Belgrade Summer School on Dynamics

July 17-22, 2021

Entropy rigidity for 3D conservative Anosov flows and dispersing billiards

Anosov Flows

- (M, g): smooth compact Riemannian manifold
- $\Phi = (\Phi^t)_{t \in \mathbb{R}}$: C^2 flow on M
- $X_\Phi : x \mapsto \frac{d}{dt}|_{t=0}\Phi^t(x)$: flow vector field

Recall that Φ Anosov if $TM = E^s_\Phi \oplus \mathbb{R}X_\Phi \oplus E^u_\Phi$, and for $C, \lambda > 0$:

$$\|D_x\Phi^t \cdot v\| \leq Ce^{-\lambda t}\|v\|, \qquad \forall x \in \Lambda,\ v \in E^s_\Phi(x),\ t \geq 0$$

$$\|D_x\Phi^{-t} \cdot v\| \leq Ce^{-\lambda t}\|v\|, \qquad \forall x \in \Lambda,\ v \in E^u_\Phi(x),\ t \geq 0$$

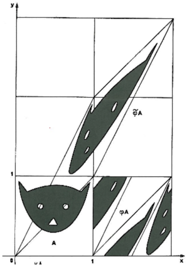
Notation: $E^{cs}_\Phi := E^s_\Phi \oplus \mathbb{R}X_\Phi$, $E^{cu}_\Phi := E^u_\Phi \oplus \mathbb{R}X_\Phi$

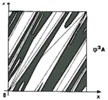
Anosov Flows

- suspensions of Anosov diffeomorphisms:

  e.g. for the cat map $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ on $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$

  $\rightsquigarrow$ flow $(x,s) \mapsto (x, s+t)$ on $\mathbb{T}^2 \times \mathbb{R}/\sim$, $(x,1) \sim (A \cdot x, 0)$





- geodesic flows on negatively curved Riemannian manifolds
- surgeries $\rightsquigarrow$ new examples (Handel-Thurston, Goodman...)

Algebraic systems

- Algebraic systems: "affine systems on homogeneous spaces"
  - G Lie group
  - $K \subset G$ compact subgroup
  - $\Gamma \subset G$ discrete cocompact subgroup (uniform lattice)
  - $M = \Gamma \backslash G / K$ homogeneous
  - algebraic flow $\Phi = (\Phi^t)_{t \in \mathbb{R}}$ on M:
    $\Phi^t \colon \Gamma g K \mapsto \Gamma g \exp(t\alpha) K$, with α in the Lie algebra of G

- Φ algebraic Anosov flow on a 3-mfd:

Tomter ('68) $\rightsquigarrow$ up to finite cover,

  - geodesic flow of a surface of constant negative curvature
  - suspension of a hyperbolic automorphism of $\mathbb{T}^2$

Anosov flows

- Anosov diffeomorphisms: conjecturally, up to topological conjugacy, algebraic models account for all Anosov diffeomorphisms
- Anosov flows $\rightsquigarrow$ rich behavior:
  - Franks-Williams ('80): there exists a closed connected 3-manifold M which admits a non-transitive Anosov flow
  - Handel-Thurston ('80): there exists a compact 3-manifold M which admits an analytic non-algebraic Anosov flow
  - Foulon-Hasselblatt ('13): there exists a hyperbolic 3-manifold M which admits a contact Anosov flow that is not topologically orbit equivalent to an algebraic flow
  - Bonatti-Béguin-Yu ('14): 3-manifolds supporting both transitive and non-transitive Anosov flows...

Locally symmetric spaces, Katok's conjecture

- (M,g): C^∞ smooth compact Riemannian manifold
- geodesic symmetry at $x \in M$: $s_x := \exp_x \circ(-\mathrm{Id}_{T_xM}) \circ \exp_x^{-1}$
- (M,g) locally symmetric if s_x is an isometry for all $x \in M$

Conjecture (Katok Entropy Conjecture)

- *(M,g): connected Riemannian manifold of negative curvature*
- *Φ: geodesic flow, μ: Liouville measure*

Then $h_{\mathrm{top}}(\Phi) = h_\mu(\Phi) \iff (M,g)$ is locally symmetric

Theorem (Katok, '82)

- *(S,g): negatively curved surface*
- *Φ: geodesic flow, μ: Liouville measure*

Then $h_{\mathrm{top}}(\Phi) = h_\mu(\Phi) \iff (S,g)$ has constant < 0 curvature

Natural invariant measures

Φ transitive Anosov flow on a compact Riemannian manifold M

- unique invariant proba. ν such that $h_{\text{top}}(\Phi) = h_\nu(\Phi)$: measure of maximal entropy (or MME)
- unique invariant proba. whose conditionals along unstable manifolds are absolutely continuous with respect to Lebesgue: Sinai-Ruelle-Bowen (or SRB) measure

Remark

When Φ preserves a smooth volume Vol, it is transitive/ergodic, and SRB measure $= \mathrm{Vol}$

Question

Let Φ be a C^∞ transitive Anosov on a 3-manifold.
If MME $=$ SRB, is Φ smoothly conjugate to an algebraic flow?

Contact Anosov flows & Foulon's question

Theorem (Foulon, '01)

Let Φ be a contact Anosov flow on a closed 3-manifold.

Then MME = contact volume
$\iff \Phi$ is, up to finite cover, smoothly conjugate to geodesic flow of a metric of constant negative curvature on a closed surface

Question (Foulon)

Let Φ be a smooth Anosov flow on a 3-manifold which preserves a smooth volume μ.
If $h_{\mathrm{top}}(\Phi) = h_\mu(\Phi)$, is Φ smoothly conjugate to an algebraic flow?

Main result: entropy rigidity for $3D$ Anosov flows

Positive answer to Foulon's question:

Theorem (De Simoi-L.-Vinhage-Yang, '20)

Let $k \geq 5$ and let Φ be a C^k Anosov flow on a compact connected 3-manifold M such that $\Phi_* \mu = \mu$ for some smooth volume μ.

Then $h_{\mathrm{top}}(\Phi) = h_\mu(\Phi) \iff \Phi$ is $C^{k-\varepsilon}$-conjugate to an algebraic flow, for $\varepsilon > 0$ arbitrarily small

Rigidity: high regularity phenomenon

Remark

Parry's synchronization procedure ('86):

- *Φ: C^2 Axiom A flow on a compact Riemannian manifold*
- *Λ: attractor whose unstable distribution is C^1*

$\rightsquigarrow$ *there exists a C^1 time change such that for the new flow, the SRB measure of the attractor coincides with the MME*

$\rightsquigarrow$ for any C^2 transitive Anosov flow Φ on a 3-manifold: Φ is C^1-orbit equivalent to an Anosov flow for which the SRB measure is equal to the MME

Remark

Adeboye-Bray-Constantine ('19): there exist systems with more geometric structure that still exhibit rigidity in low regularity

Entropy rigidity for Anosov flows

Some ingredients of the proof:

1. equality of periodic Lyapunov exponents when MME = SRB
2. expansion of Lyapunov exponents of some periodic orbits with prescribed combinatorics (Birkhoff Normal Form)
3. Anosov cocycle/class and smoothness of the invariant foliations, connection with the BNF (after Hurder-Katok)
4. orbit equivalence to algebraic flow when smooth weak stable/unstable foliations (after Ghys)
5. from orbit equivalence to flow conjugacy

1) Periodic Lyapunov exponents when MME = SRB

- M: C^∞ compact Riemannian 3-manifold
- $\Phi\colon M \to M$ conservative C^k Anosov flow, $k \geq 2$
- for any periodic orbit $\mathcal{O} = \{\Phi^t(x)\}_t$ of period $\mathcal{L}(\mathcal{O}) = \mathcal{L}(x) > 0$, the Lyapunov exponent of $\mathcal{O}$ is

$$\mathrm{LE}(\mathcal{O}) = \mathrm{LE}(x) = \frac{1}{\mathcal{L}(x)} \log J_x^u(\mathcal{L}(x)),$$

where $J_x^u(t)$: Jacobian of $D\Phi^t\colon E^u(x) \to E^u(\Phi^t(x))$

Proposition (De Simoi-L.-Vinhage-Yang)

If SRB measure = MME, then for any periodic orbit $\mathcal{O}$:

$$\mathrm{LE}(\mathcal{O}) = h_{\mathrm{top}}(\Phi)$$

Equivalently, for any $x \in \mathcal{O}$:

$$J_x^u(\mathcal{L}(x)) = e^{h_{\mathrm{top}}(\Phi)\mathcal{L}(x)}$$

Birkhoff Normal Form (BNF)

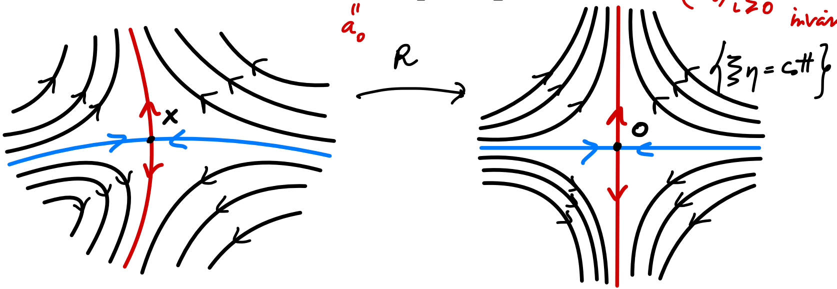
- let F be a local C^∞ conservative surface diffeomorphism
- $x = F^q(x)$ hyperbolic periodic point
- $0 < \lambda < 1 < \lambda^{-1}$: eigenvalues of DF^q_x

(Moser)-Sternberg :
there exists a local C^∞ volume-preserving map R which conjugates F^q to its Birkhoff Normal Form $N = R \circ F^q \circ R^{-1}$:

$$N = N_\Delta : (\xi, \eta) \mapsto (\Delta(\xi\eta) \cdot \xi, \Delta(\xi\eta)^{-1} \cdot \eta)$$

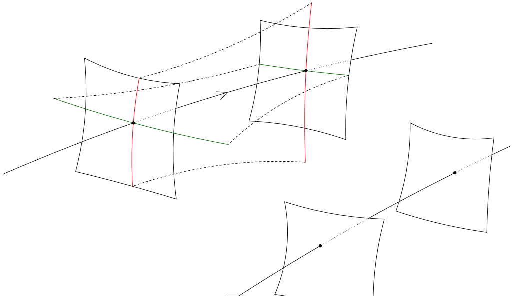
for some function $\Delta : z \mapsto \lambda + a_1 z + a_2 z^2 + \ldots$ $\leadsto (a_i)_{i \geq 0}$ Birkhoff invariants

$\lambda = a_0$



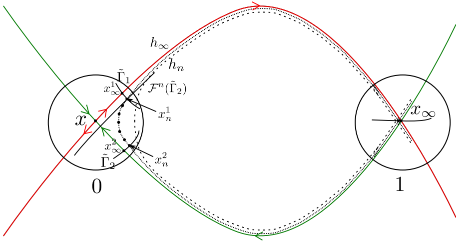
2) Expansion of periodic Lyapunov exponents (BNF)

- M: C^∞ smooth compact Riemannian 3-manifold
- let $\Phi\colon M \to M$ be a C^k Anosov flow, $k \geq 2$
- $\Phi_*\mu = \mu$ for some smooth volume measure μ
- Markov family $\mathcal{R}$ for Φ with cross section $\mathcal{S}$

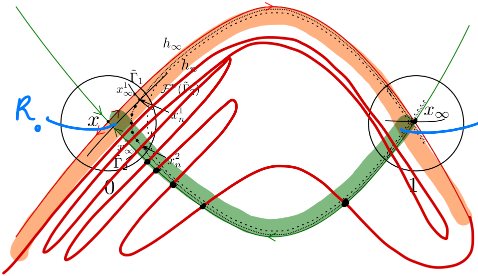


2) Expansion of periodic Lyapunov exponents (BNF)

- $\mathcal{F}$: Poincaré map induced by Φ on the cross section $\mathcal{S}$
- $x \in \mathcal{S}$: periodic point of period $\mathcal{L}(x) > 0$
- $x_\infty \in \mathcal{S}$: homoclinic point $(x_\infty \in \mathcal{W}^{cs}_\Phi(x) \pitchfork \mathcal{W}^{cu}_\Phi(x))$



2) Expansion of periodic Lyapunov exponents (BNF)



- $x_\infty \longleftrightarrow \dots 000\underset{\uparrow}{1}000 \dots$: homoclinic point
- sequence $(h_n)_{n\geq 0}$ of periodic orbits in the horseshoe:

$$h_n \longleftrightarrow \dots|\underbrace{0\dots0}_{n+1}1|\underbrace{0\dots0}_{n+1}1|\underbrace{0\dots0}_{n+1}1|\dots$$

$$x_n^1 \longleftrightarrow \dots|0\dots0\underset{\uparrow}{0}1|0\dots01|\dots$$

2) Expansion of periodic Lyapunov exponents (BNF)

- x is a saddle fixed point, with eigenvalues $0 < \lambda < 1 < \lambda^{-1}$
- near $x \in \mathcal{S}$, $\exists\, C^k$ change of coordinates R such that

$$R \circ \mathcal{F} \circ R^{-1} = N\colon (\xi, \eta) \mapsto (\Delta(\xi\eta)\xi, \Delta(\xi\eta)^{-1}\eta),$$

for some C^k function $\Delta(z) = \lambda + a_1 z + \dots$
$\rightsquigarrow a_1 \in \mathbb{R}$: first Birkhoff invariant

Proposition (De Simoi-L.-Vinhage-Yang)

As $n \to +\infty$, we have the asymptotic expansion

$$\mathrm{tr}(D\mathcal{F}^{n+2}_{x_n^1}) = C_0\lambda^{-n} + nC_1 a_1 + O(1)$$

where $C_0, C_1 \in \mathbb{R}^*$ are nonzero constants

2) Expansion of periodic Lyapunov exponents (BNF)

- let $\mathcal{L}^{(0)} = \mathcal{L}(x) > 0$ be the period of x for Φ
- for any $n \geq 0$, let $\mathcal{L}_n = \mathcal{L}(x_n^1) > 0$ be the period of x_n^1 for Φ

Lemma

For some $\mathcal{L}^{(1)} \in \mathbb{R}$, we have the following asymptotic expansion:

$$\mathcal{L}_n = n\mathcal{L}^{(0)} + \mathcal{L}^{(1)} + O(\lambda^n)$$

Lemma

If MME = SRB measure, then for each $n \geq 0$, the eigenvalues of $D\mathcal{F}_{x_n^1}^{n+2}$ are equal to $e^{\pm h_{\mathrm{top}}(\Phi)\mathcal{L}_n}$

Corollary

If MME = SRB measure, then

$$\mathrm{tr}(D\mathcal{F}_{x_n^1}^{n+2}) = C_\infty^{-1}\lambda^{-n} + O(1), \qquad C_\infty := e^{h_{\mathrm{top}}(\Phi)\mathcal{L}^{(1)}} > 0$$

2) Expansion of periodic Lyapunov exponents (BNF)

Lemma

- $\mathrm{tr}(D\mathcal{F}^{n+2}_{x_n^1}) = C_0\lambda^{-n} + nC_1 a_1 + O(1)$, *where* $C_0, C_1 \neq 0$
- *if MME = SRB measure, then*

$$\mathrm{tr}(D\mathcal{F}^{n+2}_{x_n^1}) = C_\infty^{-1}\lambda^{-n} + O(1), \qquad C_\infty := e^{h_{\mathrm{top}}(\Phi)\mathcal{L}^{(1)}} > 0$$

Corollary (De Simoi-L.-Vinhage-Yang)

If MME = SRB measure, then for any periodic orbit $\mathcal{O}$, the first Birkhoff invariant at $\mathcal{O}$ of the Poincaré map $\mathcal{F}$ vanishes

3) Anosov cocycle/class (after Hurder-Katok)

$\Phi\colon M \to M$ conservative C^k Anosov flow, $k \geq 2$

Definition (Cocycle/coboundary)

- *$C\colon M \times \mathbb{R} \to \mathbb{R}$ is a C^1 cocycle over Φ if it is of class C^1 and*

$$C(x, t+s) = C(x,t) + C(\Phi^t(x), s), \quad \forall\, x \in M, \forall\, t, s \in \mathbb{R}$$

- *a C^1 cocycle $B\colon M \times \mathbb{R} \to \mathbb{R}$ over Φ is a C^1 coboundary if there exists a C^1 function $u\colon M \to \mathbb{R}$ such that*

$$B(x,t) = u \circ \Phi^t(x) - u(x), \quad \forall\, x \in M, \forall\, t \in \mathbb{R}$$

- *C^1-cohomology class of a C^1 cocycle $C\colon M \times \mathbb{R} \to \mathbb{R}$: image of C in the group of C^1 cocycles over Φ modulo coboundaries*

3) Anosov cocycle/class (after Hurder-Katok)

- the invariant foliations $\mathcal{W}_\Phi^{cs}, \mathcal{W}_\Phi^{cu}$ are C^1
- Anosov cocycle $A_\Phi \colon M \times \mathbb{R} \to \mathbb{R}$
- Anosov class $[A_\Phi]$: "obstruction to upgraded regularity of the invariant foliations"
- for any x in a periodic orbit $\mathcal{O}$ of period $\mathcal{L}(\mathcal{O}) = \mathcal{L}(x) > 0$,

$$A_\Phi(x, \mathcal{L}(x)) = -\lambda^{-1} a_1,$$

where $0 < \lambda < 1 < \lambda^{-1}$ are the eigenvalues of Poincaré map, and $a_1 = a_1(\mathcal{O})$ is the first Birkhoff invariant

Anosov cocycle & Normal forms

Let us restrict ourselves to smooth area-preserving Anosov diffeomorphisms of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

De Latte : $\exists$ local smooth coordinate systems around each point p that depends continuously (actually C^1) on the point p which brings the diffeomorphism into the

Moser Normal Form

$$f : (x,y) \longmapsto \begin{pmatrix} \lambda_p^{-1} x & \varphi_p^{-1}(x,y) \\ \lambda_p y & \varphi_p(x,y) \end{pmatrix} \qquad \left(\varphi_p^{-1} = \frac{1}{\varphi_p}\right)$$

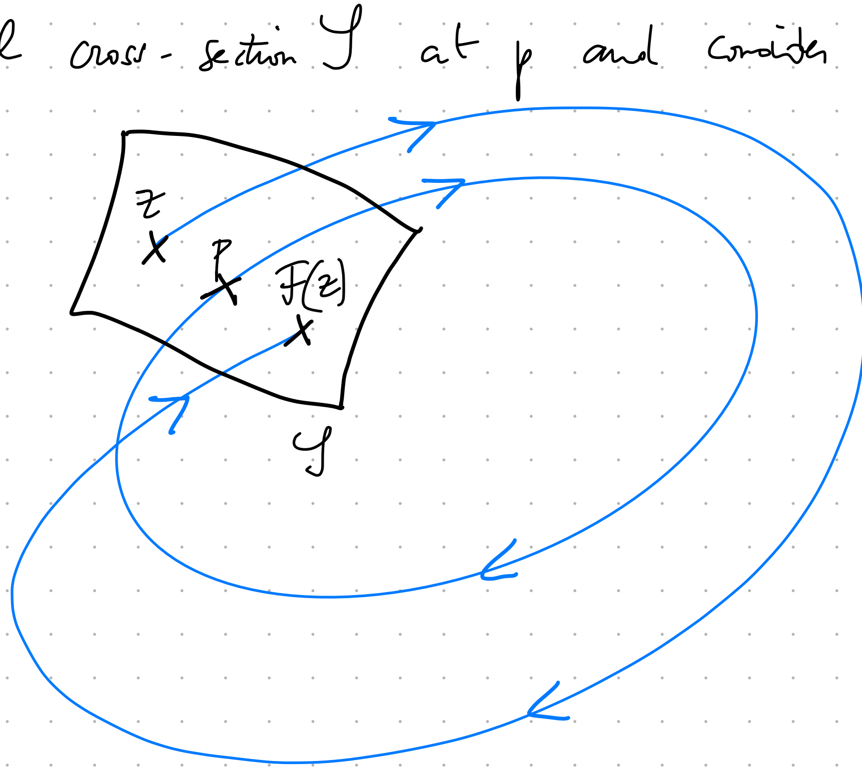
- (x,y) : local coordinates around the point p / $\hookrightarrow$ coordinates around the image of p
- $\varphi_p(x,y)$: corresponds to the resonance $\lambda_p \lambda_p^{-1} = 1$ ($\Leftarrow$ area-preservation)

  and more generally to the family of resonances $\lambda_p^{n+1} \lambda_p^{-n}$, $n \in \mathbb{Z}$

φ_p $\downarrow$ as smooth as f + $\varphi_p(0) = 1$

Let us now assume that p = periodic point of a flow

Take a local cross-section Σ at p and consider the Poincaré map F on Σ



$\rightsquigarrow$ f : Normal Form of F around p

$$\rightsquigarrow \quad df_{(x,y)} = \begin{pmatrix} \lambda^{-1} xy\,(\varphi^{-1})'(xy) + \lambda^{-1}\varphi^{-1}(xy) & \lambda^{-1} x^2 (\varphi^{-1})'(xy) \\ \lambda y^2 \varphi'(xy) & \lambda xy\,\varphi'(xy) + \lambda\varphi(xy) \end{pmatrix}$$

for $x = 0$ $\rightsquigarrow$ $df_{(0,y)} = \begin{pmatrix} \lambda^{-1} & 0 \\ \lambda y^2 \varphi'(0) & \lambda \end{pmatrix}$, $\quad \lambda \in (0,1)$.

$\rightsquigarrow$ unstable direction at $(0,y)$ is spanned by a vector $(1, a(y))$ for some function $a : \mathbb{R} \to \mathbb{R}$.

By invariance of the unstable direction under df, and since $f(0,y) = (0, \lambda y)$

$$df_{(0,y)} (1, a(y)) = \begin{pmatrix} \lambda^{-1} & 0 \\ \lambda y^2 \varphi'(0) & \lambda \end{pmatrix} \begin{pmatrix} 1 \\ a(y) \end{pmatrix} = \begin{pmatrix} \lambda^{-1} \\ \lambda y^2 \varphi'(0) + \lambda a(y) \end{pmatrix} = \kappa \begin{pmatrix} 1 \\ a(\lambda y) \end{pmatrix}$$

$$\Longrightarrow \quad a(\lambda y) = \lambda^2 y^2 \varphi'(0) + \lambda^2 a(y) \qquad (A)$$

If the unstable bundle is C^2, differentiating (A) twice with respect to y at 0:
(a is C^2)

$$\rightsquigarrow \quad \lambda^2 a''(0) = 2\lambda^2 \varphi'(0) + \lambda^2 a''(0)$$

i.e. $\varphi'(0) = 0$!

In other words, $\varphi'(0) \neq 0$ is an obstruction to the C^2 regularity of the invariant bundles (Anosov obstruction)

3) Anosov cocycle/class (after Hurder-Katok)

- $\Phi\colon M \to M$ a C^k Anosov flow on some 3-manifold M, $k \geq 5$
- $\Phi_*\mu = \mu$ for some smooth volume μ

Theorem (Hurder-Katok, '90)

The following properties are equivalent:

- *the Anosov class $[A_\Phi]$ vanishes*
- *for any periodic orbit $\mathcal{O}$,* $a_1(\mathcal{O}) = 0$
- *the weak stable/weak unstable distributions* $E^{cs}_\Phi / E^{cu}_\Phi$ *are* C^{k-3}

Recall that $[A_\Phi] = 0 \overset{\text{Livsic}}{\iff} \forall\, x$ periodic, of period $\mathcal{L}(x) = \mathcal{L}(\mathcal{O})$,

$A_\Phi(x, \mathcal{L}(x)) = 0 = -\lambda^{-1} a_1(\mathcal{O})$

4) Orbit equivalence to algebraic flow (after Ghys)

- $\Phi \colon M \to M$ a C^k Anosov flow on some 3-manifold M, $k \geq 5$
- $\Phi_* \mu = \mu$ for some smooth volume μ

Theorem (Ghys, '93)

If $\mathcal{W}^{cu}_\Phi$ and $\mathcal{W}^{cs}_\Phi$ are of class $C^{1,1}$, then Φ is C^k-orbit equivalent to an algebraic flow

Corollary (De Simoi-L.-Vinhage-Yang)

If $h_{\mathrm{top}}(\Phi) = h_\mu(\Phi)$, then

1. $[A_\Phi] = 0$
2. $E^{cs}_\Phi / E^{cu}_\Phi$ *are of class* C^{k-3}
3. Φ *is* C^k*-orbit equivalent to an algebraic model*

5) From orbit equivalence to flow conjugacy

- $\Phi\colon M \to M$ a C^k Anosov flow on some 3-manifold M, $k \geq 5$
- $\Phi_*\mu = \mu$ for some smooth volume μ
- Ghys ('87): examples of analytic Anosov flows orbit equivalent but not conjugate to an algebraic model

Theorem (De Simoi-L.-Vinhage-Yang)

If MME $= \mu$, then for any $\varepsilon > 0$, Φ is $C^{k-\varepsilon}$ conjugate to an algebraic flow

5) From orbit equivalence to flow conjugacy

- assume that MME $= \mu$ (μ smooth Φ-invariant volume)
- up to C^k conjugacy, Φ is a time change of an algebraic flow Ψ
- linear time change: $h_{\mathrm{top}}(\Phi) = h_{\mathrm{top}}(\Psi) = h > 0$

- $x \in M$ point in some periodic orbit $\mathcal{O}$:
  - period $\mathcal{L}_\Phi(\mathcal{O}) = \mathcal{L}_\Phi(x) > 0$ for Φ
  - period $\mathcal{L}_\Psi(\mathcal{O}) = \mathcal{L}_\Psi(x) > 0$ for Ψ
- as MME $= \mu$, for any periodic point $x \in M$,

$$J^u_{\Phi,x}(\mathcal{L}_\Phi(x)) = e^{h\mathcal{L}_\Phi(x)}$$
$$J^u_{\Psi,x}(\mathcal{L}_\Psi(x)) = e^{h\mathcal{L}_\Psi(x)}$$

- as Φ is a time change of Ψ, same periodic eigenvalues
  $\Rightarrow$ for any periodic orbit $\mathcal{O}$, $\mathcal{L}_\Phi(\mathcal{O}) = \mathcal{L}_\Psi(\mathcal{O})$

5) From orbit equivalence to flow conjugacy

- let $\mathcal{H}$ be a C^k orbit equivalence between Φ and Ψ:

$$X_\Phi \cdot \mathcal{H}(x) = w_\mathcal{H}(x) X_\Psi(\mathcal{H}(x)), \qquad \forall\, x \in M$$

$\rightsquigarrow$ $w_\mathcal{H} \colon M \to \mathbb{R}$ measures “speed" of $\mathcal{H}$ along flow direction

- $w_\mathcal{H} - 1$ integrates to 0 over all periodic orbits
- Livsic's theorem: there exists $u \colon M \to \mathbb{R}$ differentiable along the direction of Φ such that $w_\mathcal{H} - 1 = X_\Phi \cdot u$
- let $\mathcal{H}_0 \colon x \mapsto \Psi^{-u(x)} \circ \mathcal{H}(x)$; then flow conjugacy:

$$\mathcal{H}_0 \circ \Phi^t = \Psi^t \circ \mathcal{H}_0, \quad \forall\, t \in \mathbb{R}$$

- Lyapunov exponents of periodic orbits of Φ and Ψ are all equal to $h_{\mathrm{top}}(\Phi) = h_{\mathrm{top}}(\Psi) = h$
$\rightsquigarrow$ de la Llave ('92): $\mathcal{H}_0$ is in fact $C^{k-\varepsilon}$, for any $\varepsilon > 0$ $\square$

A few more details on the proof

1) Periodic Lyapunov exponents when MME = SRB

Φ transitive Anosov flow

Proposition

- *$p\colon M \to \mathbb{R}$ Hölder continuous $\rightsquigarrow$ unique equilibrium state μ_p:*

$$P_p := \sup_{\Phi_*\mu=\mu} \left(h_\mu(\Phi) + \int_M p\, d\mu \right) = h_{\mu_p}(\Phi) + \int_M p\, d\mu_p$$

- *SRB measure: eq. state associated for the geometric potential*

$$p^u \colon x \mapsto -\frac{d}{dt}|_{t=0} \log J_x^u(t),$$

$J_x^u(t)$: Jacobian of $D\Phi^t \colon E^u(x) \to E^u(\Phi^t(x))$; pressure: $P_{p^u} = 0$

- *MME: unique eq. state for $p = \underline{0}$; pressure: $P_{\underline{0}} = h_{\mathrm{top}}(\Phi)$*

1) Periodic Lyapunov exponents when MME = SRB

Definition (Rectangle, proper family)

• *$R \subset M$ is a rectangle if there is a closed codimension one disk $D \subset M$ transverse to Φ such that $R \subset D$, and for any $x, y \in R$,*

$$[x, y]_R := D \cap \mathcal{W}^{cs}_{\Phi,\mathrm{loc}}(x) \cap \mathcal{W}^{cu}_{\Phi,\mathrm{loc}}(y) \in R$$

• *For any rectangle R and any $x \in R$, we let*

$$\mathcal{W}^s_R(x) := R \cap \mathcal{W}^{cs}_{\Phi,\mathrm{loc}}(x), \quad \mathcal{W}^u_R(x) := R \cap \mathcal{W}^{cu}_{\Phi,\mathrm{loc}}(x)$$

• *$\mathcal{R} = \{R_1, \ldots, R_m\}$ is a proper family of size $\varepsilon > 0$ if*

1. *$M = \{\Phi^t(\mathcal{S}) : t \in [-\varepsilon, 0]\}$, where $\mathcal{S} := R_1 \cup \cdots \cup R_m$*
2. *$\mathrm{diam}(R_i) < \varepsilon$, for each $i = 1, \ldots, m$*
3. *for any $i \neq j$, $D_* \cap \{\Phi^t(D_\dagger) : t \in [0, \varepsilon]\} = \emptyset$ for $\{*, \dagger\} = \{i, j\}$*

• *Poincaré map $\mathcal{F} : \mathcal{S} \to \mathcal{S}$, $x \mapsto \Phi^{\tau(x)}(x)$, where $\tau : \mathcal{S} \to \mathbb{R}_+$ first return time on $\mathcal{S}$*

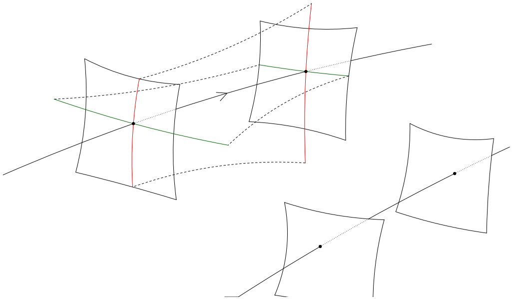
1) Periodic Lyapunov exponents when MME = SRB

A proper family $\mathcal{R} = \{R_1, \ldots, R_m\}$ is called a Markov family if for any $x \in \mathrm{int}(R_i) \cap \mathcal{F}^{-1}(\mathrm{int}(R_j))$, $i, j \in \{1, \ldots, m\}$, we have

$$\mathcal{W}^s_{R_i}(x) \subset \overline{\mathcal{F}^{-1}(\mathcal{W}^s_{R_j}(\mathcal{F}(x)))} \quad \text{and} \quad \overline{\mathcal{F}(\mathcal{W}^u_{R_i}(x))} \supset \mathcal{W}^u_{R_j}(\mathcal{F}(x))$$

Theorem

A transitive Anosov flow has a Markov family of arbitrary small size



1) Periodic Lyapunov exponents when MME = SRB

Proposition

Two equilibrium states μ_{p_1} and μ_{p_2} associated to Hölder potentials $p_1, p_2 \colon M \to \mathbb{R}$ coincide if and only if for any Markov family $\mathcal{R}$,

$$G_i \colon x \mapsto \int_0^{\tau(x)} p_i(\Phi^t(x))dt - P_{p_i} \times \tau(x), \qquad i = 1, 2,$$

are cohomologous on the cross section $\mathcal{S}$, i.e., there exists a Hölder continuous function $u \colon \mathcal{S} \to \mathbb{R}$ such that

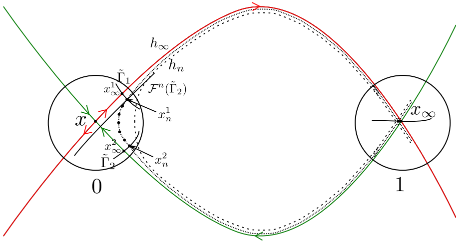
$$G_2(x) - G_1(x) = u \circ \mathcal{F}(x) - u(x), \quad \forall\, x \in \mathcal{S}$$

1) Periodic Lyapunov exponents when MME = SRB

Alternatively, when Φ is a topologically mixing Anosov flow
$\rightsquigarrow$ Margulis construction for the MME:

- family of measures ν^{cu}/ν^{s} defined on leaves of $\mathcal{W}_{\Phi}^{cu}/\mathcal{W}_{\Phi}^{s}$
- $(\Phi^t)_*\nu^{cu} = e^{h_{\mathrm{top}}(\Phi)t}\nu^{cu}$, $(\Phi^t)_*\nu^{s} = e^{-h_{\mathrm{top}}(\Phi)t}\nu^{s}$
- MME: $d\mu = d\nu^{cu} \otimes d\nu^{s}$

2) Expansion of periodic Lyapunov exponents (BNF)



- $x_\infty \longleftrightarrow \ldots 000\underset{\uparrow}{1}000 \ldots$: homoclinic point
- sequence $(h_n)_{n\geq 0}$ of periodic orbits in the horseshoe:

$$h_n \longleftrightarrow \ \ldots | \underbrace{0\ldots 0}_{n+1} 1 | \underbrace{0\ldots 0}_{n+1} 1 | \underbrace{0\ldots 0}_{n+1} 1 | \ldots$$

$$x^1_n \longleftrightarrow \ \ldots |0\ldots 00\underset{\uparrow}{1}|0\ldots 01| \ldots$$

2) Expansion of periodic Lyapunov exponents (BNF)

- x is a saddle fixed point, with eigenvalues $0 < \lambda < 1 < \lambda^{-1}$
- near $x \in \mathcal{S}$, $\exists\, C^k$ change of coordinates R such that

$$R \circ \mathcal{F} \circ R^{-1} = N\colon (\xi, \eta) \mapsto (\Delta(\xi\eta)\xi, \Delta(\xi\eta)^{-1}\eta),$$

for some C^k function $\Delta(z) = \lambda + a_1 z + \dots$

Proposition (De Simoi-L.-Vinhage-Yang)

As $n \to +\infty$, we have the asymptotic expansion

$$\mathrm{tr}(D\mathcal{F}^{n+2}_{x_n^1}) = C_0\lambda^{-n} + nC_1 a_1 + O(1)$$

where $C_0, C_1 \in \mathbb{R}^*$ are nonzero constants

2) Expansion of periodic Lyapunov exponents (BNF)

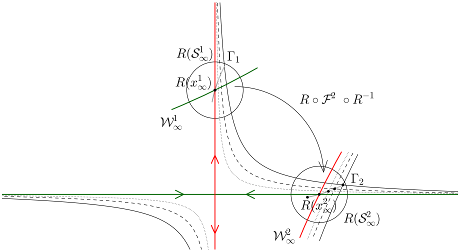
How to identify periodic orbits $(h_n)_{n\geq 0}$ in Birkhoff coordinates?

Lemma

The conjugacy R can be chosen in such a way that for $n \gg 1$,

$$R(x_n^1) = (\eta_n, \xi_n) \in \Gamma_1, \qquad R(x_n^2) = (\xi_n, \eta_n) \in \Gamma_2,$$

where $\Gamma_1, \Gamma_2 = \mathrm{Graph}(\gamma)$ are two smooth arcs which are mirror images of each other under the reflection with respect to $\{\xi = \eta\}$



2) Expansion of periodic Lyapunov exponents (BNF)

- let (ξ_n, η_n) be the coordinates of the point $R(x_n^2)$ on Γ_2:

$$\begin{cases} \xi_n &= \gamma(\eta_n) \\ \eta_n &= \Delta(\xi_n \eta_n)^n \xi_n = \Delta(\gamma(\eta_n)\eta_n)^n \gamma(\eta_n) \end{cases}$$

- replace dynamics with N and a gluing map $\mathcal{G} = R \circ \mathcal{F}^2 \circ R^{-1}$
- $\mathrm{tr}(D\mathcal{F}_{x_n^1}^{n+2}) = \mathrm{tr}(D(N^n \circ \mathcal{G})_{(\eta_n, \xi_n)}) = \mathrm{tr}(DN_{(\xi_n,\eta_n)}^n D\mathcal{G}_{(\eta_n,\xi_n)})$:

$$DN_{(\xi_n,\eta_n)} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} - 2\lambda^{-2}\xi_\infty^2 a_1 \begin{bmatrix} O(\lambda^n) & O(1) \\ O(\lambda^{2n}) & \lambda^n + O(n\lambda^{2n}) \end{bmatrix}$$

$$D\mathcal{G}_{(\eta_n,\xi_n)} = \begin{bmatrix} \gamma_1(2-\gamma_1 g_0) & \gamma_1 g_0 - 1 \\ 1-\gamma_1 g_0 & g_0 \end{bmatrix} + O(\lambda^n)$$

2) Expansion of periodic Lyapunov exponents (BNF)

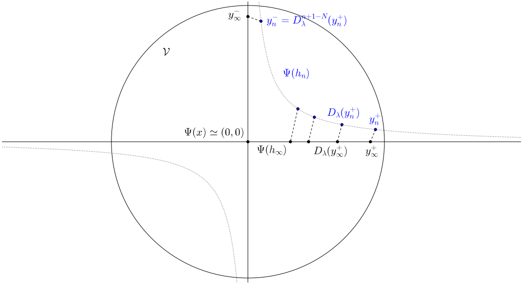
- let $\mathcal{L}^{(0)} = \mathcal{L}(x) > 0$ be the period of x for Φ
- for any $n \geq 0$, let $\mathcal{L}_n = \mathcal{L}(x_n^1) > 0$ be the period of x_n^1 for Φ

Lemma

For some $\mathcal{L}^{(1)} \in \mathbb{R}$, we have the following asymptotic expansion:

$$\mathcal{L}_n = n\mathcal{L}^{(0)} + \mathcal{L}^{(1)} + O(\lambda^n)$$

$\rightsquigarrow$ linearize the dynamics through a map Ψ



Foulon's approach for contact Anosov flows

$(\Phi^t)_{t\in\mathbb{R}}$ C^∞ Anosov flow on M:

$$TM = E^s \oplus \mathbb{R}X \oplus E^u \qquad (\mathcal{S})$$

Canonical invariant 1-form α on TM: $\begin{cases} E^s \oplus E^u = \ker \alpha \\ \alpha(X) \equiv 1 \end{cases}$

If $\alpha \wedge d\alpha$ = a volume form, we say that α is a contact form

$\rightsquigarrow$ $(\Phi^t)_t$ Reeb/contact flow $\left(d\alpha(X,\cdot) \equiv 0,\ \alpha(X) \equiv 1\right)$

- DIM $(M) = 3$: $E^{cs} = E^s \oplus \mathbb{R}X$, $E^{cu} = \mathbb{R}X \oplus E^u$ C^1
- DIM $(M) = 3$ + CONTACT FLOW : splitting $(\mathcal{S})$ is C^1

Lemma: if MME is in the Lebesgue class, the Margulis measures ν^s, ν^{cu} are Lebesgue measures: $\exists$ densities f^s, f^{cu} s.t.

$$d\nu^s = f^s\, d\lambda^s, \qquad d\nu^{cu} = f^{cu}\, d\lambda^{cu}$$

By invariance of E^s, E^u, $\exists$ positive function $\mu \in C^1(M \times \mathbb{R}, \mathbb{R})$ s.t.

$$\frac{d(\lambda^s \circ \Phi^t)}{d\lambda^s} = \mu^{-1}(\cdot, t),$$

and then, Leb. a.e., the following homological equation is satisfied:

$$\frac{f^s \circ \Phi^t(x)}{f^s(x)} = e^{-ht}\mu(x,t).$$

Livsic $\Longrightarrow$ $f^s \equiv b$ for some function $b \in C^1(M, \mathbb{R}_+)$.

Building a local algebraic structure:

① $C^1 - (SL(2,\mathbb{R}), SL(2,\mathbb{R}))$ - structure:

- ξ^s, ξ^u C^1 unit vector fields tangent to the leaves of W^s, W^u
- $X^s := b^{-1}\xi^s$, $X^u := b\,\xi^u$

Lemma: $[X, X^u] = -hX^u$, $[X, X^s] = hX^s$, $[X^s, X^u] = -X$. $(\mathcal{C})$

(h → top. entropy)

($\rightsquigarrow \Phi^t, \Phi^t_s, \Phi^t_u$ associated flows)

"Proof": $d\Phi^t(\underbrace{X^u(x)}_{b(x)\,\xi^u(x)}) = b(x)\,\mu(x,t)\,\xi^u(\Phi^t(x)) = \dfrac{b(x)}{b(\Phi^t(x))}\mu(x,t)\,X^u(\Phi^t(x))$

$= e^{ht} X^u(\Phi^t(x))$ etc. ∎

Let X_0, X_0^s, X_0^u 3 left invariant vector fields over $SL(2,\mathbb{R})$ satisfying $(\mathcal{C})$

$\rightsquigarrow \Psi^t, \Psi^t_s, \Psi^t_u$ associated flows

Lemma: $\forall x \in M$, $\exists \varepsilon > 0$ s.t. the open neighborhood $U_\varepsilon \ni \{id\}$ in $SL(2,\mathbb{R})$:

$$U_\varepsilon = \left\{ g \in SL(2,\mathbb{R}) \;:\; g = \exp(tX_0)\exp(t_s X_0^s)\exp(t_u X_0^u),\; t, t_s, t_u \in (-\varepsilon, \varepsilon)^3 \right\}$$

Satisfies that the map $F: g = e^{tX_0} e^{t_s X_0^s} e^{t_u X_0^u} \longmapsto \Phi_u^{t_u} \circ \Phi_s^{t_s} \circ \Phi^t(x)$

is a C^1 local diffeomorphism from U to $V \subset M$ and conjugates:

(i) Φ^t and Ψ^t (ii) Φ_s^t and Ψ_s^t (iii) Φ_u^t and Ψ_u^t.

$\rightsquigarrow$ local C^1-$\left(SL(2,\mathbb{R}), SL(2,\mathbb{R})\right)$-structure

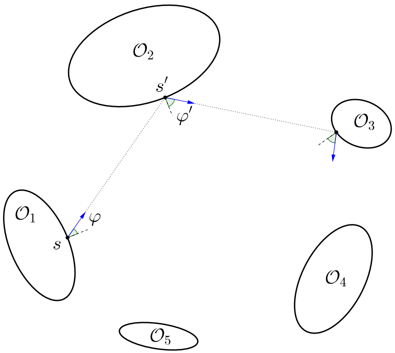
② Global algebraic structure

③ C^1 conjugacy $\Longrightarrow$ C^∞ conjugacy : de la Llave-Marco-Moriyon.

Related results for dispersing billiards

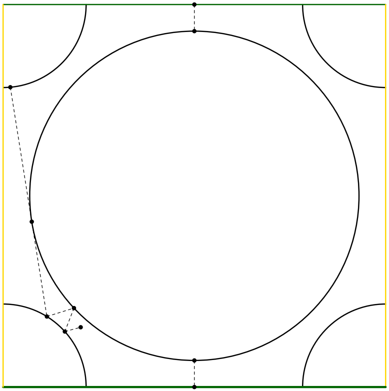
Dispersing billiards

Billiard map $\mathcal{F}\colon (s,\varphi) \mapsto (s',\varphi')$



Sinai billiards

- Sinai billiard $\mathcal{D} = \mathbb{T}^2 \setminus \cup_{i=1}^m \mathcal{O}_i$, with $\mathcal{O}_1, \ldots, \mathcal{O}_m$ convex, C^∞
- finite horizon: no trajectory makes only tangential collisions
- smooth invariant SRB proba. measure $\mu = \frac{1}{2|\partial\mathcal{D}|} \cos\varphi \, ds d\varphi$



Sinai billiards

- grazing collisions $\rightsquigarrow$ billiard map $\mathcal{F}$ has singularities
- Baladi-Demers: notion of topological entropy h_* for $\mathcal{F}$
- $\varphi_0 \in \mathbb{R}$ close to $\pi/2 \rightsquigarrow \varphi_0$-grazing collisions
- $n_0 \in \mathbb{N}$, $s_0 = s_0(\varphi_0, n_0) \in (0,1]$ smallest such that any orbit of length n_0 has at most $s_0 n_0$ many φ_0-grazing collisions

Assume that for some φ_0, n_0, it holds

$$h_* > s_0 \log 2$$

Theorem (Baladi-Demers, '18)

- *the map $\mathcal{F}$ has a unique invariant Borel probability measure μ_* of maximal entropy, i.e., $h_{\mu_*}(\mathcal{F}) = h_*$*
- *if the MME μ_* is equal to the SRB measure μ, then all the regular periodic orbits have the same Lyapunov exponent, i.e.,*

$$\mathrm{LE}(\sigma) = h_*, \quad \text{for any regular periodic orbit } \sigma$$

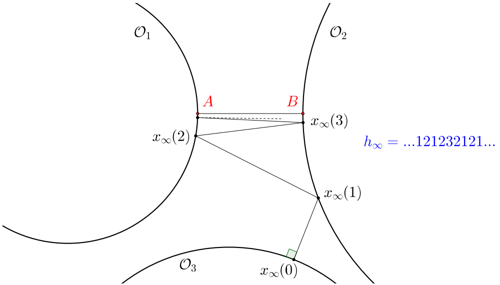
Rigidity, Sinai billiards

Theorem (De Simoi-L.-Vinhage-Yang, '20)

If the MME $\mu_* = \text{volume } \mu$, then for any regular periodic orbit $\mathcal{O}$ with a homoclinic intersection ($\exists\, x_\infty \in \mathcal{W}^s_{\mathcal{F}}(x) \pitchfork \mathcal{W}^u_{\mathcal{F}}(x)$, $x \in \mathcal{O}$), the associated Birkhoff Normal Form is linear

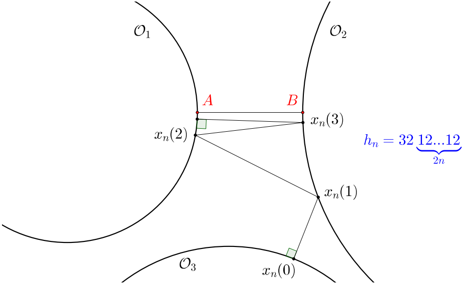
Dispersing billiards, homoclinic orbits

- non-eclipse condition
- $\sigma = 12$: period-two orbit between two obstacles $\mathcal{O}_1, \mathcal{O}_2$
- $h_\infty = \dots 1212321212 \dots$: homoclinic bouncing once on $\mathcal{O}_3$



Dispersing billiards, homoclinic orbits

- non-eclipse condition
- $\sigma = 12$: period-two orbit between two obstacles $\mathcal{O}_1, \mathcal{O}_2$
- $h_n = 32\underbrace{12\dots12}_{2n}$: sequence of shadowing periodic orbits



Sinai billiards

- x point in a regular periodic orbit, N Birkhoff Normal Form at x
- $(h_n)_{n\geq 0}$ sequence of periodic orbits as above

Lemma (De Simoi-L.-Kaloshin)

There exists a sequence of real numbers $(L_{q,p})_{\substack{p=0,\cdots,+\infty \\ q=0,\cdots,p}}$ *such that*

$$2\lambda^n \cosh((n+2)\mathrm{LE}(h_n)) = \sum_{p=0}^{+\infty} \sum_{q=0}^{p} L_{q,p} n^q \lambda^{np}$$

- Assume that the MME μ_* is equal to the SRB measure μ

$\Rightarrow$ (Baladi-Demers) for any integer $n \geq 0$, we have

$$\mathrm{LE}(h_n) = h_* > 0, \quad h_* \text{ topological entropy}$$

$\Rightarrow$ $L_{q,p} = 0$, for all $(q,p) \neq (0,0),(0,2)$

$\Rightarrow$ the Birkhoff Normal Form N is linear □

Sinai billiards

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