

Belgrade Summer School  
on Dynamics

July 17-22, 2021



## Some words on Livšic theory

$M$  compact metric space,  $f: M \rightarrow M$  continuous map.

- **Observable**  $\varphi: M \rightarrow \mathbb{R}$  vs measurements  $(\varphi \circ f^k(x))_{k \in \mathbb{Z}}$  along the orbit of  $x \in M$ .  
vs  $n^{\text{th}}$  Birkhoff average  $\frac{1}{n} S_n$ , with  $S_n := \sum_{k=0}^{n-1} \varphi \circ f^k$
- **Potential function**  $\varphi: M \rightarrow \mathbb{R}$  used to assign weights to different trajectories  
in order to select an invariant measure with specific dynamical/geometric properties.  
(orbit segment  $x, \dots, f^{n-1}(x)$  vs weight given by  $e^{S_n \varphi(x)}$ )  
→ equilibrium measure  $\mu_\varphi$  for  $(M, f, \varphi)$ : invariant probability measures maximizing  
$$h_p(f) + \int \varphi \, d\mu$$
  
(metric entropy)  
$$\left( \sup_{\mu} h_p(f) + \int \varphi \, d\mu = P(\varphi) \text{ prime} \right)$$

**Remark:**  $\mu \in M_f(M)$  (set of invariant measures for  $f: f_* \mu = \mu$ )

If  $\varphi, \psi \in C^0(M, \mathbb{R})$  differ by a coboundary (i.e.  $\varphi - \psi = h - h \circ f$ )  
for some  $h \in C^0(M, \mathbb{R})$

then  $\int q \, d\mu - \int \varphi \, d\mu = \int h \, d\mu - \int h \circ f \, d\mu = 0$ ,  $\forall \mu \in \mathcal{M}_f(M)$

hence  $(M, f, \mu)$  and  $(M, f, \varphi)$  have the same equilibrium measures.

**Question:** if  $\int q \, d\mu = \int \varphi \, d\mu$ ,  $\forall \mu \in \mathcal{M}_f(M)$ , do they differ by a coboundary?

**Remark:** in general **NO** ...

$(f = \tau_\alpha \text{ aside rotation with } \alpha \notin \mathbb{Q}/\mathbb{Z} : \mathcal{M}_f(\mathbb{R}/\mathbb{Z}) = \{\text{leb}\})$ ,

but there are many functions  $h : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$  of 0 average which are not coboundaries

... **BUT** for hyperbolic systems :

**Theorem (Lisic):** let  $M$  be a compact metric space,  $f : M \rightarrow M$  homeomorphism

satisfying the closing property + transitive ( $\exists$  dense orbit),

$q \in C^0(M, \mathbb{R})$  with the Walters Property, then  $q$  is a coboundary

if  $\forall$  periodic point  $x = f^q(x) \in M$ ,  $q \geq 1$ ,  $\int_q q(x) = 0$

Definition : (Walters property)

$\varphi \in C^0(M, \mathbb{R})$  has the Walters property if  $\forall \zeta > 0, \exists \varepsilon > 0$  s.t. for  $x, y \in M$  and  $n \geq 0$  s.t.

$$d(f^k(x), f^k(y)) < \varepsilon, \quad \forall 0 \leq k \leq n, \text{ then } |S_n \varphi(x) - S_n \varphi(y)| < \zeta.$$

Remark : if  $f$  is a  $C^1$  diffeomorphism and  $\Lambda \subset M$  is a topologically transitive locally maximal hyperbolic set, then  $f$  has the closing property and any Hölder continuous

$\varphi : M \rightarrow \mathbb{R}$  has the Walters property

Proof of Livšic Theorem :

$\Rightarrow$  clear :  $\varphi = h - h \circ f \Rightarrow \int \varphi \, d\mu = 0, \quad \forall \mu \in \text{dom}_f(M),$   
in particular for  $\mu = \frac{1}{q} \left( \delta_x + \dots + \delta_{f^{q-1}(x)} \right),$   
 $\text{if } x \text{ periodic point of period } q \geq 1$

(or just note :  $S_q \varphi(x) = h(x) - h \circ f^q(x) = 0$ )

 Conversely, assume  $\sum_q q(x) = 0$ ,  $\forall x = f^q(x) \in M$  no  $\exists? h$

Remark  $q = h - h \circ f$  determines  $h$  along the entire forward orbits.

$$(*) \quad h(f^n(y)) = h(y) + \sum_n q(y), \quad \forall y \in M$$

Transitivity: let  $(f^n(y))_n$  dense and fix  $h(y)$   $\Rightarrow h$  is determined everywhere on  $M$  by continuity.

It suffices to show that  $(*)$  defines a uniformly continuous function on  $(f^n(y))_n$ .

Given  $\zeta > 0$ , let  $\varepsilon = \varepsilon(\zeta) > 0$  be given by Walters Property and let  $\delta = \delta(\varepsilon) > 0$

be s.t. if  $x \in M$  and  $q \geq 0$  are s.t.  $d(f^q(x), x) < \delta$ , then  $\exists p = f^q(p)$  s.t.

$$d(f^k(x), f^k(p)) < \varepsilon, \quad \forall 0 \leq k < q.$$

*closing property*

Assume  $w, z \in (f^n(y))_{n \in \mathbb{Z}}$  and  $d(w, z) < \delta$ .

We have  $w = f^q(z)$  or  $z = f^q(w)$ ,  $q \geq 0$

w.l.o.g. assume  $z = f^q(w)$   $\Rightarrow d(z, f^q(z)) < \delta \Rightarrow \exists p = f^q(p)$  s.t.  $d(f^k(p), f^k(z)) < \varepsilon$   $\forall 0 \leq k < q$ .

$$\Rightarrow |h(w) - h(z)| = \left| h(f^q(z)) - h(z) \right| \stackrel{(*)}{=} \left| \int_q q(z) \right| = \left| \int_q q(z) - \int_q q(p) \right| + \underbrace{\left| \int_q q(p) \right|}_{\leq 3} \quad \text{Walters property}$$

$\Rightarrow h$  is uniformly  $C^0$  on  $(f^n(y))_{n \in \mathbb{Z}}$  and extends uniformly continuously to  $M$   
 $+ q = h - h \circ f$  on  $M$  ■

Application : synchronization of hyperbolic flows with the same periodic lengths

Proposition : let  $(\underline{\Phi}_1^t), (\underline{\Phi}_2^t)$  be two  $C^k$  flows on manifolds  $M_1, M_2$  respectively.

Assume that for  $i=1,2$ ,  $\Lambda_i \subset M_i$  is a hyperbolic set for  $(\underline{\Phi}_i^t)$  where periodic orbits are dense,  
and that there exist an orbit equivalence  $\underline{\Psi}_0 : \Lambda_1 \rightarrow \Lambda_2$  differentiable along  $(\underline{\Phi}_1^t)$ -orbits s.t.

$$(\cdot) \quad \text{Per}_{\underline{\Phi}_1}(x) = \text{Per}_{\underline{\Phi}_2}(\underline{\Psi}_0(x)), \quad \forall x \in \text{Per}(\underline{\Phi}_1^t) \cap \Lambda_1.$$

Then  $(\underline{\Phi}_1^t), (\underline{\Phi}_2^t)$  are conjugate, i.e.,  $\exists \underline{\Psi} : \Lambda_1 \rightarrow \Lambda_2$  s.t.

$$\underline{\Psi} \circ \underline{\Phi}_1^t(x) = \underline{\Phi}_2^t \circ \underline{\Psi}(x), \quad \forall (x,t) \in \Lambda_1 \times \mathbb{R}.$$

Proof: for  $i = 1, 2$ , let  $X_i$  be the flow vector field of  $(\underline{\Phi}_i^t)_t$ :  $X_i := \frac{d}{dt} \Big|_{t=0} \underline{\Phi}_i^t$ .

We let  $L_{X_1} \underline{\mathbb{I}}_0$  be the Lie derivative of  $\underline{\mathbb{I}}_0$  along  $(\underline{\Phi}_1^t)_t$ :  $L_{X_1} \underline{\mathbb{I}}_0 := \frac{d}{dt} \Big|_{t=0} (\underline{\mathbb{I}}_0 \circ \underline{\Phi}_1^t)$

As  $\underline{\mathbb{I}}_0$  sends  $(\underline{\Phi}_1^t)_t$ -orbits to  $(\underline{\Phi}_2^t)_t$ -orbit, we have

$$L_{X_1} \underline{\mathbb{I}}_0(x) = \underbrace{f(x)}_{\text{measures how } \underline{\mathbb{I}}_0 \text{ stretches time along orbits}} X_2(\underline{\mathbb{I}}_0(x)), \quad \forall x \in N.$$

measures how  $\underline{\mathbb{I}}_0$  stretches time along orbits

(\*)  $\Rightarrow \forall x \in \text{Per}(\underline{\Phi}_1^t) \cap N,$

$$\int_0^{\text{Per}_{\underline{\Phi}_1}(x)} dt = \text{Per}_{\underline{\Phi}_1}(x) = \text{Per}_{\underline{\Phi}_2}(\underline{\mathbb{I}}_0(x)) = \int_0^{\text{Per}_{\underline{\Phi}_1}(x)} f(\underline{\Phi}_1^t(x)) dt$$

$$\text{hence } \frac{1}{\text{Per}_{\underline{\Phi}_1}(x)} \int_0^{\text{Per}_{\underline{\Phi}_1}(x)} \left( f(\underline{\Phi}_1^t(x)) - 1 \right) dt = 0, \quad \forall x \in \text{Per}(\underline{\Phi}_1^t) \cap N.$$

By Ljšić theorem (for flows), we deduce that  $f - 1$  is a coboundary, i.e.,

$$f - 1 = L_{X_1} u, \quad \text{for some function } u: N \rightarrow \mathbb{R}.$$

Now set  $\tilde{\Phi} : \Lambda_1 \ni x \mapsto \underline{\Phi}_2^{-n(x)} \circ \underline{\Phi}_0(x)$  (still an orbit equivalence)

Let  $g : \Lambda_1 \rightarrow \mathbb{R}$  s.t.  $L_{x_1} \tilde{\Phi} = g \cdot X_2 \circ \tilde{\Phi}$  on  $\Lambda_1$ .

$$\rightsquigarrow \forall n \in \mathbb{N}, \quad g(n) \cdot X_2(\tilde{\Phi}(x)) = L_{x_1} \left( \underline{\Phi}_2^{-n(x)} \circ \underline{\Phi}_0(x) \right)$$

$$= -L_{x_1} n(x) \cdot X_2 \left( \underline{\Phi}_2^{-n(x)} \circ \underline{\Phi}_0(x) \right)$$

$$+ X_2 \left( \underline{\Phi}_2^{-n(x)} \circ \underline{\Phi}_0(x) \right) \cdot L_{x_1} \underline{\Phi}_0(x)$$

$$= \left( f - L_{x_1} n \right)(x) \cdot X_2(\tilde{\Phi}(x)) = X_2(\tilde{\Phi}(x))$$

i.e.,  $g = 1$ , hence  $\tilde{\Phi}$  preserves time along the orbits.  $\blacksquare$

# Rigidity of expanding endomorphisms of the circle

Let  $S^1 = \mathbb{R}/\mathbb{Z}$  be the circle.

**Definition:** for  $d \in \mathbb{N}_{\geq 2}$ . A continuous map  $f: S^1 \rightarrow S^1$  has degree  $d$  if it lifts to  $F: \mathbb{R} \rightarrow \mathbb{R}$  s.t.  $F(x+1) = F(x) + d$ ,  $\forall x \in \mathbb{R}$ .

**Proposition:** let  $f$  be an expanding endomorphism of the circle of degree  $d \geq 2$ . Then it is topologically conjugated to the map  $E_d: x \mapsto dx \bmod 1$ .

**Sketch of proof:** let  $\mathcal{E}$  be the set of increasing maps  $H \in C^0(\mathbb{R}, \mathbb{R})$  s.t.  
 $H(x+1) = H(x) + 1$ ,  $\forall x \in \mathbb{R}$ .

For all  $H \in \mathcal{E}$  let  $\bar{\Psi}(H): \begin{cases} \mathbb{R} \rightarrow \mathbb{R} \\ x \mapsto F^{-1} \circ H \circ \tilde{E}_d(x) \end{cases}$

$(\begin{array}{l} F: \mathbb{R} \rightarrow \mathbb{R} \text{ lifts } f \\ \tilde{E}_d: \mathbb{R} \rightarrow \mathbb{R} \text{ lifts } E_d \end{array})$

as  $\mathcal{E}$  is stable under  $\bar{\Psi}$ , and  $\bar{\Psi}: \mathcal{E} \rightarrow \mathcal{E}$  is a contraction ( $d(H_1, H_2) = \sup_{\mathbb{R}} |H_2 - H_1|$ )

$\Rightarrow \bar{\Psi}$  has a unique fixed point  $H_*$  which projects to  $h_*: S^1 \rightarrow S^1$  s.t.

$$f \circ h_* = h_* \circ E_d$$

$H_1$  is injective: otherwise  $\exists$  non trivial open interval  $I \subset R$  s.t.  $H_1(I) = \{pt\}$

$$\rightsquigarrow H^k, \{pt\} = F^k \circ H_1(I) = H_1 \circ \underbrace{\tilde{E}_d^k}_{\text{contains two points } y, y+1} (I)$$

contains two points  $y, y+1$  for  $k$  large

but  $H_1(y+1) = H_1(y) + 1 \neq H_1(y)$ !

$\rightsquigarrow h_1$  is a homeomorphism of  $S^1$ . 

Let  $f, g$  be two endomorphisms of the circle  $S^1$  s.t. there exists a Borel measurable bijection defined a.e., such that  $f \circ h = h \circ g$  a.e. We say that  $h$  is an absolutely continuous conjugacy if besides it is non-singular with respect to Lebesgue measure  $\lambda$ .

$$\lambda(x) = 0 \Leftrightarrow \lambda(h(x)) = 0, \quad \forall x \in B.$$

**Theorem (Shub-Sullivan '84):** Let  $n \in \mathbb{N}_{\geq 2} \cup \{\infty\} \cup \omega$ . If two orientation preserving expanding  $C^n$  endomorphisms  $f$  and  $g$  of the circle  $S^1$  are conjugate by an absolutely continuous conjugacy, then they are conjugate by a  $C^n$  diffeomorphism.

Sketch of proof of Shub-Sullivan

The proof uses the following result:

**Theorem: (Krzyżewski - Sacksteder)** Any  $C^2$  expanding map  $f: \mathbb{S}' \rightarrow \mathbb{S}'$  has a  $C^{n-1}$  smooth  $f$ -invariant measure  $\mu_f = \rho_f d\lambda$

**Remark:**  $\rho_f > 0$  since if  $\rho_f(x) = 0$  for some  $x \in \mathbb{S}'$  then  $\rho_f$  vanishes on the backwards orbit of  $x$ , which is dense in  $\mathbb{S}'$ .

**Corollary:** Let  $f: \mathbb{S}' \rightarrow \mathbb{S}'$  be a  $C^2$  expanding endomorphism. Then  $f$  is conjugate to an expanding endomorphism  $g$  of  $\mathbb{S}'$  which preserves Lebesgue measure.

**Proof:** Let  $h: \mathbb{R} \mapsto \int_0^x \rho_f d\lambda = \mu_f([0, x]) \rightsquigarrow C^2$  diffeomorphism of  $\mathbb{S}'$

$$h_* \mu_f([a, b]) = \mu_f \left\{ y \text{ s.t. } \mu_f([0, y]) \in [a, b] \right\} = b - a \rightsquigarrow h_* \mu_f = \lambda$$

$\Rightarrow h \circ f \circ h^{-1}$  preserves Lebesgue measure  $\left( (h \circ f \circ h^{-1})_* \lambda = h_* f (h^{-1})_* h_* \mu_f = h_* f \circ \mu_f = h_* \mu_f = \lambda \right)$



Lemma (bounded distortion):

Let  $f$  be a  $C^{1+\alpha}$  expanding endomorphism of  $S^1$ . Then  $\exists c > 0$  s.t. if  $I \subset S^1$  is an interval s.t.  $f^n|_I$  is injective, then

$$c^{-1} < \frac{|df^n(y)|}{|df^n(x)|} < c, \quad \forall x, y \in I.$$

Proof: Let  $\lambda > 1$  s.t.  $\inf_{S^1} |df| \geq \lambda > 1$ .

For  $x, y \in I$ , let  $d(x, y, I)$  be the distance between  $x$  and  $y$  measured within  $I$ .

$$\begin{aligned} \forall j \in [0, n], \quad d(f^j(x), f^j(y), f^j(I)) &= d((f^{n-j})^{-1}(f^n(x)), (f^{n-j})^{-1}(f^n(y)), f^j(I)) \\ &\leq \lambda^{-n+j} d(f^n(x), f^n(y), f^n(I)). \end{aligned}$$

$df$  is  $\alpha$ -Hölder + bdd away from 0  $\Rightarrow \log |df|$  is  $\alpha$ -Hölder

$$\begin{aligned} \text{so } |\log |df(f^j(x))| - \log |df(f^j(y))|| &\leq K (\lambda^{-n+j} d(f^n(x), f^n(y), f^n(I)))^\alpha \\ &\leq K' (\lambda^\alpha)^{-n+j}. \end{aligned}$$

$$\text{Thus } \left| \log \frac{|df^n(y)|}{|df^n(x)|} \right| \leq \sum_{j=0}^n \left| \log \frac{|df(f^j(y))|}{|df(f^j(x))|} \right|$$

chain rule

$$\leq k' \sum_{j=0}^n (k^\alpha)^{n-j} \leq \frac{k'}{1-k^{-\alpha}}.$$

**Corollary** : Let  $x_n = f^n(x)$ ,  $y_n = f^n(y)$  and  $D_n = |df^n(x)|$ .

If  $d(x, y) \leq (c D_n)^{-1}$ , then

$$c^{-1} D_n d(x, y) \leq d(x_n, y_n) \leq c D_n d(x, y).$$

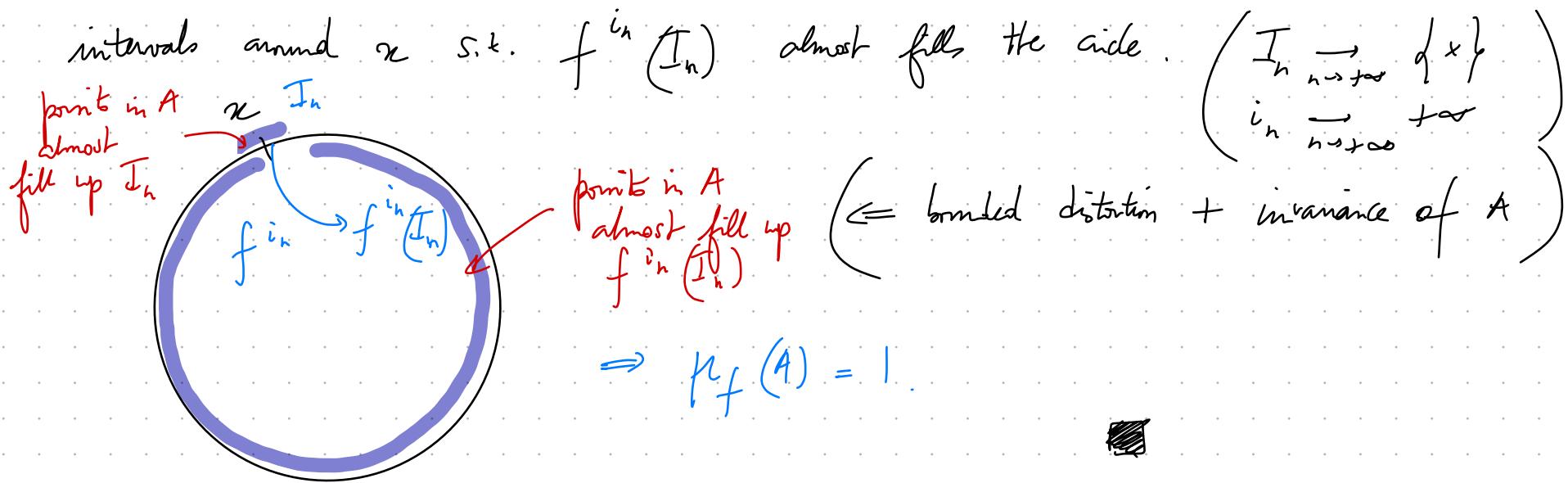
**Proof** :  $d(x, y) \leq (c D_n)^{-1} \Rightarrow f^n|_{[x,y]}$  injective.

Then apply the bounded distortion lemma ■

**Corollary** : if  $f$  is a  $C^{1+\alpha}$  expanding endomorphism of  $S^1$  then  $f$  is ergodic.

**Proof** : let  $A$  be invariant ( $f(A) \subset A$ ), with  $\mu_f(A) > 0$ .

Let  $x \in A$  be a density point and let  $(I_n)_n$  be a collection of smaller and smaller



Back to Shub - Sullivan:

- if  $h$  homeomorphism and  $f \circ h = g$ , by ergodicity  $h_* \mu_g = \mu_f$ .  
( $h_* \mu_g$  and  $\mu_f$  are a.c. and invariant under  $f$ )  
w.l.o.g.  $h(0) = 0$  w. I.f.  $I_f([0, x]) = \mu_f([0, x]) = \mu_g(h[0, x]) = \mu_g([0, h(x)]) = I_g(h(x))$   
 $I_{f/g}(x) = \int_0^x (f/g)^d\lambda$  is  $C^2$   $\Rightarrow$   $h$  is  $C^2$ .  $\blacksquare$   
 (implied function theorem)
- Otherwise:  
w.l.o.g., by the previous corollary,  $f$  and  $g$  preserve Lebesgue measure  $\lambda$ .  
 $h$  is an a.c. conjugacy  $\Rightarrow h_* \lambda$  and  $h^{-1}_* \lambda$  are a.c. invariant measures for  $f$ , resp.  $g$ .

By injectivity,  $h \circ \lambda = \lambda$ , i.e.  $h$  preserves beginning.

Completely-to-one locally non-singular maps have Jacobian derivatives

and  $|dh| = 1 \Rightarrow |df| \circ h = |dg|$ . (\*)

- $|df| \text{ const} \Leftrightarrow |dg| \text{ const}$  and in that case,  $\exists n > 1$ ,  $\alpha, \beta \in \mathbb{C}$ ,  $|\alpha| = |\beta| = 1$   
(\*)

s.t.  $g(z) = \alpha z^n$ ,  $f(z) = \beta z^n$ .

Let  $\gamma \in \mathbb{C}$  be s.t.  $\gamma^{n-1} = \frac{\alpha}{\beta}$  w.r.t.  $h: z \mapsto \gamma z$  conjugates  $f$  and  $g$

- if neither  $|df|$  nor  $|dg|$  stt:

**Proposition:**  $f, g$   $C^2$  expanding endomorphisms of  $S'$  s.t.  $f \circ \lambda = g \circ \lambda = \lambda$  and s.t.

$df, dg$  not constant and  $f \circ h = hg$  with  $h \circ \lambda = \lambda$ . Then  $\exists R$  biometry of  $S'$  s.t.  
 $h = R$  a.e.

**Corollary** : (of Shub-Sullivan)

Let  $E_d : x \mapsto dx \bmod 1$ . Let  $f : S^1 \rightarrow S^1$  be a degree  $d$  expanding endomorphism of degree  $d$   
( $\rightsquigarrow \exists h$  homeomorphism of  $S^1$  s.t.  $h \circ f = E_d \circ h$  by what we have seen).

Let  $\lambda_f$  be the Lyapunov exponent of  $f$  with respect to  $\mu_f = \rho_f^{-1} \lambda$  (a.c.i.m.)

If  $\lambda_f = \log d$  then  $f$  is smoothly conjugate to  $E_d$ .

**Proof:**  $\lambda_f = h_{\mu_f}(f) = h_{\mu_f}(E_d)$       }  $h \circ \mu_f = \lambda$  = measure of maximal entropy  
 $\log d = h_{top}(E_d) = h_\lambda(E_d)$       } (by uniqueness)

$\Rightarrow h$  is an a.c. conjugacy between  $f$  and  $E_d \Rightarrow h$  is smooth. Shub-Sullivan \blacksquare

# Rigidity of Anosov diffeomorphisms of $T^2$

Thm: (de la Llave - Marco - Moryan, '90s)

Let  $f, g : T^2 \rightarrow T^2$  be two  $C^2$  Anosov diffeomorphism ( $T^2$  is a hyperbolic set)  
s.t.  $h \circ f = g \circ h$ , for some homeomorphism  $h$ .

Assume periodic obstructions vanish, i.e.

$$(P) \quad \forall p \in f^q(p), \exists C \text{ s.t. } df^q(p) = C \cdot dg^q(h(p)) \cdot C^{-1}$$

Then  $h$  is  $C^{2-\varepsilon}$  smooth,  $\forall \varepsilon > 0$ .

Sketch of the proof:

① Equilibrium state:

Given an Anosov diffeo  $f$ , and a Hölder potential  $\varphi$ ,

$\exists!$   $f$ -invariant measure  $\mu_{f,\varphi}$  (equilibrium state) which maximizes

$$h_\mu(f) + \int \varphi \, d\mu$$

By uniqueness, if  $\varphi, \psi$  are cohomologous, then  $\mu_{f,\varphi} = \mu_{f,\psi}$ .

The equilibrium state for  $-\log \text{Jac}^n(f)$  is the SRB measure  $\mu_f$ :

~ it has a.c. conditional measures on the unstable leaves.

(2) Functionality:

If  $f = h^{-1} \circ g \circ h$  then  $h^* \mu_{g,\psi}^{\text{eq. state}} = \mu_{f,\psi \circ h}$

(by functionality of metric entropy)

(3) Smoothness along foliations:

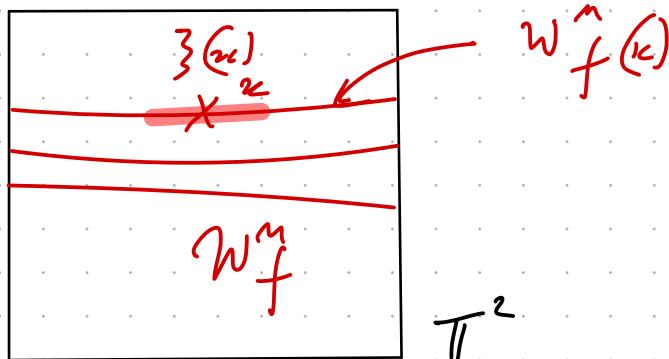
(P)  $\Rightarrow$  the potential  $\varphi = -\log \text{Jac}^n f$  is cohomologous to  $\psi \circ h$

where  $\psi = -\log \text{Jac}^n g$

~  $m_f = \mu_{f,\varphi}^{\text{def.}} = \mu_{f,\psi \circ h} = h^* \mu_{g,\psi}^{\text{def.}} = h^* m_g$

i.e.  $h$  is absolutely continuous.

④ If  $\mathcal{Z}$  is a measurable partition subordinate to  $W_f^m$  (atoms: subsets of  $W_f^m$ -leaves)



then for  $m_f$ -a.e.  $x$ ,  $h \Big|_{\mathcal{Z}(x)} : \mathcal{Z}(x) \rightarrow h(\mathcal{Z}(x))$  is a.c.  $\Rightarrow$  Smooth,  
Shub-Sullivan

i.e.  $h \in C_m^r(T^2)$  ( $C^r$ -smooth along  $W_f^m$ -leaves)

By the same argument applied to  $f^{-1}$  and  $g^{-1}$ ,  $(h \circ f^{-1} \circ g^{-1} \circ h)$

we also have  $h \in C_s^r(T^2)$  ( $C^r$ -smooth along  $W_f^s$ -leaves)

⑤  $\Leftrightarrow: C^{r-\varepsilon}(T^2) \subset C_s^r(T^2) \cap C_m^r(T^2)$   
(Joune Lemma)



## Rigidity of Anosov flows in dimension 3

If  $(\Phi^t)_t, (\Phi_e^t)_t$  are  $C^1$ -close transitive Anosov flows on  $M_1, M_2$ , then by structural stability,  $\exists$  orbit equivalence  $\bar{\Phi}_0 : M_1 \rightarrow M_2$  (which sends  $(\Phi_1^t)_t$ -orbit to  $(\Phi_e^t)_t$ -orbit preserving time direction but not time a priori).

If

$$(•) \quad \text{Per}_{\bar{\Phi}_0}(x) = \text{Per}_{\bar{\Phi}_0}(\bar{\Phi}_0(x)), \quad \forall x \in \text{Per}(\Phi_1^t)_t \cap A,$$

we have seen that  $\bar{\Phi}_0$  can be upgraded to a flow conjugacy  $\bar{\Phi}$ .

Besides, (similar to the previous result but for continuous-time dynamical systems)

**Theorem :** (de la Llave - Marco - Moriyon, Pollicott)

Assume that  $(\Phi_1^t)_t, (\Phi_e^t)_t$  are conjugate transitive Anosov flows on 3D manifolds.

Assume further that the differentials of Poincaré return maps for all periodic orbits are conjugate. Then the flows  $(\Phi_1^t)_t$  and  $(\Phi_e^t)_t$  are smoothly conjugate.