, $\lambda \in \mathbb{R}$ is the *coupling constant*, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ is the *frequency* and $\theta \in \mathbb{R}$ is the *phase*. In the following, we consider analytic potentials, i.e. $v \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$.

One particular example is the so-called *almost Mathieu operator*, which arises naturally as a physical model. It corresponds to the case where the potential v is taken to be $v\colon x \mapsto 2\cos(2\pi x)$.

In the following, we assume that the frequency α satisfies some *Diophantine condition*, which is denoted by $\alpha \in \mathrm{DC}$, if there exist two constants $K > 0$ and $\tau > 0$ such that for any integer $n \in \mathbb{Z} \setminus \{0\}$, the following lower bound holds:

$$\|n\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{K}{|n|^\tau},$$

where for $x \in \mathbb{R}$, $\|x\|_{\mathbb{R}/\mathbb{Z}} := \inf_{k \in \mathbb{Z}} |x-k|$. In this case we denote $\alpha \in \mathrm{DC}(K,\tau)$.

Let us recall the following definition.

Definition 1.1 (Resolvent set, spectrum)**.** *The resolvent set of the operator H is defined to be the subset of complex energies $E \in \mathbb{C}$ for which the operator $H - E \times \mathrm{id}$ admits a bounded inverse.*

We call the complement of this set the spectrum of H. In our case, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and then the spectrum does not depend on the choice of the phase θ, so we denote it by $\Sigma = \Sigma_{\lambda,\alpha}$. Recall that the spectrum Σ is a nonempty compact subset of $\mathbb{R}$.

For a phase $\theta \in \mathbb{R}$ and a function $f \in \ell^2(\mathbb{Z})$, we let $\mu^f = \mu^f_{\lambda v,\alpha,\theta}$ be the spectral measure of $H_{\lambda v,\alpha,\theta}$ associated with f. It is defined by the relation

$$\langle (H - E \times \mathrm{id})^{-1} f, f \rangle_{\ell^2(\mathbb{Z})} = \int_{\mathbb{R}} \frac{1}{E' - E} d\mu^f(E')$$

for any $E \in \mathbb{C} \setminus \Sigma$ in the resolvent set.

Let $\{e_n\}_{n \in \mathbb{Z}}$ denote the canonical basis of $\ell^2(\mathbb{Z})$. It is possible to see that $\{e_{-1}, e_0\}$ is a generating basis of $\ell^2(\mathbb{Z})$, that is, there is no proper subset of $\ell^2(\mathbb{Z})$ which is invariant by H and contains $\{e_{-1}, e_0\}$. Let us define *the* spectral measure $\mu := \mu^{e_{-1}} + \mu^{e_0}$. It follows that the support of μ is Σ.

Let $E \in \Sigma$. Although the eigenvalue equation

$$Hu = Eu$$

does not always admit a solution in $\ell^2(\mathbb{Z})$, there exist a lot of solutions with bounded growth. It is the content of the theorem of Berezansky that we now recall.

Theorem 1.2 (Berezansky's theorem)**.** *For μ-almost every $E \in \Sigma$, there exists a solution $u = (u_n)_{n \in \mathbb{Z}}$ such that for every $n \in \mathbb{Z}$,*

$$u_{n+1} + u_{n-1} + \lambda v(\theta + n\alpha) u_n = E u_n, \tag{1}$$

and $|u_n| = O((1+|n|)^{1/2+\varepsilon})$, $\varepsilon > 0$. Conversely, if u has at most polynomial growth and the previous equation holds, then $E \in \Sigma$.

This shows that the spectrum Σ is intimately related to the existence of generalized eigenfunctions of the operator H. Note that for a sequence $u = (u_n)_{n \in \mathbb{Z}}$, Equation (1) can be rewritten as follows:

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \begin{pmatrix} E - \lambda v(\theta + n\alpha) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}.$$

Therefore there is a strong connection between the spectral properties of the Schrödinger operator H and the dynamical properties of the *Schrödinger cocycle* $(\alpha, S_{\lambda v,E}) \colon \mathbb{R}/\mathbb{Z} \times \mathbb{C}^2 \to \mathbb{R}/\mathbb{Z} \times \mathbb{C}^2$, where

$$(\alpha, S_{\lambda v,E}) \colon (x, w) \mapsto \left(x + \alpha, \begin{pmatrix} E - \lambda v(x) & -1 \\ 1 & 0 \end{pmatrix} \cdot w \right).$$

1.2. **Cocycles and hyperbolicity.** Let us denote $A := S_{\lambda v,E}$. Since v is taken analytic, we have $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}_2(\mathbb{C}))$. Then the iterates of the previous cocycle have the form $(\alpha, A)^n = (n\alpha, A_n)$, where

$$A_n(x) := \begin{cases} A(x+(n-1)\alpha)\cdots A(x+\alpha)A(x) & \text{for} \quad n \geq 0, \\ A^{-1}(x+n\alpha)A^{-1}(x+(n+1)\alpha)\cdots A^{-1}(x-\alpha) & \text{for} \quad n < 0. \end{cases}$$

The *Lyapunov exponent* is given by the formula

$$L(\alpha, A) := \lim_{n\to\infty} \frac{1}{n} \int \ln \|A_n(x)\| dx.$$

Since we take α irrational, $x \mapsto x+\alpha$ is uniquely ergodic and we have

$$L(\alpha, A) = \lim_{n\to\infty} \sup_{x\in\mathbb{R}/\mathbb{Z}} \frac{1}{n} \ln \|A_n(x)\|.$$

Since $A = S_{\lambda v,E}$, the Lyapunov exponent depends on the energy E, so we denote $L(E) := L(\alpha, S_{\lambda v,E})$.

The cocycle (α, A) is *uniformly hyperbolic* if for every $x \in \mathbb{R}/\mathbb{Z}$ there exists a continuous splitting $\mathbb{C}^2 = E^s(x) \oplus E^u(x)$ such that for some constants $C > 0$, $c > 0$, and for every $n \geq 0$,

$$\begin{aligned} \|A_n(x) \cdot w\| &\leq Ce^{-cn}\|w\|, \quad w \in E^s(x), \\ \|A_n(x)^{-1} \cdot w\| &\leq Ce^{-cn}\|w\|, \quad w \in E^u(x+n\alpha). \end{aligned}$$

This splitting is invariant by the dynamics, which means that for every $x \in \mathbb{R}/\mathbb{Z}$, $A(x) \cdot E^*(x) = E^*(x+\alpha)$, for $* = s, u$. In this case, it is clear that we have $L(\alpha, A) > 0$. Uniform hyperbolicity can be characterized by a cone-field criterion and is a robust property: the set $\mathcal{UH} \subset \mathbb{R} \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}_2(\mathbb{C}))$ of uniformly hyperbolic cocycles is open.

One important link between the spectrum of $H_{\lambda v,\alpha,\theta}$ and the dynamics of $(\alpha, S_{\lambda,E})$ is the fact that $\Sigma_{\lambda v,\alpha}$ corresponds to the set of energies E such that the cocycle $(\alpha, S_{\lambda,E})$ is not uniformly hyperbolic.

Let us also recall the definition given in [A2] and [A3], which distinguishes between different kinds of growth of the iterates of the cocycle $(\alpha, S_{\lambda,E})$ in the case it is not uniformly hyperbolic.

Definition 1.3. *When $E \in \Sigma$, cocycles $(\alpha, A) = (\alpha, S_{\lambda,E})$ are classified in three different regimes:*

1. *supercritical, or nonuniformly hyperbolic, when the cocycle (α, A) exhibits a positive Lyapunov exponent;*
2. *subcritical, if the cocycle iterates $\|A_n(x)\|$ are uniformly subexponentially bounded through some strip $|\mathrm{Im}(x)| < c$, $c > 0$;*
3. *critical otherwise.*

1.3. **Conjugacies and (almost) reducibility.** Given two analytic cocycles $(\alpha, A^{(1)})$ and $(\alpha, A^{(2)}) \in \mathbb{R} \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}_2(\mathbb{C}))$, but not necessarily of Schrödinger type, an *analytic (complex) conjugacy* between them is an analytic cocycle $(0, Z)$, where $Z \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}_2(\mathbb{C}))$, and

$$(0, Z)^{-1} \circ (\alpha, A^{(1)}) \circ (0, Z) = (\alpha, A^{(2)}).$$

In other terms, for every $x \in \mathbb{R}/\mathbb{Z}$,

$$Z(x+\alpha)^{-1} A^{(1)}(x) Z(x) = A^{(2)}(x).$$

When the function A has values in $\mathrm{SL}_2(\mathbb{R})$, there is an analogous notion of real conjugacy, i.e. in the case where $Z \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}_2(\mathbb{R}))$. Many dynamical properties are preserved by conjugacies, in particular the Lyapunov exponent.

Definition 1.4 (Reducibility)**.** *The analytic cocycle (α, A) is (C^ω-)reducible if it can be analytically conjugated to a cocycle (α, A_*) for some constant matrix A_*.*

Eliasson [E] has shown that when the frequency α is Diophantine and A is close to a constant, the associate cocycle is typically reducible in a measure-theoretic sense.

For real cocycles, the interest of allowing the function Z to take values in $\mathrm{PSL}_2(\mathbb{R})$ is illustrated for instance by the following reducibility result in the uniformly hyperbolic case: indeed, the conjugacy involves the stable and unstable directions E^*, $* = s, u$, and it is not always possible to take it with values in $\mathrm{SL}_2(\mathbb{R})$ instead of $\mathrm{PSL}_2(\mathbb{R})$.

Theorem 1.5. *Let (α, A) be a uniformly hyperbolic cocycle, with α Diophantine and A analytic. Then there exists an analytic map $Z\colon \mathbb{R}/\mathbb{Z} \to \mathrm{PSL}_2(\mathbb{R})$ such that $x \mapsto Z(x+\alpha)^{-1}A(x)Z(x)$ is constant.*

Avila and Krikorian also introduced a weak variant of the notion of reducibility.

Definition 1.6 (Almost reducibility)**.** *An analytic cocycle (α, A) is (C^ω-)almost reducible if it can be conjugated as close as we want to a constant cocycle: for any $\varepsilon > 0$, there exists a constant matrix A_* and an analytic conjugacy Z such that*

$$\sup_{x\in\mathbb{R}/\mathbb{Z}} \|Z(x+\alpha)^{-1}A(x)Z(x) - A_*\| \leq \varepsilon.$$

1.4. **The fibered rotation number.** In this part, we restrict ourselves to the case of a real cocycle $(\alpha, A) \in (\mathbb{R}\setminus\mathbb{Q}) \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}_2(\mathbb{R}))$. Such a cocycle acts naturally on the circle; this allows to define a notion of rotation number, which is intimately related to spectral gaps as we will see.

We introduce the projective skew-product $F_A\colon \mathbb{R}/\mathbb{Z} \times \mathbb{S}^1 \to \mathbb{R}/\mathbb{Z} \times \mathbb{S}^1$, where

$$F_A(x, w) := \left(x + \alpha, \frac{A(x)\cdot w}{\|A(x)\cdot w\|}\right).$$

If the map $A\colon \mathbb{R}/\mathbb{Z} \to \mathrm{SL}_2(\mathbb{R})$ is homotopic to the identity, then it is also the case of F_A. It is therefore possible to lift the latter to a map $\widetilde{F}_A\colon \mathbb{R}/\mathbb{Z}\times\mathbb{R} \to \mathbb{R}/\mathbb{Z}\times\mathbb{R}$ of the form $\widetilde{F}_A(x,y) = (x+\alpha, y + \psi_x(y))$, and for every $x \in \mathbb{R}/\mathbb{Z}$, ψ_x is $\mathbb{Z}$-periodic. If $\pi\colon (\mathbb{R}/\mathbb{Z})\times\mathbb{R} \to (\mathbb{R}/\mathbb{Z})\times\mathbb{S}^1$ denotes the projection $(x,y) \mapsto (x, e^{2i\pi y})$, then

$$F_A \circ \pi = \pi \circ \widetilde{F}_A.$$

The map $\psi\colon (\mathbb{R}/\mathbb{Z})\times(\mathbb{R}/\mathbb{Z}) \to \mathbb{R}$ is called a *lift* of A. Let μ be any probability on $\mathbb{R}/\mathbb{Z}\times\mathbb{R}$ invariant by the map $\widetilde{F}_A$, and whose projection on the first coordinate is given by Lebesgue measure. Then the number

$$\rho(\alpha, A) := \int \psi_x(y)\ d\mu(x,y) \bmod \mathbb{Z}$$

does not depend on the choices neither of the lift ψ nor of the measure μ, and is called the *fibered rotation number* of (α, A) (see [JM] for more details).

The fibered rotation number is invariant under real conjugacies which are homotopic to the identity. In fact, a more general result also holds. Recall that the fundamental group of $\mathrm{SL}_2(\mathbb{R})$ is isomorphic to $\mathbb{Z}$. Let

$$R_\theta := \begin{pmatrix} \cos(2\pi\theta) & -\sin(2\pi\theta) \\ \sin(2\pi\theta) & \cos(2\pi\theta) \end{pmatrix}.$$

Any $A\colon \mathbb{R}/\mathbb{Z} \to \mathrm{SL}_2(\mathbb{R})$ is homotopic to a map $x \mapsto R_{nx}$ for some $n =: \deg(A) \in \mathbb{Z}$ called the *degree* of A. Let $(\alpha, A^{(1)})$ and $(\alpha, A^{(2)})$ be two real cocycles homotopic to the identity, and assume they are conjugated by some map $Z\colon \mathbb{R}/\mathbb{Z} \to \mathrm{PSL}_2(\mathbb{R})$, i.e., $Z(\cdot+\alpha)^{-1}A^{(1)}(\cdot)Z(\cdot) = A^{(2)}(\cdot)$. Let k denote the degree of Z; equivalently,

Z is homotopic to $x \mapsto R_{\frac{kx}{2}}$ because now, Z takes value in $\mathrm{PSL}_2(\mathbb{R})$. Since both $(\alpha, A^{(1)})$ and $(\alpha, A^{(2)})$ are homotopic to the identity, they have well-defined fibered rotation numbers, and the following formula relates one to the other:

$$\rho(\alpha, A^{(1)}) = \rho(\alpha, A^{(2)}) + \frac{k\alpha}{2}. \tag{2}$$

We introduce the *frequency module* $\mathcal{M} := \alpha\mathbb{Z} \oplus \mathbb{Z}$.

Theorem 1.7 (Gap-labelling theorem). *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. If the cocycle (α, A) is uniformly hyperbolic, then $2\rho(\alpha, A) \in \mathcal{M}$.*

More general instances of this result can be found for example in [ABD], when the base dynamics is not necessarily an irrational rotation but only a minimal transformation.

In the case of Schrödinger cocycles $(\alpha, A) = (\alpha, S_{\lambda v,E})$, the rotation number depends on the energy E in a continuous fashion, and we denote $\rho(E) := \rho(\alpha, S_{\lambda v,E})$. Theorem 1.7 implies that if the energy $E \in \mathbb{R}$ belongs to the resolvent set, then $2\rho(E) \in \mathcal{M}$. Moreover, Johnson and Moser [JM] have shown that the function ρ is constant precisely on the set $\mathbb{R} \setminus \Sigma$.

By definition, a *spectral gap* is a connected component of $\mathbb{R} \setminus \Sigma$. Since the rotation number depends continuously on the energy, it follows from the gap-labelling theorem that on a spectral gap G, there exists an integer $m \in \mathbb{Z}$ such that $2\rho_{|G} \equiv m\alpha \mod \mathbb{Z}$; we call m the *label* of the gap G. We will then denote by $G_0 := (-\infty, \underline{E})$ and $G_m := (E_m^-, E_m^+)$, $m \neq 0$ the spectral gaps, where $2\rho_{|G_m} \equiv m\alpha \mod \mathbb{Z}$.

1.5. **Dual Schrödinger operators and (almost) localization.** We introduce the *dual Schrödinger operator* $\hat{H} = \hat{H}_{\lambda v,\alpha,\theta}$ defined on $\ell^2(\mathbb{Z})$ by the following formula: for every $n \in \mathbb{Z}$,

$$(\hat{H}\hat{u})_n := \sum_{k\in\mathbb{Z}} \lambda \hat{v}_k \hat{u}_{n-k} + 2\cos(2\pi(\theta + n\alpha))\hat{u}_n,$$

where the $\hat{v}_k$'s are the Fourier coefficients of the analytic map $v\colon x \mapsto \sum \hat{v}_k e^{2i\pi kx}$.

In the particular case of the almost Mathieu operator, that is when $v\colon x \mapsto 2\cos(2\pi x)$, we have $\hat{H}_{\lambda v,\alpha,\theta} = \lambda H_{\lambda^{-1}v,\alpha,\theta}$; this reflects an important symmetry known as *Aubry duality* and whose main features we are going to detail in the following.

In the case of a general analytic potential v, Aubry duality implies that the spectrum $\hat{\Sigma}$ of $\hat{H}_{\lambda v,\alpha,\theta}$ coincides with the spectrum Σ of $H_{\lambda v,\alpha,\theta}$. Since $\Sigma = \hat{\Sigma}$ is the union of the supports of the spectral measures, it follows from Berezansky's theorem:

Theorem 1.8. *For any $\theta \in \mathbb{R}$, there exists a dense set of $E \in \Sigma_{\lambda v,\alpha}$ such that there exists a nonzero solution $\hat{H}_{\lambda v,\alpha,\theta}\hat{u} = E\hat{u}$ with $|\hat{u}_k| \leq 1 + |k|$, $k \in \mathbb{Z}$.*

We now come to the notion of *localization.*

Definition 1.9 (Anderson localization). *We say that the operator $\hat{H}_{\lambda v,\alpha,\theta}$ displays Anderson localization if it has pure point spectrum with exponentially decaying eigenvectors.*

The notion of localization is defined similarly for a Schrödinger operator H but we have stated it here for the dual model since it is in this setting that we are going to use it in the following. Let us recall the definition of a resonance.

Definition 1.10 (Resonance). *Let $\theta \in \mathbb{R}$, $\varepsilon_0 > 0$. We say that $k \in \mathbb{Z}$ is an ε_0-resonance if $\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq e^{-\varepsilon_0|k|}$ and $\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} = \min_{|j|\leq|k|} \|2\theta - j\alpha\|_{\mathbb{R}/\mathbb{Z}}$.*

In particular, 0 is always a resonance. Moreover, if α is Diophantine, the first condition automatically implies the second for $|k|$ sufficiently large.

Definition 1.11 (Resonant phase)**.** *We order the ε_0-resonances: $0 = n_0 < |n_1| \leq |n_2| \leq \dots$ We say that θ is ε_0-resonant if the set of resonances is infinite. If θ is non-resonant, with resonances $\{n_0, \dots, n_J\}$, we set $n_{J+1} := \infty$.*

The Diophantine condition on α implies exponential repulsion of resonances:

Lemma 1.12. *If α is Diophantine, then there exists a constant $c > 0$ such that $|n_{j+1}| \geq c\|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}}^{-c} \geq ce^{c\varepsilon_0|n_j|}$.*

We say that a phase θ is *rational* when $2\theta = n\alpha \mod \mathbb{Z}$ for some $n \in \mathbb{Z}$. In this case, we have an especially strong resonance at n, and θ is non-resonant since there is no resonance n' with $|n'| > |n|$.

Definition 1.13 (Almost localization)**.** *The family $\{\hat{H}_{\lambda v,\alpha,\theta}\}_{\theta\in\mathbb{R}}$ exhibits almost localization if for some $\varepsilon_0 > 0$, the eigenvectors decay exponentially fast in definite intervals between two successive ε_0-resonances.*

More precisely, denote by $\{n_j\}_j$ the set of ε_0-resonances of θ. Then $\{\hat{H}_{\lambda v,\alpha,\theta}\}_{\theta\in\mathbb{R}}$ is almost localized if for $\varepsilon_0 > 0$ sufficiently small, there exist constants $C_0 > 0$, $C_1 > 0$ and $\varepsilon_1 > 0$ such that for every solution $\hat{u}$ of $\hat{H}_{\lambda v,\alpha,\theta}\hat{u} = E\hat{u}$ satisfying $\hat{u}_0 = 1$ and $|\hat{u}_k| \leq 1 + |k|$, and for every $C_0(1 + |n_j|) < k < C_0^{-1}|n_{j+1}|$, we have $|\hat{u}_k| \leq C_1 e^{-\varepsilon_1 k}$.

Remark 1.14. *From Berezansky's theorem applied to $\hat{H}_{\lambda v,\alpha,\theta}$, we see that almost localization implies localization for non-resonant phases θ.*

Recall that we assume that α is Diophantine and v is analytic. Under these assumptions, Avila and Jitomirskaya have obtained:

Theorem 1.15 (Theorem 3.2, [AJ2])**.** *There exists $\lambda_0 = \lambda_0(v) > 0$ such that for $|\lambda| < \lambda_0$, the family $(\hat{H}_{\lambda v,\alpha,\theta})_{\theta\in\mathbb{R}}$ is almost localized. In the case of the almost Mathieu operator, i.e. when $v\colon x \mapsto 2\cos(2\pi x)$, we have $\lambda_0 = 1$.*

1.6. **Localization and reducibility.** As mentioned previously, there are strong connections between localization properties of dual Schrödinger operators and reducibility of Schrödinger cocycles. This fact is often referred to as *Aubry duality* in the literature. We give here the precise statement of this remarkable property as it is formulated in Theorem 2.5, [AJ2], in the case of a given energy. Recall that $\mathcal{M} := \alpha\mathbb{Z} \oplus \mathbb{Z}$ denotes the frequency module.

Theorem 1.16 (Classical Aubry duality)**.** *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $v\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be an analytic potential. Let $\theta \in \mathbb{R}$, $E \in \mathbb{R}$ be such that there exists a nonzero exponentially decaying solution $\hat{u}$ to $\hat{H}_{\lambda v,\alpha,\theta}\hat{u} = E\hat{u}$, and let $A := S_{\lambda v,E}$.*

1. *If $2\theta \notin \mathcal{M}$, then there exists $Z \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}_2(\mathbb{R}))$ such that for every $x \in \mathbb{R}/\mathbb{Z}$, $Z(x+\alpha)^{-1}A(x)Z(x) = R_{\pm\theta}$, i.e. (α, A) is reducible to a constant rotation cocycle. In particular, $\|A_n(x)\| = O(1)$ for every $x \in \mathbb{R}/\mathbb{Z}$.*
2. *If $2\theta \in \mathcal{M}$, then there exist $Z \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}_2(\mathbb{R}))$ and $\kappa \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ such that for every $x \in \mathbb{R}/\mathbb{Z}$, $Z(x + \alpha)^{-1}A(x)Z(x) = \begin{pmatrix} \pm 1 & \kappa(x) \\ 0 & \pm 1 \end{pmatrix}$. In particular $\|A_n(x)\| = O(n)$ for every $x \in \mathbb{R}/\mathbb{Z}$. When α is Diophantine, κ can be chosen to be constant and (α, A) is thus reducible to a constant cocycle (α, B) for some parabolic matrix $B = \begin{pmatrix} \pm 1 & \kappa \\ 0 & \pm 1 \end{pmatrix}$.*

In particular, this together with property (2) imply that when E is an energy for which $\hat{H}$ is localized, then $2\rho = \pm 2\theta \mod \mathcal{M}$.

The proof of this result involves an algebraic relation between the families of operators $\{H_{\lambda v,\alpha,\theta}\}_{\theta\in\mathbb{R}}$ and $\{\hat{H}_{\lambda v,\alpha,\theta}\}_{\theta\in\mathbb{R}}$ which associates a *Bloch wave* with every eigenvector. This is based on the following identity: assume $u\colon \mathbb{R}/\mathbb{Z}\to\mathbb{C}$ is an $\ell^2(\mathbb{Z})$ function whose Fourier series satisfies $\hat{H}_{\lambda v,\alpha,\theta}\hat{u} = E\hat{u}$. Define $U\colon x\mapsto \begin{pmatrix} e^{2i\pi\theta}u(x) \\ u(x-\alpha)\end{pmatrix}$; then for any $x\in\mathbb{R}/\mathbb{Z}$,

$$S_{\lambda,E}(x)\cdot U(x) = e^{2i\pi\theta}U(x+\alpha). \tag{3}$$

In [AJ2], Avila and Jitomirskaya also show that an analogous duality holds between the almost localization properties of $\hat{H}$ and the almost reducibility of associate Schrödinger cocycles. Using Theorem 1.15, they get the following result:

Theorem 1.17 (Theorem 1.4, [AJ2]). *Assume that the frequency α is Diophantine and that the potential $v\colon \mathbb{R}/\mathbb{Z}\to\mathbb{R}$ is analytic. Then for $0<|\lambda|<\lambda_0(v)$, the cocycles associated with $\{H_{\lambda v,\alpha,\theta}\}_{\theta\in\mathbb{R}}$ are almost reducible. If moreover $v\colon x\mapsto 2\cos(2\pi x)$, then $\lambda_0(v)=1$.*

In fact they obtain a more precise statement: under the same assumptions,

Theorem 1.18 (Theorem 4.1, [AJ2]). *Assume $0<|\lambda|<\lambda_0(v)$. Denote $A := S_{\lambda v,E}$. Then there exists $c=c(\lambda,v,\alpha)>0$ with the following property:*

(1) *If $\rho(E)$ is c-resonant, then there exists a sequence $Z^{(n)}\in C^\omega(\mathbb{R}/\mathbb{Z},\mathrm{SL}_2(\mathbb{R}))$ such that $x\mapsto Z^{(n)}(x+\alpha)^{-1}A(x)Z^{(n)}(x)$ converges to a constant rotation uniformly in $\{|\mathrm{Im}(x)|<c\}$.*
(2) *If $\rho(E)$ is not c-resonant and $2\rho\notin\mathcal{M}$, then there exists $Z\in C^\omega(\mathbb{R}/\mathbb{Z},\mathrm{SL}_2(\mathbb{R}))$ which can be analytically extended to $\{|\mathrm{Im}(x)|<c\}$ and such that $x\mapsto Z(x+\alpha)^{-1}A(x)Z(x)$ is a constant rotation.*
(3) *If $2\rho(E)\in\mathcal{M}$, then A is (analytically) reducible.*

For a global picture on reducibility results, we refer to the paper of You and Zhou [YZ].

2. Reducibility on the boundary of a spectral gap

2.1. **Reducibility to a constant parabolic cocycle.**

Let us consider the spectral gap G_m associated with a rotation number ρ satisfying $2\rho = m\alpha \mod \mathbb{Z}$. We consider an energy $E\in\Sigma$ on the boundary of this gap. By continuity of the rotation number, we know that $2\rho(E) = m\alpha \mod \mathbb{Z}$. In [AJ2], Avila and Jitomirskaya derive the following result from Theorem 1.8.

Theorem 2.1 (Theorem 3.3, [AJ2]). *If $E'\in\Sigma$ then there exists $\theta=\theta(E')\in\mathbb{R}$ and a bounded solution $\hat{u}$ to $\hat{H}_{\lambda v,\alpha,\theta}\hat{u} = E'\hat{u}$ with $\hat{u}_0=1$ and $|\hat{u}_n|\le 1$, $n\in\mathbb{Z}$.*

In the following we choose some [3] phase $\theta=\theta(E)$ satisfying the conclusions of the last theorem. Taking $|\lambda|<\lambda_0(v)$, we know from Theorem 1.15 that $\hat{H}_{\lambda v,\alpha,\theta}$ is almost localized.

Using Theorem 1.16 and Theorem 1.18, Avila and Jitomirskaya show the following relation between the fibered rotation number $\rho(E)$ and the phase $\theta(E)$:

Corollary 2.2. *Let $\varepsilon_0>0$. Fix $E'\in\Sigma$ and a phase $\theta(E')$ as before. If $\theta(E')$ is not ε_0-resonant, then $2\rho(E') = \pm 2\theta(E') \mod \mathcal{M}$. On the contrary, if $\theta(E')$ is ε_0-resonant, then $\rho(E')$ is c-resonant for some $c>0$.*

In our case, $\rho(E)$ is rational, hence non-resonant, so from Corollary 2.2 we see that $\theta(E)$ is also non-resonant. Then Remark 1.14 implies that $\hat{H}_{\lambda v,\alpha,\theta(E)}$ is in fact localized and Theorem 1.16 therefore applies. The previous discussion can thus be summarized as follows:

3. As explained in [AJ2], such a phase is not necessarily unique.

Proposition 2.3. *For an energy $E \in \Sigma$ on the boundary of a spectral gap, the Schrödinger cocycle $(\alpha, S_{\lambda v,E})$ is reducible to a constant*[4] *cocycle (α, B) for some parabolic matrix* $B = \begin{pmatrix} \pm 1 & \kappa \\ 0 & \pm 1 \end{pmatrix}$.

In the following, we will recall the results obtained in [AJ2] and [A1], which provide us with a uniform control on κ and on the conjugacies with respect to the last resonance of the phase θ.

2.2. **Lower bounds on truncated Bloch waves.** In [AJ2], Avila and Jitomirskaya consider truncated Bloch waves, obtained by cutting off high frequency modes. The index where the truncation is performed is chosen between two successive resonances. These truncated Bloch waves behave almost as well as the usual ones under the action of the cocycle: the error term decays exponentially fast with the index of the truncation due to almost localization of the dual Schrödinger operator. Moreover the estimates are uniform in E; indeed they are based on almost localization results, and the constants that appear in Definition 1.13 are independent of E. Thanks to the truncated Bloch waves, they construct conjugacies that allow them to prove some almost reducibility result. Their estimates start by rough bounds on the growth of the cocycle that come from subcriticality. Then they use a bootstrap argument to gradually improve the estimates on the cocycle and on the Bloch waves, which result in a good control on the matrix obtained after conjugation, and on the conjugacies.

Recall that we work with a Diophantine frequency, i.e., $\alpha \in \mathrm{DC}$. We assume that the potential is small, that is, we take λ with $|\lambda| < \lambda_0(v)$ as in Theorem 1.15. This implies almost localization of the family $\{\hat{H}_{\lambda v,\alpha,\theta}\}_{\theta\in\mathbb{R}}$. We also fix $\varepsilon_0 > 0$ small.

Let us consider an energy $E \in \Sigma$ in the spectrum and denote $A := S_{\lambda,E}$. We choose some phase $\theta = \theta(E)$ and a bounded solution of $\hat{H}\hat{u} = E\hat{u}$ as in Theorem 2.1. We denote by $\{n_j\}_j$ the set of ε_0-resonances of θ. In the following, C, respectively c denote large, respectively small constants whose actual value may change in the course of calculations, but which are uniform in the energy; this is crucial for the result we want to show on the size of spectral gaps since we will later have to let E vary. Avila and Jitomirskaya obtained the following result in this case:

Theorem 2.4 (Theorem 6.2, [AJ2]). *We have $L(\alpha, A) = 0$.*

In our context, this implies that the growth of the iterates of the cocycle is subexponential on a strip: for any $\delta > 0$, there exist $c_\delta > 0$ and $C_\delta > 0$ which may depend on λ, α but not on E, and such that for any $k \in \mathbb{N}$,

$$\sup_{|\mathrm{Im}(x)|<c_\delta} \|A_k(x)\| \leq C_\delta e^{\delta k}. \tag{4}$$

Let $\left(\frac{p_n}{q_n}\right)_n$ denote the continued fraction approximants to α. We have the following properties:

$$\|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} = \inf_{1\leq k\leq q_{n+1}-1} \|k\alpha\|_{\mathbb{R}/\mathbb{Z}},$$
$$1 \geq q_{n+1}\|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq 1/2.$$

We also define:

$$\beta = \beta(\alpha) := \limsup_{n\to\infty} \frac{\ln(q_{n+1})}{q_n}.$$

In the case we consider where α is Diophantine, we are in the *subexponential* regime, that is, $\beta = 0$. This implies

$$q_{n+1} \leq e^{o(q_n)}.$$

4. Recall that we work with a Diophantine frequency α.

Given a sequence of coefficients $\hat{w} = (\hat{w}_k)_{k\in\mathbb{Z}}$ and an interval $I \subset \mathbb{Z}$, we let $w^I : x \mapsto \sum_{k\in I} \hat{w}_k e^{2i\pi kx}$. We denote the length of an interval $I = [a, b]$ by $|I| := b-a$. A trigonometrical polynomial $p\colon \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ has *essential degree* at most k if its Fourier coefficients outside an interval I of length k are vanishing.

We choose n between two resonances of the phase θ, i.e., $C_0|n_j| < n < C_0^{-1}|n_{j+1}|$, $j \in \mathbb{N}$, where n is of the form $n = rq_l - 1 < q_{l+1}$. Let $I := [-[n/2], n - [n/2]]$, and with the notation introduced before, define $u := u^I$. We consider the truncated Bloch wave $U\colon x \mapsto \begin{pmatrix} e^{2i\pi\theta}u(x) \\ u(x-\alpha)\end{pmatrix}$. Relation (3) becomes now:

$$A(x)\cdot U(x) - e^{2i\pi\theta}U(x+\alpha) = e^{2i\pi\theta}\begin{pmatrix} h(x) \\ 0\end{pmatrix}, \tag{5}$$

where for $k \in \mathbb{Z}$,

$$\hat{h}_k = \chi_I(k)\,[E - 2\cos(2\pi(\theta + k\alpha))]\,\hat{u}_k - \sum_{j\in\mathbb{Z}} \chi_I(k-j)\lambda\hat{v}_j\hat{u}_{k-j}.$$

Since $\hat{H}\hat{u} = E\hat{u}$, we then get

$$-\hat{h}_k = \chi_{\mathbb{Z}\setminus I}(k)\,[E - 2\cos(2\pi(\theta + k\alpha))]\,\hat{u}_k - \sum_{j\in\mathbb{Z}} \chi_{\mathbb{Z}\setminus I}(k-j)\lambda\hat{v}_j\hat{u}_{k-j}.$$

Almost localization of $\hat{H}$ implies that for $C_0^{-1}n < |k| < C_0 n$, we have $|\hat{u}_k| \leq C_1 e^{-\varepsilon_1|k|}$, $C_1, \varepsilon_1 > 0$. We also know that for every k, $|\hat{u}_k| \leq 1$. Moreover, by analyticity of the potential, the coefficients $\hat{v}_k$ decay exponentially fast with $|k|$. We thus obtain:

$$\|h\|_c \leq Ce^{-cn}, \tag{6}$$

where the constants $C, c > 0$ are uniform with respect to the energy. Let us recall the following result, which gives a control on the norm of a trigonometrical polynomial by the values it takes at a finite number of points:

Theorem 2.5 (Theorem 6.1, [AJ2])**.** *Let $1 \leq r \leq [q_{n+1}/q_n]$. If p has essential degree $k = rq_n - 1$, and $x_0 \in \mathbb{R}/\mathbb{Z}$, then*

$$\|p\|_0 \leq Cq_{n+1}^{Cr} \sup_{0\leq j\leq k} |p(x_0 + j\alpha)|.$$

Since in our case $\beta = 0$, we then get:

$$\|p\|_0 \leq Ce^{o(k)} \sup_{0\leq j\leq k} |p(x_0 + j\alpha)|. \tag{7}$$

In the following result, δ and δ_0 are two positive constants chosen much smaller than c.

Theorem 2.6 (Theorem 3.7, [A1])**.** *We have* $\inf_{|\mathrm{Im}(x)|<\delta_0} \|U(x)\| \geq ce^{-\delta n}$.

Proof. Otherwise for any $\delta > 0$, there exists $x_0 \in \mathbb{R}/\mathbb{Z}$ with $\mathrm{Im}(x_0) = t$, $|t| < \delta_0$, such that $\|U(x_0)\| < ce^{-\delta n}$. Iterating relation (5), we get for $k \geq 0$,

$$U(x_0+k\alpha) = e^{-2ik\pi\theta}A_k(x_0)\cdot U(x_0) - \sum_{l\leq k-1} e^{-2il\pi\theta}A_l(x_0+\alpha)\begin{pmatrix} h(x_0 + (k-l-1)\alpha) \\ 0\end{pmatrix}.$$

Using (4) and (6), we thus get that for every $0 \leq k \leq n$, $|u(x_0 + k\alpha)| \leq ce^{-\delta n/2}$. Then estimate (7) implies that $\|u_t\|_0 \leq ce^{-\delta/5}$, where for $x \in \mathbb{R}/\mathbb{Z}$, $u_t(x) := u(x + ti)$. This contradicts the fact that $\int_{\mathbb{R}/\mathbb{Z}} u_t(x)dx = 1$. □

The following result is due to Uchiyama; it tells us that whenever $\widetilde{U}\colon \mathbb{R}/\mathbb{Z} \to \mathbb{C}^2$ is an analytic function bounded away from 0, it is possible to construct an analytic map with values in $\mathrm{SL}_2(\mathbb{C})$ which has $\widetilde{U}$ as first column.

Theorem 2.7. *Let* $\widetilde{U}\colon \mathbb{R}/\mathbb{Z} \to \mathbb{C}^2$ *be an analytic function such that for every* x, $|\mathrm{Im}(x)| < a$, *we have* $\delta_1 \le \|\widetilde{U}(x)\| \le \delta_2^{-1}$, $\delta_1, \delta_2 > 0$. *Then there exists* $\widetilde{B}\colon \mathbb{R}/\mathbb{Z} \to \mathrm{SL}_2(\mathbb{C})$ *with first column* $\widetilde{U}$ *and such that* $\|\widetilde{B}\|_a \le C\delta_1^{-2}\delta_2^{-1}(1-\ln\delta_1\delta_2)$.

Applying this result to the truncated Bloch wave U, Avila was able to improve the upper bound on the growth of the cocycle (α, A):

Theorem 2.8. [A1] *For some constants* $C, c > 0$,

$$\sup_{0 \le s \le ce^{cn}} \|A_s\|_{\delta_0} \le Ce^{cn}.$$

Recall the following classical result, which states repulsion of resonances in the case of subexponential regime.

Lemma 2.9. *If* $\beta = 0$, *then* $o(|n_{j+1}|) \ge -\ln\|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge c|n_j|$.

Let $m \ge C$; choose n minimal such that $m \le ce^{cn}$, $C|n_j| < n < C^{-1}|n_{j+1}|$ and $n = rq_l - 1 < q_{l+1}$ for some j, l, r. By Lemma 2.9, we see that $n \le C\ln(m)$ so we finally conclude that the iterates of the cocycle (α, A) grow at most polynomially:

Theorem 2.10 (Theorem 3.4, [A1])**.** *We have* $\|A_m\|_c \le Cm^C$.

Thanks to these estimates, Avila improves the previous lower bound on the truncated Bloch wave U:

Theorem 2.11 (See the proof of Theorem 3.8, [A1])**.** *Fix some* $n := |n_j|$ *and let* $N := |n_{j+1}|$ *if defined, otherwise let* $N := +\infty$. *As above, we consider the truncated Bloch wave* $U = U^I$, *where* $I := [-cN, cN]$. *Then the following lower bound holds: for some* $c > 0$ *independent of the energy* E,

$$\inf_{|\mathrm{Im}(x)|<c} \|U(x)\| \ge e^{-o(n)}. \tag{8}$$

Proof. Let $u := u^I$, where $I := [-cN, cN]$. Choose $rq_l > Cn_j$ minimal such that $rq_l - 1 < q_{l+1}$ and let $J := [-[rq_l/2], rq_l - 1 - [rq_l/2]]$. As before we consider $U = U^I$, and we define U^J in a similar way. Then using the polynomial bound obtained in Theorem 2.10 instead of estimate (4) in the proof of Theorem 2.6, its conclusion can be improved to $\inf_{|\mathrm{Im}(x)|<c} \|U^J(x)\| \ge e^{-o(n)}$. But $\|U^I - U^J\|_c \le e^{-cn}$ so we get

$$\inf_{|\mathrm{Im}(x)|<c} \|U(x)\| \ge e^{-o(n)}. \tag{9}$$

□

2.3. **Quantitative reducibility on the boundary of a spectral gap.** In this part, we consider an energy E located on the boundary of a spectral gap. As we have recalled above, we know that the associate Schrödinger cocycle can be reduced to a constant parabolic cocycle. We will prove here a quantitative version of this fact; more precisely, we show that the off-diagonal coefficient of the parabolic matrix is exponentially small in terms of the label of the gap, while the conjugacy can be taken subexponentially big. This result is a consequence of the precise estimates recalled above, which give a lower bound on the Bloch waves intervening in the definition of the conjugacies. In particular, we will use the fact that these are not too small with respect to the label of the gap. It is important to note that the (implicit) constants that appear in the following are independent of the spectral gap we work with; this is a consequence of the uniformity in the previous results.

The ideas here are based on the methods developed in [AJ2] and [A1]. In a first time, we use the parametrization by the phase $\theta(E)$, and the estimates we get involve its last resonance. We then show how they can be translated in terms of the label of the spectral gap we consider.

Fix some $\varepsilon_0 > 0$. Let then $m \in \mathbb{Z} \setminus \{0\}$ and $E := E_m^+ \in \Sigma$ be the energy on the right boundary of the spectral gap G_m labeled by m. We know that $2\rho(E) = m\alpha \mod \mathbb{Z}$. Let $\theta = \theta(E)$ be chosen as usual. As explained in Subsection 2.1, $2\rho(E) = \pm 2\theta(E) \mod \mathcal{M}$; in particular, $2\theta(E) = \widetilde{n}\alpha \mod \mathbb{Z}$ for some $\widetilde{n} \in \mathbb{Z}$. Then θ is non-resonant, with an especially strong resonance at $\widetilde{n}$. We denote the set of ε_0-resonances of θ by $\{n_j\}_{0 \le j \le J}$, where $n_J = \widetilde{n}$, and we let $n := |\widetilde{n}|$. The main result of this part is the following. It tells us that we can conjugate our cocycle to a constant parabolic cocycle with a good control both on the conjugacy and the off-diagonal coefficient. The parametrization is given here in terms of the last resonance $\widetilde{n}$, but we will see that it is possible to relate it to the integer m.

Theorem 2.12. *There exist a map $Z = Z_m \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}_2(\mathbb{R}))$ as well as $\kappa = \kappa_m \in \mathbb{R}$ such that for every $x \in \mathbb{R}/\mathbb{Z}$,*

$$Z(x+\alpha)^{-1} S_{\lambda v, E_m^+}(x) Z(x) = \begin{pmatrix} \pm 1 & \kappa_m \\ 0 & \pm 1 \end{pmatrix} =: B_m. \tag{10}$$

Moreover, there exist constants $C, c > 0$ independent of m such that $|\kappa| \le Ce^{-cn}$, and $\|Z\|_0 \le e^{o(n)}$. We also have

$$|m| \le C' n \tag{11}$$

for some uniform constant $C' > 0$.

Of course a similar statement is true for values of the energy on the left boundary of a spectral gap. Let us start with the following (complex) triangularization result; it is a consequence of Theorem 3.8 in [A1] with the choice $N := +\infty$. It will be useful to obtain a good control on the off-diagonal coefficient of the parabolic matrix given by Theorem 2.12. As usual, we set $A := S_{\lambda v, E}$.

Theorem 2.13. [A1] *There exist $c > 0$ uniform in E and $Z^{(1)} \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}_2(\mathbb{C}))$ with $\|Z^{(1)}\|_c \le e^{o(n)}$ such that for any $x \in \mathbb{R}/\mathbb{Z}$,*

$$Z^{(1)}(x+\alpha)^{-1} A(x) Z^{(1)}(x) = \begin{pmatrix} e^{2i\pi\theta} & b(x) \\ 0 & e^{-2i\pi\theta} \end{pmatrix},$$

where $\|b\|_c \le e^{-cn}$. In particular, for every $0 \le k \le e^{cn}$,

$$\|A_k\|_c \le Ce^{o(n)}. \tag{12}$$

The following result is a first step towards Theorem 2.12.

Proposition 2.14. *There exist a map $Z \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}_2(\mathbb{R}))$ and $\kappa \in \mathbb{R}$ such that for every $x \in \mathbb{R}/\mathbb{Z}$,*

$$Z(x+\alpha)^{-1} A(x) Z(x) = \begin{pmatrix} \pm 1 & \kappa \\ 0 & \pm 1 \end{pmatrix}. \tag{13}$$

Moreover, $|\kappa| \le e^{o(n)}$, and $\|Z\|_0 \le e^{o(n)}$.

Proof. As previously, let $\hat{u}$ be a solution to $\hat{H}\hat{u} = E\hat{u}$. We define $u(x) := \sum_k \hat{u}_k e^{2i\pi kx}$ and $U(x) := \begin{pmatrix} e^{2i\pi\theta} u(x) \\ u(x-\alpha) \end{pmatrix}$. We take $n := |\widetilde{n}|$ and $N := +\infty$ in Theorem 2.11. In this case note that U corresponds to the Bloch wave U^I considered in the statement of this theorem. In particular, it follows from (8) that $\inf_{|\mathrm{Im}(x)| < c} \|U(x)\| \ge e^{-o(n)}$.

We let $\widetilde{\theta} := \theta - \frac{\widetilde{n}\alpha}{2}$, so that $2\widetilde{\theta} \in \mathbb{Z}$. Define

$$\widetilde{U}(x) := e^{i\pi\widetilde{n}x} U(x), \tag{14}$$

so that

$$A(x) \cdot \widetilde{U}(x) = e^{2i\pi\widetilde{\theta}} \widetilde{U}(x+\alpha) = \pm \widetilde{U}(x+\alpha).$$

Set $S := \mathrm{Re}(\widetilde{U})$ and $T := \mathrm{Im}(\widetilde{U})$; for any $x \in \mathbb{R}/\mathbb{Z}$ we get

$$A(x) \cdot S(x) = \pm S(x+\alpha), \qquad A(x) \cdot T(x) = \pm T(x+\alpha). \tag{15}$$

Since $\inf_{x\in\mathbb{R}/\mathbb{Z}} \|\widetilde{U}(x)\| \geq \inf_{|\mathrm{Im}(x)|<c} \|U(x)\| \geq e^{-o(n)}$, and by (15), we can choose $V = S$ or T such that $\inf_{x\in\mathbb{R}/\mathbb{Z}} \|V(x)\| \geq e^{-o(n)}$ holds as well.

Let $R_{1/4}$ denote the rotation $(x, y) \mapsto (-y, x)$. Since V does not vanish, we can define $Z^{(1)} \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}_2(\mathbb{R}))$, where for every $x \in \mathbb{R}/\mathbb{Z}$,

$$Z^{(1)}(x) := \begin{pmatrix} V(x) & R_{1/4}\frac{V(x)}{\|V(x)\|^2} \end{pmatrix}. \tag{16}$$

From $\inf_{x\in\mathbb{R}/\mathbb{Z}} \|V(x)\| \geq e^{-o(n)}$, it follows that $\|Z^{(1)}\|_0 \leq e^{o(n)}$. For $x \in \mathbb{R}/\mathbb{Z}$, we also have $Z^{(1)}(x)^{-1} = \begin{pmatrix} \frac{{}^{\mathrm{t}}V(x)}{\|V(x)\|^2} \\ {}^{\mathrm{t}}V(x)R_{-1/4} \end{pmatrix}$. For every $x \in \mathbb{R}/\mathbb{Z}$, we obtain

$$Z^{(1)}(x+\alpha)^{-1}A(x)Z^{(1)}(x) = \begin{pmatrix} \pm 1 & \kappa^{(1)}(x) \\ 0 & \pm 1 \end{pmatrix},$$

where $\kappa^{(1)} \colon x \mapsto \frac{{}^{\mathrm{t}}V(x+\alpha)}{\|V(x+\alpha)\|^2} A(x) R_{1/4} \frac{V(x)}{\|V(x)\|^2}$ satisfies $\|\kappa^{(1)}\|_0 \leq e^{o(n)}$. But α is Diophantine so we can solve the cohomological equation

$$\pm\phi(\cdot+\alpha) \mp \phi(\cdot) = \kappa^{(1)}(\cdot) - \int_0^1 \kappa^{(1)}(x)dx$$

with $\int_0^1 \phi(x)dx = 0$ in $\mathbb{R}/\mathbb{Z}$. We denote $\kappa := \int_0^1 \kappa^{(1)}(x)dx$. Letting $Z(x) := Z^{(1)}(x)\begin{pmatrix} 1 & \phi(x) \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{R}/\mathbb{Z}$, we get

$$Z(x+\alpha)^{-1}A(x)Z(x) = \begin{pmatrix} \pm 1 & \kappa \\ 0 & \pm 1 \end{pmatrix},$$

with $|\kappa| \leq e^{o(n)}$, as claimed. □

The following lemma improves the estimate on the off-diagonal coefficient κ.

Lemma 2.15. *Let κ be as in Proposition 2.14. We have in fact*

$$|\kappa| \leq Ce^{-cn}, \qquad C > 0,$$

where $c > 0$ is chosen as in Theorem 2.13.

Proof. If $l \in \mathbb{N}$, then iterating the previous relation, we obtain: for every $x \in \mathbb{R}/\mathbb{Z}$,

$$A_l(x) = Z(x+l\alpha)\begin{pmatrix} \pm 1 & \kappa l \\ 0 & \pm 1 \end{pmatrix} Z(x)^{-1}.$$

Recall that $\|Z\|_0 \leq e^{o(n)}$. Take $l := \lfloor e^{cn} \rfloor$ in the previous expression; we have $\|A_l\|_0 \sim |\kappa|e^{cn}$. But we also know from (12) that $\|A_l\|_0 \leq Ce^{o(n)}$; therefore, $|\kappa|e^{cn} \lesssim Ce^{o(n)}$ and the result follows. □

Let us now estimate the topological degree of the conjugacy map Z. This will enable us to link the last resonance $\widetilde{n}$ of the phase θ to the label m of the spectral gap we are studying.

Lemma 2.16. *Recall that the map U is analytic on a strip $|\mathrm{Im}(x)| < c$, $c > 0$. Then there exists $C' = C'(c) > 0$ such that the degree of Z satisfies*

$$|\deg(Z)| \leq C'n.$$

Proof. Since $x \mapsto \begin{pmatrix} 1 & \phi(x) \\ 0 & 1 \end{pmatrix}$ is homotopic to the identity, it is enough to get a bound on $\deg(Z^{(1)})$, where $Z^{(1)}$ is the map appearing in the proof of Proposition 2.14. For this we will estimate the degree of its first column V, seen here as a map $V\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2 \setminus \{0\}$. As we have noted before,

$$\inf_{x \in \mathbb{R}/\mathbb{Z}} \|V(x)\| \geq e^{-o(n)}. \tag{17}$$

Consider the Fourier expansion $V(x) = \sum_{k \in \mathbb{Z}} V_k e^{2i\pi kx}$ of V, and its truncation $\widetilde{V}(x) := \sum_{|k| \leq k_0} V_k e^{2i\pi kx}$ for some integer $k_0 \in \mathbb{N}$ to be chosen later. Since V is defined after the map U which is analytic on the strip $|\mathrm{Im}(x)| < c$, we have $|V_k| = o(e^{-2\pi|k|c})$. Hence, for $x \in \mathbb{R}/\mathbb{Z}$, $\|\widetilde{V}(x) - V(x)\| \lesssim e^{-2\pi k_0 c}$. Comparing with (17), we get a constant $C' = C'(c) > 0$ and $k_0 \leq C'n$ such that for every $x \in \mathbb{R}/\mathbb{Z}$,

$$\|\widetilde{V}(x) - V(x)\| \leq \|V(x)\|.$$

By Rouche's theorem, we deduce that $\deg(V) = \deg(\widetilde{V})$. Consider a coordinate of $\widetilde{V}$ which is not identically vanishing. It is a trigonometric polynomial of degree less than $C'n$, so it has at most $C'n$ zeroes in $\mathbb{R}/\mathbb{Z}$, and we conclude

$$|\deg(V)| \leq C'n.$$

□

Recall that $2\rho(E) = m\alpha \bmod \mathbb{Z}$ and $2\theta(E) = \widetilde{n}\alpha \bmod \mathbb{Z}$ with $n = |\widetilde{n}|$. Using (13), the upper bound on the degree of Z given by Lemma 2.16, and formula (2), which estimates how much the rotation number changes after conjugation, we obtain a link between the parametrization by the last resonance of θ and the label of the spectral gap we consider:

Corollary 2.17. *There exists $C' > 0$ uniform in m such that the following holds:*

$$|m| \leq C'n.$$

Combining the results obtained in Proposition 2.14, Lemma 2.15, and Corollary 2.17, the proof of Theorem 2.12 is now complete.

2.4. **Perturbation of the cocycle near the boundary of a spectral gap.** Let us assume that $(\alpha, S_{\lambda v,E})$ is reducible to a constant cocycle (α, B) for some parabolic matrix $B = \begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix}$ with $|\kappa| \ll 1$ through an analytic conjugacy $Z \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}_2(\mathbb{R}))$, where $Z = (z_{ij})_{1 \leq i,j \leq 2}$. In other terms, for every $x \in \mathbb{R}/\mathbb{Z}$,

$$Z(x+\alpha)^{-1} S_{\lambda v,E}(x) Z(x) = B. \tag{18}$$

Lemma 2.18. *The coefficients (z_{ij}) satisfy the following relations: for any $x \in \mathbb{R}/\mathbb{Z}$,*

$$\begin{cases} z_{21}(x+\alpha) = z_{11}(x), \\ z_{22}(x+\alpha) = z_{12}(x) - \kappa z_{11}(x), \end{cases} \tag{19}$$

and

$$z_{11}(x+\alpha) z_{12}(x) - z_{11}(x) z_{12}(x+\alpha) = 1 + \kappa z_{11}(x+\alpha) z_{11}(x). \tag{20}$$

For a function f defined on $\mathbb{R}/\mathbb{Z}$, we denote by $[f]$ its average and $\|f\|_{\mathrm{L}^2} := \sqrt{[f^2]}$. Then we also have:

$$\|z_{11}\|_{\mathrm{L}^2} = \|z_{21}\|_{\mathrm{L}^2} \geq (2\|Z\|_0)^{-1}. \tag{21}$$

Proof. Equation (18) gives: for any $x \in \mathbb{R}/\mathbb{Z}$,

$$\begin{pmatrix} (E-\lambda v(x))z_{11}(x) - z_{21}(x) & (E-\lambda v(x))z_{12}(x) - z_{22}(x) \\ z_{11}(x) & z_{12}(x) \end{pmatrix} =$$
$$= \begin{pmatrix} z_{11}(x+\alpha) & \kappa z_{11}(x+\alpha) + z_{12}(x+\alpha) \\ z_{21}(x+\alpha) & \kappa z_{21}(x+\alpha) + z_{22}(x+\alpha) \end{pmatrix}.$$

and we get the first two relations. The third one follows by taking the determinant, since det $(S_{\lambda v,E}) \equiv 1$, det $(Z) \equiv 1$ and det $(B) = 1$. Now, if u_1, u_2 denote the columns of Z, the fact that det $(Z) \equiv 1$ implies $\|u_1\|_{\mathrm{L}^2}\|u_2\|_{\mathrm{L}^2} > 1$, and then,

$$\|u_1\|_{\mathrm{L}^2} = \|z_{11}\|_{\mathrm{L}^2} + \|z_{21}\|_{\mathrm{L}^2} = 2\|z_{11}\|_{\mathrm{L}^2} > \|u_2\|_{\mathrm{L}^2}^{-1} \geq (\|Z\|_0)^{-1}.$$

□

Let us move a little the energy from E to $E + \varepsilon$ for some $\varepsilon \in \mathbb{R}$, but keeping the conjugacy Z.

Lemma 2.19. *For any $x \in \mathbb{R}/\mathbb{Z}$, $Z(x+\alpha)^{-1} S_{\lambda v, E+\varepsilon}(x) Z(x) = B_\varepsilon(x)$, where*

$$B_\varepsilon(x) := \begin{pmatrix} 1 + \varepsilon(z_{11}(x)z_{12}(x) - \kappa z_{11}(x)^2) & \kappa + \varepsilon(z_{12}(x)^2 - \kappa z_{11}(x)z_{12}(x)) \\ -\varepsilon z_{11}(x)^2 & 1 - \varepsilon z_{11}(x)z_{12}(x) \end{pmatrix}.$$

Proof. We have $S_{\lambda v, E+\varepsilon}(x) = S_{\lambda v,E}(x) + \begin{pmatrix} \varepsilon & 0 \\ 0 & 0 \end{pmatrix}$. We deduce from (18) that $S_{\lambda v, E+\varepsilon}(x)$ is conjugated to

$$\begin{aligned} B_\varepsilon(x) &:= Z(x+\alpha)^{-1} S_{\lambda v, E+\varepsilon}(x) Z(x) \\ &= B + \varepsilon Z(x+\alpha)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Z(x) \\ &= B + \varepsilon \begin{pmatrix} z_{22}(x+\alpha)z_{11}(x) & z_{22}(x+\alpha)z_{12}(x) \\ -z_{21}(x+\alpha)z_{11}(x) & -z_{21}(x+\alpha)z_{12}(x) \end{pmatrix}. \end{aligned}$$

Thanks to the relations obtained in (19), we get $B_\varepsilon(x) = B + \varepsilon \widetilde{B}(x)$, where

$$\widetilde{B}(x) := \begin{pmatrix} z_{11}(x)z_{12}(x) - \kappa z_{11}(x)^2 & z_{12}(x)^2 - \kappa z_{11}(x)z_{12}(x) \\ -z_{11}(x)^2 & -z_{11}(x)z_{12}(x) \end{pmatrix}.$$

□

3. Exponential decay of the size of the gaps: the averaging method

In this part, we give a first proof of the upper bounds on the size of spectral gaps, using the method of averaging. We look for a cocycle $(\alpha, \widetilde{Z}_\varepsilon)$ near the identity that would conjugate (α, B_ε) to a cocycle closer to be constant. Then we examine the type of the resulting matrix in $\mathrm{SL}_2(\mathbb{R})$ (hyperbolic, parabolic, elliptic) to estimate the size of the gap. The idea is that the left boundary of the gap corresponds approximately to a value of the energy for which the averaged matrix becomes elliptic again. In the following we use the notations introduced in the last part.

3.1. **Preliminary results.** We recall the following definition.

Definition 3.1. *We say that $x \in \mathbb{R}$ is Diophantine with respect to α if there exist constants $K, \tau > 0$ such that for every $m \in \mathbb{Z}$,*

$$\|x - m\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{K}{|m|^\tau}.$$

In this case we denote $x \in \mathrm{DC}_\alpha(K, \tau)$.

Recall that $\alpha \in \mathrm{DC}(K,\tau)$. Take $B_0 \in \mathrm{SL}_2(\mathbb{R})$; its eigenvalues are of the form $e^{\pm i\theta_0}$ with $\theta_0 \in \mathbb{R} \cup i\mathbb{R}$. Assume that $\mathrm{Re}(\theta_0) \in \mathrm{DC}_\alpha(K,\tau)$. The following lemma is classical.

Lemma 3.2 (see Proposition 1, [A]). *Recall that $[\cdot]$ denotes the average of a function defined on the torus $\mathbb{R}/\mathbb{Z}$. Fix $c > 0$. For any $G\colon \{|\mathrm{Im}(x)| < c\} \to \mathrm{sl}_2(\mathbb{R})$, the cohomological equation*

$$Y(\cdot + \alpha)B_0 - B_0 Y(\cdot) = B_0(G(\cdot) - [G]) \tag{22}$$

admits a unique solution $Y\colon \mathbb{R}/\mathbb{Z} \to \mathrm{sl}_2(\mathbb{R})$. Moreover, there exists a constant $C(\tau) > 0$ that only depends on τ such that for any $0 < \delta \leq c$,

$$\|Y\|_0 \leq C(\tau)\frac{\|G\|_\delta}{M},$$

where $M = M(K,\tau,B_0,\delta) > 0$ satisfies $\frac{1}{M} := \left(\frac{\|B_0\|^2}{K\delta^{\tau+2}}\right)^3$.

This equation is solved in Fourier series. We will use the previous result in the case where $B_0 = B = \begin{pmatrix} 1 & \kappa_m \\ 0 & 1 \end{pmatrix}$ for some $m \in \mathbb{Z}$. In particular, its eigenvalues satisfy the previous assumptions and $\|B_0\|$ is uniformly bounded because $|\kappa_m| \ll 1$. Therefore the constant $M > 0$ can be taken uniform in m, that is, it only depends on K, τ, δ, where $\alpha \in \mathrm{DC}(K,\tau)$. With the notations introduced in the last part, we can then conjugate the cocycle (α, B_ε) to a cocycle with smaller non-constant part:

Lemma 3.3 (see Proposition 2, [A]). *We assume that Z is defined on the strip $|\mathrm{Im}(x)| < c$, $c > 0$; in the following we take $0 < \delta \leq c$ as before. Then there exists a constant $C'(\tau) > 0$ depending only on τ such that if $\varepsilon_2 > 0$ satisfies*

$$\|Z\|_\delta^2 \varepsilon_2 \leq C'(\tau) M(K,\tau,\delta), \tag{23}$$

then for every ε, $|\varepsilon| \leq \varepsilon_2$, we can find $\widetilde{Z}_\varepsilon \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}_2(\mathbb{R}))$, $B'_\varepsilon \in \mathrm{SL}_2(\mathbb{R})$ and $\widetilde{B}' = \widetilde{B}'(\varepsilon)\colon \mathbb{R}/\mathbb{Z} \to \mathrm{SL}_2(\mathbb{R})$ depending analytically both on ε and x such that for any $x \in \mathbb{R}/\mathbb{Z}$,

$$\widetilde{Z}_\varepsilon(x+\alpha)^{-1}(B + \varepsilon\widetilde{B}(x))\widetilde{Z}_\varepsilon(x) = B'_\varepsilon + \varepsilon^2 \widetilde{B}'(x),$$

and for any $0 < \delta \leq c$,

$$\begin{cases} \|\widetilde{Z}_\varepsilon - \mathrm{id}\|_0 & \leq & C''(\tau)\frac{\|Z\|_\delta^2}{M}|\varepsilon|, \\ \|B'_\varepsilon - B\| & \leq & C''(\tau)\|Z\|_\delta^2|\varepsilon|, \\ \|\widetilde{B}'\|_0 & \leq & C''(\tau)\frac{\|Z\|_\delta^4}{M^2}. \end{cases}$$

Moreover the constant $C''(\tau) > 0$ only depends on τ.

Remark 3.4. *Although the conjugacy $\widetilde{Z}_\varepsilon$ depends on ε, it does not change the rotation number of the cocycle; indeed, we may choose $\varepsilon_2 > 0$ sufficiently small such that for $|\varepsilon| \leq \varepsilon_2$, the map $\widetilde{Z}_\varepsilon$ is homotopic to* id. *We will assume this in the following.*

Proof. We follow the proof given in [A]. Define $G\colon x \mapsto B^{-1}\widetilde{B}(x) - \frac{\mathrm{tr}(B^{-1}\widetilde{B}(x))}{2}\mathrm{id}$ and let Y be the solution to (22) with this choice of G. We also assume that $\|Z\|_\delta^2\varepsilon_2 \ll 1$. For $0 < |\varepsilon| \leq \varepsilon_2$, let then $\widetilde{Z}_\varepsilon := \exp(\varepsilon Y)$, $B'_\varepsilon := B\exp(\varepsilon[G])$ and

$$\widetilde{B}'(x) := \frac{1}{\varepsilon^2}\left[\widetilde{Z}_\varepsilon(x+\alpha)^{-1}(B + \varepsilon\widetilde{B}(x))\widetilde{Z}_\varepsilon(x) - B'_\varepsilon\right].$$

From Lemma 3.2, the first and second inequalities are clearly satisfied, since $\|Y\|_0 \leq C(\tau)\frac{\|G\|_\delta}{M}$, and $\|G\|_\delta \lesssim \|\widetilde{B}\|_\delta \lesssim \|Z\|_\delta^2$ by the definitions of G and $\widetilde{B}$ (recall that B

is a parabolic matrix close to id). For the third one, we estimate

$$\begin{aligned}\varepsilon^2\|\widetilde{B}'\|_0 &= \|\widetilde{Z}_\varepsilon(x+\alpha)^{-1}(B+\varepsilon\widetilde{B}(x))\widetilde{Z}_\varepsilon(x) - B'_\varepsilon\|_0\\ &\leq \|-Y(x+\alpha)B+BY(x)+B(G(x)-[G])\|_0 \times |\varepsilon|\\ &+ \left\|\sum_{k+l\geq 2} \varepsilon^{k+l}\frac{(-Y(x+\alpha))^k}{k!}B\frac{(Y(x))^l}{l!}\right\|_0\\ &+ \left\|\sum_{k+l\geq 1} \varepsilon^{k+l+1}\frac{(-Y(x+\alpha))^k}{k!}\widetilde{B}(x)\frac{(Y(x))^l}{l!}\right\|_0\\ &+ \left\|B\sum_{k\geq 2}\frac{\varepsilon^k[G]^k}{k!}\right\|.\end{aligned}$$

The first term vanishes, and in the three remaining ones, we can factor by ε^2. Since $\sum_{k+l=m}\frac{1}{k!l!}=\frac{2^m}{m!}$, the second term is smaller than $(e^{2\|Y\|_0|\varepsilon|}-1-2\|Y\|_0|\varepsilon|)\|B\| = O(\|Y\|_0^2\varepsilon^2)$. The result follows since $\|Y\|_0 \leq C(\tau)\frac{\|G\|_\delta}{M} \lesssim C(\tau)\frac{\|Z\|_\delta^2}{M}$, $\|\widetilde{B}\|_0 \lesssim \|Z\|_0^2$, and $\|[G]\| \lesssim \|Z\|_\delta^2$. ☐

We define two matrices $b_0, b_1 \in \mathrm{sl}_2(\mathbb{R})$, where

$$b_0 := \begin{pmatrix} 0 & \kappa\\ 0 & 0\end{pmatrix}, \qquad b_1 := \begin{pmatrix} [z_{11}z_{12}] - \frac{\kappa}{2}[z_{11}^2] & [z_{12}^2] - \kappa[z_{11}z_{12}]\\ -[z_{11}^2] & -[z_{11}z_{12}] + \frac{\kappa}{2}[z_{11}^2]\end{pmatrix}.$$

Corollary 3.5. *Let $\varepsilon_2 > 0$ be chosen as in* (23)*. Then for any $|\varepsilon| \leq \varepsilon_2$, there exist $C'''(\tau) > 0$ depending only on τ and $Z'_\varepsilon \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}_2(\mathbb{R}))$ with $\deg(Z'_\varepsilon) = \deg(Z)$, such that for any $x \in \mathbb{R}/\mathbb{Z}$,*

$$Z'_\varepsilon(x+\alpha)^{-1}S_{\lambda v,E+\varepsilon}(x)Z'_\varepsilon(x) = \exp(b_0+\varepsilon b_1+\varepsilon^2 r(x)),$$

where r depends analytically on ε and x, and $\|r\|_0 \leq C'''(\tau)\frac{\|Z\|_\delta^4}{M^2}$.

Proof. With the previous notations, we have $\mathrm{tr}(B^{-1}[\widetilde{B}]) = \mathrm{tr}([\widetilde{B}]) - \kappa[z_{11}^2] = 0$, which implies that $[G] = B^{-1}[\widetilde{B}] - \frac{\mathrm{tr}(B^{-1}[\widetilde{B}])}{2}\mathrm{id} = B^{-1}[\widetilde{B}]$. For every $x \in \mathbb{R}/\mathbb{Z}$, we then get

$$\begin{aligned}B'_\varepsilon + \varepsilon^2\widetilde{B}'(x) &= Be^{\varepsilon[G]} + \varepsilon^2\widetilde{B}'(x)\\ &= B + \varepsilon[\widetilde{B}] + \varepsilon^2 R(x)\\ &= \exp(b_0+\varepsilon b_1+\varepsilon^2 r(x)),\end{aligned}$$

where both $R\colon \mathbb{R}/\mathbb{Z} \to \mathrm{SL}_2(\mathbb{R})$ and $r\colon \mathbb{R}/\mathbb{Z} \to \mathrm{sl}_2(\mathbb{R})$ depend analytically on ε and x, and $\|r\|_0 \leq C'''(\tau)\frac{\|Z\|_\delta^4}{M^2}$. It remains to set $Z'_\varepsilon(\cdot) := Z(\cdot)\widetilde{Z}_\varepsilon(\cdot)$. The estimate on the degree follows from the fact that $\widetilde{Z}_\varepsilon(\cdot)$ is homotopic to id. ☐

The following result was shown by Puig in [P1].

Proposition 3.6. [P1] *If $\kappa \neq 0$, the energy E lies on the boundary of a non-collapsed spectral gap. More precisely, if $|\varepsilon|$ is small, then Z'_ε conjugates $(\alpha, S_{\lambda v,E+\varepsilon})$ to a cocycle close to the constant cocycle $(\alpha, \exp(b_0+\varepsilon b_1))$, where the matrix $\exp(b_0+\varepsilon b_1)$ is hyperbolic if $\kappa\varepsilon < 0$, respectively elliptic if $\kappa\varepsilon > 0$. Moreover, we have the following asymptotics:*

$$\lim_{\varepsilon\to 0,\ \kappa\varepsilon<0}\frac{L(E+\varepsilon)}{\sqrt{|\varepsilon|}} = \lim_{\varepsilon\to 0,\ \kappa\varepsilon>0}\frac{|\rho(E+\varepsilon)-\rho(E)|}{\sqrt{|\varepsilon|}} = \sqrt{[z_{11}^2]|\kappa|} > 0.$$

Let us recall the proof of this result since such ideas will be useful in the following.

Proof. Let $M = M(\varepsilon)$ be the matrix

$$M := b_0 + \varepsilon b_1 = \begin{pmatrix} m_1 & m_2 \\ m_3 & -m_1 \end{pmatrix},$$

whose determinant is $d = d(\varepsilon) = -m_1^2 - m_2 m_3$. If $\kappa \neq 0$, this determinant becomes

$$d = \kappa\varepsilon[z_{11}^2] + O(\varepsilon^2).$$

We then choose the sign of ε such that $d(\varepsilon) < 0$ for $|\varepsilon|$ small enough. We diagonalize the matrix M; define

$$P := \begin{pmatrix} m_2 & m_2 \\ -m_1 + \sqrt{-d} & -m_1 - \sqrt{-d} \end{pmatrix},$$

where $|\det(P(\varepsilon))| = 2|\kappa|\sqrt{-\kappa\varepsilon[z_{11}^2]} + O(\varepsilon)$. We then obtain $MP = PD$, where

$$D = \begin{pmatrix} \sqrt{-d} & 0 \\ 0 & -\sqrt{-d} \end{pmatrix}.$$

Using the change of variables defined by P, we can conjugate the cocycle $(\alpha, \exp(b_0 + \varepsilon b_1 + \varepsilon^2 r))$ to $(\alpha, \exp(D + s))$ with $s(\varepsilon, x) = \varepsilon^2 P^{-1} r(\varepsilon, x) P$, which is uniformly $O(|\varepsilon|^{3/2})$ in x. We get

$$D + s = \sqrt{-\kappa\varepsilon[z_{11}^2]} \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + O(|\varepsilon|) \right),$$

so if $|\varepsilon|$ is sufficiently small and $d(\varepsilon)$ is negative, then the cocycle $(\alpha, S_{\lambda v, E+\varepsilon})$ is uniformly hyperbolic since it is conjugated to a perturbation of a constant cocycle associated with some hyperbolic matrix. Moreover, its Lyapunov exponent satisfies:

$$\lim_{\varepsilon \to 0} \frac{L(E + \varepsilon)}{\sqrt{|\varepsilon|}} = \sqrt{[z_{11}^2]|\kappa|}.$$

Now if we choose the sign of ε such that $d(\varepsilon) > 0$ for $|\varepsilon|$ small enough, define

$$P := \begin{pmatrix} m_2 & m_2 \\ -m_1 + i\sqrt{d} & -m_1 - i\sqrt{d} \end{pmatrix}.$$

We see that $|\det(P(\varepsilon))| = 2|\kappa|\sqrt{\kappa\varepsilon[z_{11}^2]} + O(\varepsilon)$. We then obtain $MP = PD$, where

$$D = \begin{pmatrix} i\sqrt{d} & 0 \\ 0 & -i\sqrt{d} \end{pmatrix}.$$

Using the change of variables defined by P, we can conjugate the cocycle $(\alpha, \exp(b_0 + \varepsilon b_1 + \varepsilon^2 r))$ to $(\alpha, \exp(D + s))$ with $s(\varepsilon, x) = \varepsilon^2 P^{-1} r(\varepsilon, x) P$, which is uniformly $O(|\varepsilon|^{3/2})$ in x. We get

$$D + s = \sqrt{\kappa\varepsilon[z_{11}^2]} \left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + O(|\varepsilon|) \right),$$

so if $|\varepsilon|$ is sufficiently small and $d(\varepsilon)$ is positive, then the cocycle $(\alpha, S_{\lambda v, E+\varepsilon})$ is conjugated to a perturbation of a constant cocycle associated with a complex rotation of angle $\sqrt{d}$. In this case, we obtain the following asymptotics:

$$\lim_{\varepsilon \to 0} \frac{|\rho(E + \varepsilon) - \rho(E)|}{\sqrt{|\varepsilon|}} = \sqrt{[z_{11}^2]|\kappa|}.$$

□

Remark 3.7. *This tells us that when the energy $E \in \Sigma$ is such that $(\alpha, S_{\lambda v, E})$ is conjugated to a constant cocycle (α, B) for some parabolic matrix B with non diagonal coefficient $\kappa \neq 0$, then E is on the boundary of a non-collapsed spectral gap. Indeed, for $|\varepsilon| > 0$ small, $E \pm \varepsilon$ belongs either to the spectrum or the resolvent set according to the sign of the perturbation. In the following, we will mostly consider* *the case where E is on the right boundary of a non-collapsed spectral gap; what precedes then implies that $\kappa \geq 0$.*

3.2. **Exponential bounds: transition from hyperbolic to elliptic matrices.** In this part we combine previous results on quantitative reducibility to a constant parabolic cocycle and the facts recalled above about perturbations of the Schrödinger cocycle near the boundary of a spectral gap. We show that up to an exponentially small change of the energy, the perturbed cocycle has different rotation number, which gives an upper bound on the size of the gap.

Let $E \in \Sigma$ be an energy located on the boundary of a spectral gap with rotation number $\rho(E)$, where $2\rho(E) = m\alpha \mod \mathbb{Z}$. Again we choose a phase $\theta(E)$ with $2\theta = \widetilde{n}\alpha \mod \mathbb{Z}$, and we set $n := |\widetilde{n}|$. Theorem 2.12 provides us with a map $Z \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}_2(\mathbb{R}))$ such that for every $x \in \mathbb{R}/\mathbb{Z}$,

$$Z(x+\alpha)^{-1}A(x)Z(x) = \begin{pmatrix} \pm 1 & \kappa \\ 0 & \pm 1 \end{pmatrix}.$$

Moreover $|\kappa| \leq Ce^{-cn}$ and $\|Z\|_0 \leq e^{o(n)}$ for some $C, c > 0$ independent of m.

The case where κ vanishes is handled in Remark 3.9 below, so let us assume that $\kappa \neq 0$; as we have seen, this implies that the gap is non-collapsed. We consider the energy $E := E_m^+$ corresponding to the right boundary of the spectral gap. From Proposition 3.6, we are then in the case where $\kappa > 0$.

From Corollary 3.5, we can conjugate the cocycle $(\alpha, S_{\lambda v, E+\varepsilon})$ to the cocycle $(\alpha, \exp(b_0 + \varepsilon b_1 + \varepsilon^2 r))$, where $b_0,\ b_1 \in \mathrm{sl}_2(\mathbb{R})$ are of the form

$$b_0 = \begin{pmatrix} 0 & \kappa \\ 0 & 0 \end{pmatrix}, \qquad b_1 = \begin{pmatrix} [z_{11}z_{12}] - \frac{\kappa}{2}[z_{11}^2] & [z_{12}^2] - \kappa[z_{11}z_{12}] \\ -[z_{11}^2] & -[z_{11}z_{12}] + \frac{\kappa}{2}[z_{11}^2] \end{pmatrix},$$

and $\|r\|_0 \leq C'''(\tau)\frac{\|Z\|_\delta^4}{M^2}$. In fact, we will need one more step of averaging: again by Corollary 3.5, there exist $b_2 \in \mathrm{sl}_2(\mathbb{R})$ and $r' \colon \mathbb{R}/\mathbb{Z} \to \mathrm{sl}_2(\mathbb{R})$, where $\|b_2\| \leq C_4(\tau)\frac{\|Z\|_\delta^4}{M^2}$ and $\|r'\|_0 \leq C_5(\tau)\frac{\|Z\|_\delta^6}{M^3}$ for some constants $C_4(\tau), C_5(\tau) > 0$, such that $(\alpha, S_{\lambda v, E+\varepsilon})$ is conjugated to the cocycle

$$(\alpha, \exp(b_0 + \varepsilon b_1 + \varepsilon^2 b_2 + \varepsilon^3 r'))$$

through a conjugacy whose degree does not depend on ε for $|\varepsilon|$ sufficiently small. In particular, (2) implies that $\rho(E+\varepsilon)$ and $\rho(\alpha, \exp(b_0 + \varepsilon b_1 + \varepsilon^2 b_2 + \varepsilon^3 r'))$ differ by some quantity which is independent of ε when $|\varepsilon| \ll 1$.

As in the proof of Proposition 3.6, we define

$$M = M(\varepsilon) := b_0 + \varepsilon b_1 + \varepsilon^2 b_2 = \begin{pmatrix} m_1 & m_2 \\ m_3 & -m_1 \end{pmatrix} \in \mathrm{sl}_2(\mathbb{R}).$$

We also set $d = d(\varepsilon) := \det(M(\varepsilon))$.

Letting ε vary, we get a path in $\mathrm{sl}_2(\mathbb{R})$. For $\varepsilon < 0$ with $|\varepsilon|$ small enough, the energy $E+\varepsilon$ lies in the spectral gap. Moreover, we know that for any $\varepsilon < 0$ such that $E+\varepsilon$ belongs to the gap, the rotation number of the cocycle $(\alpha, \exp(M(\varepsilon) + \varepsilon^3 r'))$ is identically vanishing; indeed, on the right boundary of the gap, i.e. when $\varepsilon = 0$, the matrix $\exp(M(0)) = \exp(b_0)$ is parabolic, and $\rho(\alpha, \exp(b_0)) = 0$.

We want to show that we can choose $\varepsilon < 0$, $|\varepsilon|$ exponentially small with respect to n, such that $E+\varepsilon$ does not belong to the gap anymore, which gives an exponential bound on the size of the latter. For this, it is sufficient to show that for such an ε, the cocycle $(\alpha, \exp(M(\varepsilon) + \varepsilon^3 r'))$ has nonzero rotation number. The idea is to look at the sign of $d(\varepsilon)$. Indeed, when this determinant is positive, $\exp(M(\varepsilon) + \varepsilon^3 r')$ is a perturbation of the elliptic matrix $\exp(M(\varepsilon))$, whose rotation number is nonzero.

We then have to look for an energy $E + \varepsilon$ where the sign of the determinant changes. We have seen previously that such a transition exists when ε changes sign. This transition was for values of $|\varepsilon|$ much smaller than κ, so it was enough to look at first order terms in ε. To see that a second transition exists, we will also have to consider higher order terms. Actually this is possible because the coefficient κ itself is exponentially small. We compute:

$$\begin{aligned} d(\varepsilon) &= -\varepsilon^2 \left([z_{11}z_{12}] - \frac{\kappa}{2}[z_{11}^2]\right)^2 + \varepsilon[z_{11}^2]\left(\kappa + \varepsilon[z_{12}^2] - \varepsilon\kappa[z_{11}z_{12}]\right) + O(\kappa\varepsilon^2\|b_2\|^2, \varepsilon^3\|b_2\|^2) \\ &= \varepsilon\left([z_{11}^2]\kappa + \varepsilon\left([z_{11}^2][z_{12}^2] - [z_{11}z_{12}]^2\right) + O(\kappa\varepsilon\|Z\|_\delta^8, \varepsilon^2\|Z\|_\delta^8)\right), \end{aligned} \tag{24}$$

where by the previous estimate on $\|b_2\|$, the hidden constants only depend on K, τ, δ. In particular, they are independent of m, and hence of the spectral gap we consider.

Lemma 3.8. *By Cauchy-Schwarz inequality, we always have* $[z_{11}^2][z_{12}^2] - [z_{11}z_{12}]^2 \geq 0$. *Moreover, the quantity* $[z_{11}^2][z_{12}^2] - [z_{11}z_{12}]^2$ *vanishes if and only if the potential* v *is constant.*

Proof. By the equality case in Cauchy-Schwarz inequality, $[z_{11}^2][z_{12}^2] - [z_{11}z_{12}]^2 = 0$ if and only if z_{11} and z_{12} are proportional, that is, $z_{12} = \mu z_{11}$ for some $\mu \in \mathbb{R}$. From (20), we deduce that for every $x \in \mathbb{R}/\mathbb{Z}$, $\kappa z_{11}(x+\alpha)z_{11}(x) = -1$. It follows that z_{11} is 2α-periodic, hence constant since α is irrational. Since $z_{12} = \mu z_{11}$, and from (19), this implies that the conjugacy Z itself is constant. But the parabolic matrix to which we conjugate is also constant; therefore, $S_{\lambda v,E}$, thus v, are constant too. □

Remark 3.9. *Assume that the potential is constant, that is,* $v \equiv v_0 \in \mathbb{R}$. *Then for any* $E \in \mathbb{R}$, $S_{\lambda v,E}$ *is a constant matrix in* $\mathrm{SL}_2(\mathbb{R})$. *Since* $\mathrm{tr}(S_{\lambda v,E}) = E - \lambda v_0$, *we see that the spectrum is equal to* $[-2+\lambda v_0, 2+\lambda v_0]$; *in particular, there is no spectral gap and the statement we aim to show is vacuously true.*

If v *is not constant, but* $\kappa = 0$, *we see that* $d(\varepsilon) = \varepsilon^2([z_{11}^2][z_{12}^2] - [z_{11}z_{12}]^2) + O(\varepsilon^3)$, *which is positive for* $|\varepsilon| \neq 0$ *small. Arguing as in Proposition 3.6, we deduce that the gap is collapsed and there is nothing to prove in this case.*

In the following, we assume that the potential v is not constant, and that $\kappa > 0$. Recall that we associate with $m \in \mathbb{Z} \setminus \{0\}$ an integer $n := |\tilde{n}|$, where $2\theta(E_m^+) = \tilde{n}\alpha \mod \mathbb{Z}$.

Proposition 3.10. *If* $(z_{ij})_{1\leq i,j\leq 2} = (z_{ij}^m)_{1\leq i,j\leq 2}$ *denote the coefficients of the conjugacy* Z_m *given by Theorem 2.12, then*

$$\frac{[z_{11}^2]}{[z_{11}^2][z_{12}^2] - [z_{11}z_{12}]^2} \leq e^{o(n)}. \tag{25}$$

We also have

$$[z_{11}^2][z_{12}^2] - [z_{11}z_{12}]^2 \geq e^{-o(n)}. \tag{26}$$

Proof. Suppose it is not the case. Then we may find $\sigma > 0$ such that for any $m_0 \geq 0$, there exists $m \geq m_0$ for which the coefficients $(z_{ij})_{1\leq i,j\leq 2} = (z_{ij}^m)_{1\leq i,j\leq 2}$ of Z_m satisfy:

$$\frac{[z_{11}^2]}{[z_{11}^2][z_{12}^2] - [z_{11}z_{12}]^2} \geq e^{\sigma n},$$

where n is associated with m as we explained above. As in the proof of Cauchy-Schwarz inequality, we consider the polynomial

$$P(\mu) := [(z_{12} - \mu z_{11})^2] = \mu^2[z_{11}^2] - 2\mu[z_{11}z_{12}] + [z_{12}^2].$$

It is minimal for $\mu = \mu_0$, where $\mu_0 := \frac{[z_{11}z_{12}]}{[z_{11}^2]}$, and we have

$$[(z_{12} - \mu_0 z_{11})^2] = \frac{[z_{11}^2][z_{12}^2] - [z_{11}z_{12}]^2}{[z_{11}^2]} \leq e^{-\sigma n}.$$

This implies that $z_{12} = \mu_0 z_{11} + h$, where $[h^2] \leq e^{-\sigma n}$. Moreover, we know from (20) that for every $x \in \mathbb{R}/\mathbb{Z}$,

$$z_{11}(x+\alpha)h(x) - z_{11}(x)h(x+\alpha) = 1 + \kappa z_{11}(x+\alpha)z_{11}(x). \tag{27}$$

Since $\|Z_m\|_0 = e^{o(n)}$ and $|\kappa| \leq Ce^{-cn}$, we know that

$$\|\kappa z_{11}(\cdot+\alpha)z_{11}(\cdot)\|_0 = O(e^{-cn}). \tag{28}$$

From Cauchy-Schwarz inequality, we also have

$$[|z_{11}(\cdot+\alpha)h(\cdot) - z_{11}(\cdot)h(\cdot+\alpha)|] = O(e^{-\frac{\sigma n}{2}}). \tag{29}$$

Combining (27), (28) and (29), and since n can be taken arbitrarily big, we get $1 = o(1)$, a contradiction.

On the other hand, the lower bound (26) follows from (25) and (21). $\square$

Corollary 3.11. *Let $\varepsilon_2 > 0$ be as in (23). Then there exist constants $C', c' > 0$ independent of m, as well as an energy $\varepsilon'_m < 0$ with $|\varepsilon'_m| \leq \varepsilon_2$,*

$$|\varepsilon'_m| \lesssim \frac{[z_{11}^2]}{[z_{11}^2][z_{12}^2] - [z_{11}z_{12}]^2}\kappa_m \leq C'e^{-c'|m|}, \tag{30}$$

such that for $\varepsilon < 2\varepsilon'_m$ and $|m|$ sufficiently large, $\rho(\alpha, \exp(M(\varepsilon) + \varepsilon^3 r'))$ is nonzero. In particular, we deduce that the size of spectral gaps decays exponentially fast with respect to their label, as desired:

$$E_m^+ - E_m^- \leq 2C'e^{-c'|m|}.$$

Proof. Recall that $M = M(\varepsilon) = b_0 + \varepsilon b_1 + \varepsilon^2 b_2$, and $d = d(\varepsilon) = \det(M(\varepsilon))$. Let $C, c > 0$ be such that the Bloch wave U constructed in the proof of Proposition 2.14 is defined on $|\mathrm{Im}(x)| < c$, and assume that for any m, $0 \leq \kappa_m \leq Ce^{-cn}$. Take $\delta := c/100$. Since by (8), $\inf_{|\mathrm{Im}(x)|<c} \|U(x)\| \geq e^{-o(n)}$, (14) and (16) imply that

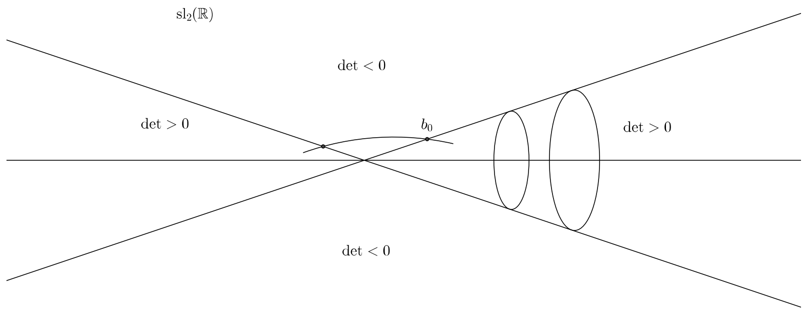
$$\|Z\|_\delta \leq Ce^{\pi n \delta}. \tag{31}$$

We see from (24) that $d(\varepsilon)$ actually changes sign (and becomes positive) for $\varepsilon < \varepsilon'_m$, where $\varepsilon'_m = -\frac{[z_{11}^2]}{[z_{11}^2][z_{12}^2]-[z_{11}z_{12}]^2}\kappa + O(\kappa\varepsilon\|Z\|_\delta^8, \varepsilon^2\|Z\|_\delta^8)$. From (25) and the fact that $\kappa_m \leq Ce^{-cn}$ and $8\pi\delta \leq c/2$, we deduce that

$$|\varepsilon'_m| \leq Ce^{-\frac{cn}{2}}. \tag{32}$$

Inequality (30) then follows from (32) and the relation between m and n obtained in (11). Note that $\|Z\|_\delta^2|\varepsilon'_m| = O(e^{-\frac{cn}{3}})$ is indeed very small; in particular, it legitimates the use of the perturbative expansion given by Corollary 3.5. We can also take $\varepsilon_2 > 0$ with $|\varepsilon'_m| \leq \varepsilon_2$, $m \in \mathbb{Z}$, such that (23) holds.

Let us now check that for $\varepsilon < 2\varepsilon'_m$, and provided that $|m|$ is big enough, the rotation number of the cocycle obtained after perturbation by $\varepsilon^3 r'$ is indeed nonzero. From what precedes, the path in $\mathrm{sl}_2(\mathbb{R})$ given by $\varepsilon \mapsto M(\varepsilon)$, $\varepsilon < 0$, is as in the following picture. The general idea is based on the fact that small perturbations preserve the transversality of this path.



Recall that

$$d = \varepsilon(\varepsilon - \varepsilon'_m)\left([z_{11}^2][z_{12}^2] - [z_{11}z_{12}]^2\right) + O(\kappa\varepsilon^2\|Z\|_\delta^8, \varepsilon^3\|Z\|_\delta^8).$$

We know that $\kappa \leq Ce^{-cn}$, $|\varepsilon'_m| \leq Ce^{-\frac{cn}{2}}$, and by (26), we also have $[z_{11}^2][z_{12}^2] - [z_{11}z_{12}]^2 \geq e^{-o(n)}$. Let us take $\varepsilon = 2\varepsilon'_m$ and look at what happens when $|m|$ gets big; since $\|Z\|_\delta \leq Ce^{\pi n\delta}$, we have

$$d \gtrsim |\varepsilon'_m|^2\left([z_{11}^2][z_{12}^2] - [z_{11}z_{12}]^2\right).$$

Let us set

$$P = P(\varepsilon) := \begin{pmatrix} m_2 & m_2 \\ -m_1 + i\sqrt{d} & -m_1 - i\sqrt{d} \end{pmatrix},$$

so that $MP = PD$ for some complex rotation $D = D(\varepsilon)$ of angle $\sqrt{d}$. Note that $|\det(P)| = 2|m_2|\sqrt{d}$. The matrix P defines a (complex) conjugacy between the cocycles $(\alpha, \exp(M + \varepsilon^3 r'))$ and $(\alpha, \exp(D + s))$, where $s(\varepsilon, x) := \varepsilon^3 P^{-1} r'(\varepsilon, x) P$. Recall that $\|r'\|_0 \leq C_5(\tau)\frac{\|Z\|_\delta^6}{M^3} = O(e^{\frac{cn}{2}})$, while $|\varepsilon| \leq Ce^{-cn}$; we obtain

$$D + s = \sqrt{d}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + O(e^{-\frac{cn}{2}})\right) = \sqrt{d}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + O(e^{-c'|m|})\right).$$

If we consider the corresponding real conjugacy, we see that it is the perturbation of a rotation of angle $\sqrt{d}$. By looking at its action on the circle, we conclude that for $\varepsilon = 2\varepsilon'_m$, and when $|m|$ is sufficiently large, the cocycle $(\alpha, \exp(M(\varepsilon) + \varepsilon^3 r'))$ has nonzero rotation number, as claimed.

□

4. Exponential decay of the size of the gaps: the monotonicity method

In this part, we give another proof of the upper bounds on the size of spectral gaps. It is based on monotonicity arguments; we refer to [AK] for more details in this direction. Recall that the fibered rotation number is defined as the average with respect to some invariant measure μ of a certain quantity, that we refer to as the "drift" in what follows. It is defined by the natural action of a $SL_2(\mathbb{R})$-cocycle on the circle: for every matrix of the cocycle, we ask how much it rotates each point of the circle.

Schrödinger cocycles depend on the energy E; we know that the fibered rotation number is monotonic with respect to E. In particular, the derivative of the drift with respect to the energy is always nonpositive. Yet it is not strictly monotonic: indeed, we have seen that the rotation number is constant on spectral gaps. This property is linked to the fact that the measure μ may be supported on fixed points that are independent of the energy E. In particular we see that $(0, 1)$ is mapped to

$(-1,0)$ independently of E. In fact this is essentially the only obstruction to strict monotonicity. When we iterate once, we see that the derivative with respect to the energy becomes negative.

The proof proceeds as follows. We start with an energy E_m^+ on the right boundary of the spectral gap labeled by $m \in \mathbb{Z} \setminus \{0\}$. We know that for this value of the energy, we can conjugate the cocycle to some constant parabolic cocycle. We show that the initial drift is nonpositive, but exponentially small with respect to m. Monotonicity for Schrödinger cocycles translates into monotonicity for the conjugate cocycles; in fact, the derivative of the second iterate of the drift is negative, subexponentially small in terms of the label. Then we see that after an exponentially small perturbation of the energy, we have left the spectral gap, which concludes.

4.1. **Monotonicity for Schrödinger cocycles.** Let us recall that for an energy $E \in \mathbb{R}$ and for $x \in \mathbb{R}/\mathbb{Z}$, we denote by

$$S_{\lambda v,E}(x) := \begin{pmatrix} E - \lambda v(x) & -1 \\ 1 & 0 \end{pmatrix}$$

the associate matrix. The Schrödinger cocycle $(\alpha, S_{\lambda v,E})$ acts on $\mathbb{R}/\mathbb{Z} \times \mathbb{S}^1$ by:

$$(x, e^{2i\pi y}) \mapsto \left(x + \alpha, \frac{S_{\lambda v,E}(x) \cdot e^{2i\pi y}}{\|S_{\lambda v,E}(x) \cdot e^{2i\pi y}\|} \right).$$

This map admits a lift to $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$, given by $(x,y) \mapsto (x+\alpha, y + \phi_x^E(y))$, where the "drift" ϕ_x^E satisfies $\frac{S_{\lambda v,E}(x) \cdot e^{2i\pi y}}{\|S_{\lambda v,E}(x) \cdot e^{2i\pi y}\|} = e^{2i\pi(y+\phi_x^E(y))}$. The spectrum Σ of Schrödinger operators is compact; let $E_0 > 0$ be such that $\Sigma \subset [-E_0, E_0]$. Recall the following result, which corresponds to monotonicity for the second iterate of the Schrödinger cocycle:

Proposition 4.1. *There exists $\eta_0 > 0$ such that for any $E \in [-E_0, E_0]$, and any $(x,y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$, we have*

$$\partial_E[\phi_x^E(y) + \phi_{x+\alpha}^E(y + \phi_x^E(y))] \leq -\eta_0 < 0.$$

Proof. The drift is characterized by the following formula: for any $(x,y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$,

$$\tan(2\pi(y + \phi_x^E(y))) = \frac{1}{E - \lambda v(x) - \tan(2\pi y)}.$$

Iterating once the previous relation, we obtain:

$$\tan(2\pi(y + \phi_x^E(y) + \phi_{x+\alpha}^E(y + \phi_x^E(y)))) = \theta_x^E(y) = \frac{N_x^E(y)}{D_x^E(y)},$$

where

$$N_x^E(y) := E - \lambda v(x) - \tan(2\pi y),$$
$$D_x^E(y) := E^2 - E(\lambda v(x) + \lambda v(x+\alpha) + \tan(2\pi y)) + \lambda v(x+\alpha)(\lambda v(x) + \tan(2\pi y)) - 1.$$

We get

$$\partial_E[\phi_x^E(y) + \phi_{x+\alpha}^E(y + \phi_x^E(y))] = \frac{1}{2\pi} \times \frac{\partial_E \theta_x^E(y)}{1 + \theta_x^E(y)^2} = -\frac{1}{2\pi} \times \frac{(N_x^E(y))^2 + 1}{(N_x^E(y))^2 + (D_x^E(y))^2}.$$

We deduce that there exists $\eta_0 > 0$ such that for any $E \in [-E_0, E_0]$ and any $(x,y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$,

$$\partial_E[\phi_x^E(y) + \phi_{x+\alpha}^E(y + \phi_x^E(y))] \leq -\eta_0 < 0.$$

□

4.2. **Analysis of the drift near the boundary of a spectral gap.** For some integer $m \in \mathbb{Z} \setminus \{0\}$, we look at the spectral gap G_m labeled by m. Recall that we denote $G_m = (E_m^-, E_m^+)$. We consider the energy E_m^+ on its right boundary. As previously, we take θ with $2\theta = \widetilde{n}\alpha \mod \mathbb{Z}$, and we set $n := |\widetilde{n}|$. From Theorem 2.12, we know that there exist a conjugacy $Z_m = (z_{ij})_{1 \leq i,j \leq 2} \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}_2(\mathbb{R}))$ and some parabolic matrix

$$B_m = \begin{pmatrix} \pm 1 & \kappa_m \\ 0 & \pm \end{pmatrix}$$

such that for any $x \in \mathbb{R}/\mathbb{Z}$,

$$S_{\lambda v, E_m^+}(x) = Z_m(x+\alpha) B_m Z_m(x)^{-1}.$$

We also have the estimates $0 \leq \kappa_m \leq Ce^{-cn}$ and $\|Z_m\|_0 \leq e^{o(n)}$, $C, c > 0$.

For every $\varepsilon \in \mathbb{R}$, and for any $x \in \mathbb{R}/\mathbb{Z}$, we have

$$S_{\lambda v, E_m^+ + \varepsilon}(x) = Z_m(x+\alpha) B_{m,\varepsilon}(x) Z_m(x)^{-1}, \tag{33}$$

where $B_\varepsilon(\cdot) = B_{m,\varepsilon}(\cdot)$ has the following form:

$$B_\varepsilon(\cdot) = \begin{pmatrix} 1 + \varepsilon(z_{11}z_{12} - \kappa_m z_{11}^2) & \kappa_m + \varepsilon(z_{12}^2 - \kappa_m z_{11} z_{12}) \\ -\varepsilon z_{11}^2 & 1 - \varepsilon z_{11} z_{12} \end{pmatrix}.$$

Let d_m be the degree of the map Z_m. Recall that for an energy $E \in \mathbb{R}$, we denote $\rho(E) := \rho(\alpha, S_{\lambda v, E})$. We also let $\rho_m(\varepsilon)$ be the fibered rotation number of the cocycle $(\alpha, B_{m,\varepsilon})$. We know from (33) that

$$\rho(E_m^+ + \varepsilon) = \rho_m(\varepsilon) + \frac{d_m \alpha}{2}. \tag{34}$$

We want to see that for $\varepsilon < 0$ exponentially small, $\rho(E_m^+ + \varepsilon) \neq \rho(E_m^+)$; from the previous formula, we see that it is enough to show that $\rho_m(\varepsilon) \neq \rho_m(0)$.

We denote by $y \mapsto \psi_\cdot^\varepsilon(y)$ the "drift" associated with the matrix $B_\varepsilon(\cdot)$. It is given by the formula: for any $(x, y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$,

$$\frac{B_\varepsilon(x) \cdot e^{2i\pi y}}{\|B_\varepsilon(x) \cdot e^{2i\pi y}\|} = e^{2i\pi(y + \psi_x^\varepsilon(y))}.$$

For any $y \in \mathbb{R}$, we get:

$$2\pi\psi_\cdot^\varepsilon(y) = \arctan\left(\frac{-\varepsilon z_{11}^2 + (1 - \varepsilon z_{11}z_{12})\tan(2\pi y)}{1 + \varepsilon(z_{11}z_{12} - \kappa_m z_{11}^2) + (\kappa_m + \varepsilon(z_{12}^2 - \kappa_m z_{11}z_{12}))\tan(2\pi y)}\right) - 2\pi y.$$

For any $(x, y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$, we see that

$$\psi_x^\varepsilon(y) = -\frac{1}{2\pi}\arctan\left(\frac{P_\varepsilon(x, \tan(2\pi y))}{Q_\varepsilon(x, \tan(2\pi y))}\right), \tag{35}$$

where $P_\varepsilon(\cdot, X)$ and $Q_\varepsilon(\cdot, X)$ correspond to two quadratic polynomials

$$P_\varepsilon(\cdot, X) := (\kappa_m + \varepsilon(z_{12}^2 - \kappa_m z_{11}z_{12}))X^2 + \varepsilon(2z_{11}z_{12} - \kappa_m z_{11}^2)X + \varepsilon z_{11}^2,$$

$$Q_\varepsilon(\cdot, X) := (1 - \varepsilon z_{11}z_{12})X^2 + (\kappa_m + \varepsilon[(z_{12}^2 - z_{11}^2) - \kappa_m z_{11}z_{12}])X + 1 + \varepsilon(z_{11}z_{12} - \kappa_m z_{11}^2).$$

We denote their respective discriminants by Δ_1^ε and Δ_2^ε. It is easy to see that

$$\Delta_1^\varepsilon = \kappa\varepsilon z_{11}^2(\kappa\varepsilon z_{11}^2 - 4), \qquad \Delta_2^\varepsilon = -4 + (\kappa + \varepsilon(z_{11}^2 + z_{12}^2 - \kappa z_{11}z_{12}))^2.$$

Since for $\varepsilon = 0$, the cocycle (α, B_0) is constant, the map $\psi_x^0 \equiv \psi^0$ does not depend on x. Besides, the following lemma tells us that the initial drift is exponentially small.

Lemma 4.2. *There exists $C > 0$ such that for any $(x, y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$,*

$$0 \leq -\psi_x^0(y) \leq Ce^{-cn}. \tag{36}$$

Proof. If we take $\varepsilon = 0$ in (35), we see that for every $(x, y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$,

$$\psi_x^0(y) = -\frac{1}{2\pi} \arctan\left(\frac{\kappa_m \tan^2(2\pi y)}{\tan^2(2\pi y) + \kappa_m \tan(2\pi y) + 1}\right).$$

In particular, the previous quantity is always nonpositive; it vanishes for $y = 0$ and attains its minimum when $\tan(2\pi y) = -2/\kappa_m$. Therefore, for any $(x, y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$,

$$\psi_x^0(y) \geq -\frac{1}{2\pi} \arctan\left(\frac{\kappa_m}{1 - (\kappa_m/2)^2}\right).$$

Since $0 \leq \kappa_m \leq Ce^{-cn}$, the result follows. □

4.3. **Some qualitative remarks.** We know that the rotation number $\rho_m(\varepsilon)$ is obtained by averaging $(x, y) \mapsto \psi_x^\varepsilon(y)$ with respect to a measure $\mu = \mu_{m,\varepsilon}$ invariant by $(x, y) \mapsto (x + \alpha, y + \psi_x^\varepsilon(y))$:

$$\rho_m(\varepsilon) = \int \psi_x^\varepsilon(y)\; d\mu(x, y) \mod \mathbb{Z}. \tag{37}$$

For $\varepsilon = 0$, we have $B_0 \equiv B \in \mathrm{SL}_2(\mathbb{R})$. The parabolic matrix B has a unique fixed point $\begin{pmatrix}1\\0\end{pmatrix}$; accordingly, 0 is the unique fixed point of $y \mapsto y + \psi^0(y)$. Therefore, any measure invariant by $(x, y) \mapsto (x + \alpha, y + \psi^0(y))$ has support in the non-wandering set, which is here reduced to $\mathbb{R}/\mathbb{Z} \times \{0\}$. We deduce that the rotation number vanishes: $\rho_m(0) = 0$.

Assume that $\varepsilon > 0$, $|\varepsilon| \ll 1$. Since we consider the right boundary of a spectral gap, we know that $\kappa \geq 0$. We assume $\kappa > 0$. We then see that for $\varepsilon > 0$ very small, $\Delta_1^\varepsilon < 0$ and $\Delta_2^\varepsilon < 0$, so both P_ε and Q_ε are positive. Therefore, for any $x \in \mathbb{R}/\mathbb{Z}$ and any $y \in \mathbb{R}$, the drift $\psi_x^\varepsilon(y)$ is negative; in particular, the rotation number $\rho_m(\varepsilon)$ becomes strictly negative: we have left the spectral gap.

Let us now consider the case where $\varepsilon < 0$, $|\varepsilon| \ll 1$. From (33), we see that for any $x \in \mathbb{R}/\mathbb{Z}$,

$$\mathrm{tr}(B_\varepsilon(x)) = 2 - \kappa\varepsilon z_{11}^2(x) \geq 2,$$

hence these matrices are hyperbolic unless $z_{11}(x)$ vanishes. The action of every matrix $B_\varepsilon(x)$ on the circle has two fixed points, one attractive and one repulsive, each of which is very close to $\begin{pmatrix}1\\0\end{pmatrix}$, and the dynamics is of type north pole – south pole. Moreover, the fixed points correspond to those values of y for which $\psi_x^\varepsilon(y) = 0$, that is when $P_\varepsilon(x, \tan(2\pi y)) = 0$, which yields

$$\tan(2\pi y) = \frac{-\varepsilon(2z_{11}(x)z_{12}(x) - \kappa z_{11}^2(x)) \pm \sqrt{-\kappa\varepsilon z_{11}^2(x)(4 - \kappa\varepsilon z_{11}^2(x))}}{2(\kappa + \varepsilon(z_{12}^2(x) - \kappa z_{11}(x)z_{12}(x)))}. \tag{38}$$

Since $|\kappa| \ll 1$, and when $|\varepsilon|$ is very small, we obtain

$$y \simeq \frac{1}{2\pi} \arctan\left(\frac{-\varepsilon z_{11}(x)z_{12}(x) \pm \sqrt{-\kappa\varepsilon z_{11}^2(x)}}{\kappa + \varepsilon z_{12}^2(x)}\right).$$

In a first time, we have $|\varepsilon| \sup_{x\in\mathbb{R}/\mathbb{Z}} (z_{11}^2(x) + z_{12}^2(x)) \ll \kappa$, and the previous expression is approximately equal to $\frac{\pm 1}{2\pi}\sqrt{\frac{-\varepsilon z_{11}^2(x)}{\kappa}}$. We see that in this case, the two fixed points are very close to each other (and to 0). In particular, there is a big wandering domain, corresponding to the complement of the interval containing all the fixed points obtained for different values of x, and where the drift ψ^ε is always negative. When $|\tan(2\pi y)| \gg 1$, we see that ψ^ε is approximately equal to $\frac{-1}{2\pi}(\kappa + \varepsilon(z_{12}^2 - \kappa z_{11}z_{12}))$. Between the two fixed points, i.e. for $y \simeq 0$, ψ^ε is positive, of order

$\frac{-1}{2\pi}\varepsilon z_{11}^2$. By previous arguments, we know that when $\kappa > 0$, the spectral gap is non-degenerate. In particular, for $\varepsilon < 0$ with $|\varepsilon|$ very small, the energy $E+\varepsilon$ belongs to the spectral gap, that is, the cocycle $(\alpha, S_{\lambda v,E+\varepsilon})$ is uniformly hyperbolic, as well as its conjugate (α, B_ε).

We are interested in finding $\varepsilon < 0$, $|\varepsilon|$ exponentially small, such that the average of the drift becomes positive. We get a flavour of this fact by remarking that for $\varepsilon z_{12}^2 \simeq -\kappa$, the denominator of the expression in (38) becomes very small. The values of y for which $\psi^\varepsilon(y)$ vanishes correspond to $|\tan(2\pi y)| \gg 1$, and between them, the drift is always positive. In fact, we could make this quantitative by looking at the second iterate of the drift; again some quadratic forms appear, and the transition occurs for a value of $\varepsilon < 0$ for which the corresponding discriminants become negative. In what follows, we explain another way to see this transition.

4.4. **Estimates on the derivative of the second iterate of the drift.** We show here that the derivative of the second iterate of ψ^ε is negative, subexponentially small; this follows from monotonicity of Schrödinger cocycles, that we detailed above, through the conjugacy Z_m.

Recall that $\phi_{(\cdot)}^E$ is the drift associated with the Schrödinger cocycle $(\alpha, S_{\lambda v,E})$; we also denote by $\Phi_{(\cdot)}^E : y \mapsto y + \phi_{(\cdot)}^E(y)$ the associate map. The cocycle $(0, Z_m)$ acts on $\mathbb{R}/\mathbb{Z} \times \mathbb{S}^1$ as follows:

$$(x, e^{2i\pi y}) \mapsto \left(x, \frac{Z_m(x)\cdot e^{2i\pi y}}{\|Z_m(x)\cdot e^{2i\pi y}\|}\right).$$

The previous map admits a lift to $\mathbb{R}/\mathbb{Z}\times\mathbb{R}$, given by

$$H^m : (x,y) \mapsto (x, H_x^m(y)).$$

In this case, the map $(x,y) \mapsto (x, (H_x^m)^{-1}(y))$ is also a lift for the action of $(Z_m)^{-1}$. We have the following lemma.

Lemma 4.3. *For any $(x,y) \in \mathbb{R}/\mathbb{Z}\times\mathbb{R}$,*

$$\begin{aligned}&\partial_\varepsilon|_{\varepsilon=0}\left[\psi_x^\varepsilon(y) + \psi_{x+\alpha}^\varepsilon(y+\psi_x^\varepsilon(y))\right]\\ &= [(H_{x+2\alpha}^m)^{-1}]'(\Phi_{x+\alpha}^{E_m^+}\circ\Phi_x^{E_m^+}\circ H_x^m(y))\cdot\partial_E|_{E=E_m^+}\left[\Phi_{x+\alpha}^E\circ\Phi_x^E\right](H_x^m(y)).\end{aligned}$$

Proof. The conjugacy relation (33) translates as follows in terms of the lifted dynamics (see [A] for instance): for any $\varepsilon\in\mathbb{R}$, and any $(x,y)\in\mathbb{R}/\mathbb{Z}\times\mathbb{R}$,

$$y + \psi_x^\varepsilon(y) = (H_{x+\alpha}^m)^{-1}\circ\Phi_x^{E_m^++\varepsilon}\circ H_x^m(y).$$

Iterating once, we obtain:

$$y+\psi_x^\varepsilon(y)+\psi_{x+\alpha}^\varepsilon(y+\psi_x^\varepsilon(y)) = (H_{x+2\alpha}^m)^{-1}\circ\Phi_{x+\alpha}^{E_m^++\varepsilon}\circ\Phi_x^{E_m^++\varepsilon}\circ H_x^m(y).$$

We get:

$$\begin{aligned}&\partial_\varepsilon|_{\varepsilon=0}\left[\psi_x^\varepsilon(y) + \psi_{x+\alpha}^\varepsilon(y+\psi_x^\varepsilon(y))\right]\\ &= [(H_{x+2\alpha}^m)^{-1}]'(\Phi_{x+\alpha}^{E_m^+}\circ\Phi_x^{E_m^+}\circ H_x^m(y))\cdot\partial_E|_{E=E_m^+}\left[\Phi_{x+\alpha}^E\circ\Phi_x^E\right](H_x^m(y)).\end{aligned}$$

□

Lemma 4.4. *The quantities $(H_{(\cdot)}^m)'$, $[(H_{(\cdot)}^m)^{-1}]'$ are always positive. Moreover, we have the following estimates:*

$$\begin{aligned}\inf_{x\in\mathbb{R}/\mathbb{Z}}\ \inf_{y\in\mathbb{R}}(H_x^m)'(y) &\geq e^{-o(n)},\\ \inf_{x\in\mathbb{R}/\mathbb{Z}}\ \inf_{y\in\mathbb{R}}[(H_x^m)^{-1}]'(y) &\geq e^{-o(n)}.\end{aligned} \tag{39}$$

Although we will not need this in the following, note that we also have

$$\sup_{y\in\mathbb{R}} \|(H_x^m)'(y)\|_0 \le e^{o(n)}, \qquad \sup_{y\in\mathbb{R}} \|[(H_x^m)^{-1}]'(y)\|_0 \le e^{o(n)}. \tag{40}$$

Proof. The quantity $H_x^m(y)$ is characterized by the following equation:

$$\tan(2\pi H_x^m(y)) = \frac{z_{21}(x)+z_{22}(x)\tan(2\pi y)}{z_{11}(x)+z_{12}(x)\tan(2\pi y)}.$$

Set $t=\tan(2\pi y)$. Since det $(Z_m)=1$, we get

$$\begin{aligned}(H_{(\cdot)}^m)'(y) &= \frac{1+\tan(2\pi y)^2}{(1+\tan(2\pi H_x^m(y))^2)(z_{11}+z_{12}\tan(2\pi y))^2}\\ &= \frac{1+t^2}{(z_{12}^2+z_{22}^2)t^2+2(z_{11}z_{12}+z_{21}z_{22})t+(z_{11}^2+z_{21}^2)}.\end{aligned}$$

The denominator is equal to $(z_{11}+z_{12}t)^2+(z_{21}+z_{22}t)^2$. Set

$$\begin{cases} a_m &:= & z_{12}^2+z_{22}^2,\\ b_m &:= & 2(z_{11}z_{12}+z_{21}z_{22}),\\ c_m &:= & z_{11}^2+z_{21}^2,\end{cases}$$

so that $(H_{(\cdot)}^m)'(y)=\frac{1+t^2}{a_mt^2+b_mt+c_m}$. Since det $(Z_m)=1$, we have $b_m^2-4a_mc_m=-4$. In particular, the discriminant of the denominator is equal to -4, and $(H_{(\cdot)}^m)'(y)$ is always positive. Set $M_m:=\frac{2|b_m|}{a_m}$. For any $t\in[-M_m,M_m]$, we obtain

$$\frac{1+t^2}{a_mt^2+b_mt+c_m} \ge \frac{1}{a_mM_m^2+|b_m|M_m+c_m} \ge \frac{1}{25c_m}.$$

For any $|t|\ge M_m$, we have $a_mt^2\ge 2|b_mt|$, hence

$$\frac{1+t^2}{a_mt^2+b_mt+c_m} \ge \frac{1+t^2}{2a_mt^2+c_m} \ge \min\left(\frac{1}{c_m},\frac{1}{2a_m}\right).$$

We know that $\|Z_m\|_0\le e^{o(n)}$. We deduce:

$$\inf_{x\in\mathbb{R}/\mathbb{Z}}\inf_{y\in\mathbb{R}}(H_x^m)'(y) \ge \frac{1}{100\|Z_m\|_0^2} \ge e^{-o(n)}.$$

The polynomial $(a_mt^2+b_mt+c_m)$ is minimal when $t=t_m:=-\frac{b_m}{2a_m}$, and it is then equal to $\frac{1}{a_m}$; note that $t_m\in[-M_m,M_m]$. For any $t\in[-M_m,M_m]$, we have

$$\frac{1+t^2}{a_mt^2+b_mt+c_m} \le a_m(1+M_m^2) \le a_m+16c_m.$$

For any $|t|\ge M_m$, we have $a_mt^2\ge 2|b_mt|$, hence

$$\frac{1+t^2}{a_mt^2+b_mt+c_m} \le \frac{2(1+t^2)}{a_mt^2+2c_m} \le \max(2/a_m,1/c_m).$$

Recall that $\|Z_m\|_0\le e^{o(n)}$. Moreover a_m and c_m correspond to the squares of the norms of the columns of Z_m, hence $\inf_{\mathbb{R}/\mathbb{Z}} a_m, \inf_{\mathbb{R}/\mathbb{Z}} c_m \ge e^{-o(n)}$ and we deduce

$$\sup_{y\in\mathbb{R}}\|(H_x^m)'(y)\|_0 \le e^{o(n)}.$$

The map $y\mapsto (H_x^m)^{-1}(y)$ is defined in the same way for the matrix $Z_m(x)^{-1}$. In particular, for any $(x,y)\in\mathbb{R}/\mathbb{Z}\times\mathbb{R}$, we have

$$\tan(2\pi(H_x^m)^{-1}(y)) = \frac{-z_{21}(x)+z_{11}(x)\tan(2\pi y)}{z_{22}(x)-z_{12}(x)\tan(2\pi y)},$$

hence

$$[(H^m_{(\cdot)})^{-1}]'(y) = \frac{1+t^2}{(z_{12}^2+z_{11}^2)t^2 - 2(z_{22}z_{12}+z_{21}z_{11})t + (z_{22}^2+z_{21}^2)},$$

and the estimates are proved similarly. □

Recall that $\psi^\varepsilon = \psi^\varepsilon_m$ denotes the drift associated with the cocycle $(\alpha, B_{m,\varepsilon})$. Combining Proposition 4.1, Lemma 4.3 and Lemma 4.4, we deduce the following lower bound on the derivative of its second iterate:

Proposition 4.5. *There exists $\eta_0 > 0$ such that for any $(x,y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$, we have*

$$\partial_\varepsilon|_{\varepsilon=0} \left[\psi^\varepsilon_{m,x}(y) + \psi^\varepsilon_{m,x+\alpha}(y+\psi^\varepsilon_{m,x}(y))\right] \leq -\eta_0 e^{-o(n)} < 0. \tag{41}$$

Proof. We have seen in Lemma 4.3 that for any $(x,y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$,

$$\begin{aligned}&\partial_\varepsilon|_{\varepsilon=0} \left[\psi^\varepsilon_{m,x}(y) + \psi^\varepsilon_{m,x+\alpha}(y+\psi^\varepsilon_{m,x}(y))\right]\\ &= [(H^m_{x+2\alpha})^{-1}]'(\Phi^{E_m^+}_{x+\alpha} \circ \Phi^{E_m^+}_x \circ H^m_x(y)) \cdot \partial_E|_{E=E_m^+} \left[\Phi^E_{x+\alpha} \circ \Phi^E_x\right](H^m_x(y)).\end{aligned} \tag{42}$$

Now, we know from Proposition 4.1 that there exists $\eta_0 > 0$ such that for any $(x,y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$,

$$\partial_E|_{E=E_m^+} \left[\Phi^E_{x+\alpha} \circ \Phi^E_x\right](y) = \partial_E|_{E=E_m^+} [\phi^E_x(y) + \phi^E_{x+\alpha}(y+\phi^E_x(y))] \leq -\eta_0. \tag{43}$$

From Lemma 4.4 we also have:

$$\inf_{x\in\mathbb{R}/\mathbb{Z}} \inf_{y\in\mathbb{R}} [(H^m_x)^{-1}]'(y) \geq e^{-o(n)}. \tag{44}$$

Equation (41) then follows from (42), (43) and (44). □

4.5. **End of the proof.** Let us see how the previous results imply exponential decay of the size of spectral gaps.

For the spectral gap G_m with label $m \in \mathbb{Z}\setminus\{0\}$, we have the following expansion with respect to ε: there exist continuous functions $\delta^i_m\colon (x,y) \mapsto \delta^i_{m,x}(y)$, $i = 0, 1$, such that for any $(x,y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$,

$$\psi^\varepsilon_{m,x}(y) + \psi^\varepsilon_{m,x+\alpha}(y+\psi^\varepsilon_{m,x}(y)) = \delta^0_{m,x}(y) + \varepsilon\delta^1_{m,x}(y) + O(\varepsilon^2);$$

from (36) and (41), we have uniform bounds on them: for any $(x,y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$,

$$0 \leq -\delta^0_{m,x}(y) \leq Ce^{-cn}, \qquad \delta^1_{m,x}(y) \leq -\eta_0 e^{-o(n)}.$$

By compactness, there exists $\varepsilon''_m > 0$,

$$\varepsilon''_m \lesssim \frac{\sup_{x,y} |\delta^0_{m,x}(y)|}{\inf_{x,y} |\delta^1_{m,x}(y)|} \leq C\eta_0 e^{-n(c+o(1))}, \tag{45}$$

such that for $\varepsilon < -\varepsilon''_m$, we may find $\omega^\varepsilon_m > 0$ satisfying: for any $(x,y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$,

$$\psi^\varepsilon_{m,x}(y) + \psi^\varepsilon_{m,x+\alpha}(y+\psi^\varepsilon_{m,x}(y)) \geq \omega^\varepsilon_m > 0. \tag{46}$$

Take $\varepsilon < -\varepsilon''_m$, and let μ be a measure invariant by $(x,y) \mapsto (x+\alpha, y+\psi^\varepsilon_{m,x}(y))$. In particular, we deduce from (46) that

$$\rho_m(\varepsilon) = \int \psi^\varepsilon_{m,x}(y)\, d\mu(x,y) \bmod \mathbb{Z} \geq \omega^\varepsilon_m/2 > 0 = \rho_m(0). \tag{47}$$

Thanks to (34) and the remark that follows it, we deduce that $E_m^+ + \varepsilon \notin G_m$; therefore $E_m^+ - E_m^- \leq \varepsilon''_m$. From (45) and since $|m| \leq C'n$ by (11), we obtain

$$E_m^+ - E_m^- \leq Ce^{-c|m|},$$

which concludes the proof. □

5. Application: homogeneity of the spectrum

Let us recall the definition of a homogeneous set.

Definition 5.1 (Homogeneous set). *A closed set $\mathcal{S} \subset \mathbb{R}$ is called homogeneous if there exist $\chi > 0$ and $\sigma_0 > 0$ such that for any $E \in \mathcal{S}$ and any $0 < \sigma \leq \sigma_0$, we have*

$$\mathrm{Leb}((E-\sigma, E+\sigma) \cap \mathcal{S}) > \chi\sigma.$$

In the case of Schrödinger operators, homogeneity of the spectrum $\mathcal{S}$ has a lot of consequences, in particular from the point of view of inverse spectral theory [5]. Fix a set $\mathcal{S}$ and consider the set of potentials V such that the associate Schrödinger operator has spectrum $\mathcal{S}$ and is reflectionless on it. Then under the hypothesis of finite total gap length, Gesztesy and Yuditskii [GY] have shown that homogeneity of $\mathcal{S}$ ensures that all the potentials associated with it are almost periodic, and the corresponding spectral measures are purely absolutely continuous.

In what follows we are interested in direct spectral analysis: we show that the spectrum of the quasiperiodic Schrödinger operators we are considering is homogeneous. The proof uses our previous estimates on the size of spectral gaps, and mimics the arguments of [DGL1].

Theorem 5.2. *Let $\alpha \in \mathrm{DC}(K,\tau)$, let $v \in C^\omega(\mathbb{R}/\mathbb{Z},\mathbb{R})$ be an analytic potential and take $0 < |\lambda| < \lambda_0(v)$. Then the spectrum Σ of the operator H is homogeneous with $\chi = 1/2$.*

Let us recall the following theorem of Avila and Jitomirskaya: [6]

Theorem 5.3 (Theorem 1.6, [AJ2]). *Let $\alpha \in \mathrm{DC}$, $v \in C^\omega(\mathbb{R}/\mathbb{Z},\mathbb{R})$ and $0 < |\lambda| < \lambda_0(v)$. Then the rotation number is 1/2-Hölder: there exists $\widetilde{C} > 0$ such that for any $E, E' \in \mathbb{R}$,*

$$|\rho(E') - \rho(E)| \leq \widetilde{C}|E'-E|^{1/2}. \tag{48}$$

Recall that if $m \in \mathbb{Z} \setminus \{0\}$, we denote by $G_m := (E_m^-, E_m^+)$ the gap labeled by m, and we let $G_0 := (-\infty, \underline{E})$. We have seen that there exist constants $C, \gamma > 0$ such that for every $m \in \mathbb{Z}$,

$$E_m^+ - E_m^- \leq Ce^{-\gamma|m|}.$$

Lemma 5.4 (see Theorem G, [DGL1]). *We have lower bounds on the distance between distinct spectral gaps:*

(1) *For every $m, m' \in \mathbb{Z} \setminus \{0\}$ with $m \neq m'$, $|m'| \geq |m|$, we have*

$$\mathrm{dist}([E_m^-, E_m^+], [E_{m'}^-, E_{m'}^+]) \geq \frac{K'}{|m'|^{\tau'}}$$

for some constants $K', \tau' > 0$ depending on $K, \tau, \widetilde{C}$.

(2) *Analogously, for every $m \neq 0$, we have*

$$E_m^- - \underline{E} \geq \frac{K'}{|m|^{\tau'}}.$$

Proof. Recall that for any $m \in \mathbb{Z} \setminus \{0\}$, $\|m\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{K}{|m|^\tau}$ by the Diophantine condition on the frequency α. Let $m' \neq m \in \mathbb{Z} \setminus \{0\}$, $|m'| \geq |m|$. Then

$$\frac{\|(m-m')\alpha\|_{\mathbb{R}/\mathbb{Z}}}{2} \geq \frac{K}{2|m'-m|^\tau} \geq \frac{K_1}{|m'|^\tau}$$

5. This corresponds to the case where one has some information on the spectrum and one tries to recover information about the potential.

6. In their paper, they state it for the integrated density of states N, but $N(E) = 1 - 2\rho(E)$.

where $K_1 := K/2^{\tau+1}$. Assume that $E_{m'}^- > E_m^+$; in particular $E_{m'}^- - E_m^+ = \operatorname{dist}([E_m^-, E_m^+], [E_{m'}^-, E_{m'}^+])$. Thanks to (48), we obtain

$$E_{m'}^- - E_m^+ \geq \frac{\|(m-m')\alpha\|_{\mathbb{R}/\mathbb{Z}}^2}{4\widetilde{C}^2} \geq \frac{K_1^2}{\widetilde{C}^2|m'|^{2\tau}} \geq \frac{K'}{|m'|^{\tau'}}$$

with $K' := \frac{K_1^2}{\widetilde{C}^2}$ and $\tau' := 2\tau$. The other inequality is proved in the same way. □

Proof of Theorem 5.2. Let $E \in \Sigma$ and $\sigma > 0$. We define

$$\mathfrak{C}(E, \sigma) := \{m \in \mathbb{Z} \setminus \{0\},\ (E_m^-, E_m^+) \cap (E - \sigma, E + \sigma) \neq \emptyset\}.$$

It corresponds to the set of all nonzero labels associated with spectral gaps whose intersection with the interval we consider is nonempty.

With the previous notations, assume first that $(-\infty, \underline{E}) \cap (E - \sigma, E + \sigma) = \emptyset$. We define $m_0 := m_0(E, \sigma)$ so that $|m_0| = \min_{m \in \mathfrak{C}(E,\sigma)} |m|$. For any $m \in \mathfrak{C}(E, \sigma)$, $m \neq m_0$, we have

$$\operatorname{dist}([E_m^-, E_m^+], [E_{m_0}^-, E_{m_0}^+]) \leq 2\sigma. \tag{49}$$

On the other hand, we know from Lemma 5.4 that

$$\operatorname{dist}([E_m^-, E_m^+], [E_{m_0}^-, E_{m_0}^+]) \geq \frac{K'}{|m|^{\tau'}}. \tag{50}$$

It follows from (49) and (50) that $|m| \geq \beta\sigma^{-\rho}$ for some constants $\beta, \rho > 0$ depending on $K, \tau, \widetilde{C}$. Moreover we know that

$$E_m^+ - E_m^- < Ce^{-\gamma|m|}.$$

We thus obtain

$$\begin{aligned}\sum_{m \in \mathfrak{C}(E,\sigma)\setminus\{m_0\}} \operatorname{Leb}((E_m^-, E_m^+) \cap (E - \sigma, E + \sigma)) &\leq \sum_{m \in \mathfrak{C}(E,\sigma)\setminus\{m_0\}} E_m^+ - E_m^- \\ &\leq C \sum_{|m| \geq \beta\sigma^{-\rho}} e^{-\gamma|m|} \\ &\leq \sigma/2\end{aligned}$$

provided that $\sigma \leq \sigma_0$ for some $\sigma_0 = \sigma_0(K, \tau, \widetilde{C}, \gamma) > 0$. Moreover, since $E \in \Sigma$, we have $E \notin (E_{m_0}^-, E_{m_0}^+)$. Hence

$$\operatorname{Leb}((E_{m_0}^-, E_{m_0}^+) \cap (E - \sigma, E + \sigma)) \leq \sigma.$$

Combining the previous inequalities, we finally get

$$\begin{aligned}\operatorname{Leb}((E - \sigma, E + \sigma) \cap \mathcal{S}) &\geq 2\sigma - \operatorname{Leb}((-\infty, \underline{E}) \cap (E - \sigma, E + \sigma)) - \\ &\quad - \operatorname{Leb}((E_{m_0}^-, E_{m_0}^+) \cap (E - \sigma, E + \sigma)) - \\ &\quad - \sum_{m \in \mathfrak{C}(E,\sigma)\setminus\{m_0\}} \operatorname{Leb}((E_m^-, E_m^+) \cap (E - \sigma, E + \sigma)) \\ &\geq 2\sigma - 0 - \sigma - \sigma/2 \\ &= \sigma/2\end{aligned}$$

provided that $\sigma \leq \sigma_0(K, \tau, \widetilde{C}, \gamma)$.

Now assume that $(-\infty, \underline{E}) \cap (E - \sigma, E + \sigma) \neq \emptyset$. For any $m \in \mathfrak{C}(E, \sigma)$, we have $0 \leq E_m^- - \underline{E} \leq 2\sigma$. We also have from Lemma 5.4 that $E_m^- - \underline{E} \geq \frac{K'}{|m|^{\tau'}}$, so that $|m| \geq \beta\sigma^{-\rho}$. We then get

$$\sum_{m \in \mathfrak{C}(E,\sigma)} \operatorname{Leb}((E_m^-, E_m^+) \cap (E - \sigma, E + \sigma)) \leq C \sum_{|m| \geq \beta\sigma^{-\rho}} e^{-\gamma|m|} \leq \sigma/2$$

provided that $\sigma \leq \sigma_0(K,\tau,\widetilde{C},\gamma)$. Moreover, since $E \in \Sigma$, $E \notin (-\infty,\underline{E})$, hence $\mathrm{Leb}((-\infty,\underline{E}) \cap (E-\sigma,E+\sigma)) \leq \sigma$. Similarly, we obtain

$$\begin{aligned}\mathrm{Leb}((E-\sigma,E+\sigma)\cap\mathcal{S}) &\geq 2\sigma - \mathrm{Leb}((-\infty,\underline{E})\cap(E-\sigma,E+\sigma)) - \\ &- \sum_{m\in\mathfrak{C}(E,\sigma)} \mathrm{Leb}((E_m^-,E_m^+)\cap(E-\sigma,E+\sigma)) \\ &\geq 2\sigma - \sigma - \sigma/2 \\ &= \sigma/2\end{aligned}$$

provided that $\sigma \leq \sigma_0(K,\tau,\widetilde{C},\gamma)$. □

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