

Belgrade Summer School
on Dynamics

July 17-22, 2021



Introduction

- Dynamics :
- discrete (diffeomorphism / endomorphism $f: M \rightarrow M$) via iterations $f^n, n \in \mathbb{Z}$ ($n \in \mathbb{N}$)
 - continuous (flows) : 1-parameter group of diffeomorphisms $t \mapsto \underline{\Phi}^t, t \in \mathbb{R}$
 $(\underline{\Phi}^{s+t} = \underline{\Phi}^t \circ \underline{\Phi}^s)$

One instance of rigidity in dynamics is :

- Given two smooth systems $f, g / (\underline{\Phi}^t)_t, (\underline{\Psi}^t)_t$ which are *topologically conjugate*,
 $(\exists h \text{ homeomorphism s.t. } h \circ f = g \circ h / \text{s.t. } h \circ \underline{\Phi}^t = \underline{\Psi}^t \circ h)$
- when can we show that they are *smoothly conjugate*?
- In particular, if there exist models to which the dynamics is C^0 -conjugate,
 $(C^2 \text{ circle diffeomorphisms no rotations} / 3D Anosov flows \rightsquigarrow \text{algebraic models} / \text{symbolic systems...})$
with irrational rotation number

which is it *smoothly conjugate* to such a system?

- general obstructions : periodic orbits

$$\left(\begin{array}{l} f^n(x) = x, n \geq 1 \\ \text{smallest such } n : \text{period} \end{array} \right) / \left(\begin{array}{l} \phi^T(x) = x, T > 0 \\ \text{smallest such } T : \text{period} \end{array} \right)$$

- obstruction to C^0 conjugacy for flows : periods
 assume $(\phi^t)_t$ and $(\psi^t)_t$ are orbit equivalent : $\exists h$ which sends $(\phi^t)_t$ -orbits to $(\psi^t)_t$ -orbits

then for h to be a conjugacy, necessary to have :

x periodic of period $T > 0$ for $(\phi^t)_t$, $h(x)$ should be T -periodic for $(\psi^t)_t$

- obstruction to C^1 conjugacy : lyapunov exponents

assume $\exists h$ C^1 diffeomorphism s.t. $h \circ f = g \circ h$

x periodic for f of period $n \geq 1$ \Rightarrow differentiate $h \circ f^n = g^n \circ h$:

$$dh(x) df^n(x) (dh(x))^{-1} = dg^n(h(x))$$

$\Rightarrow df^n(x)$ and $dg^n(h(x))$ have the same eigenvalues (Lyapunov exponents)

- even when there are no periodic points (minimal dynamics ...), related obstructions coming from "almost periodic" points may persist (Liouville numbers ...)

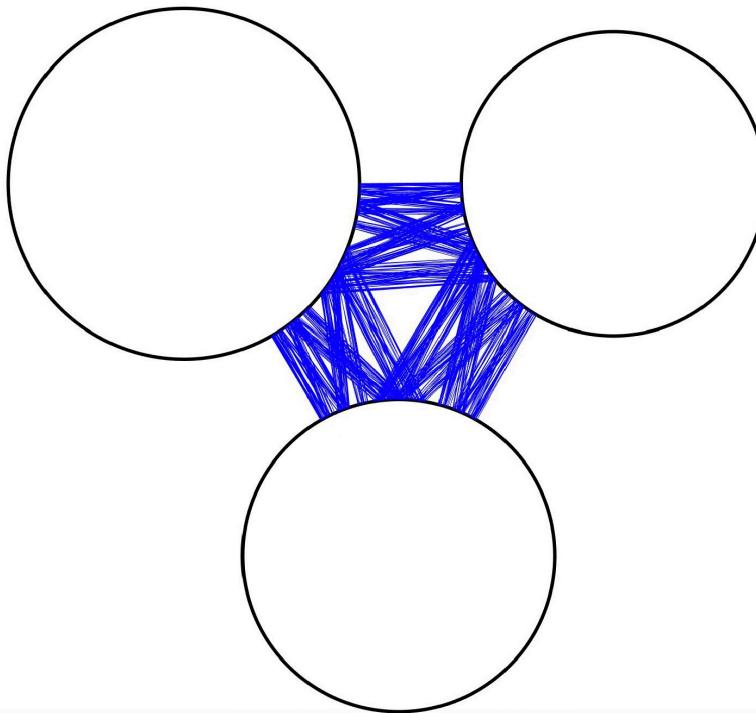
- Circle diffeomorphisms (renormalization techniques ...) : assume that the rotation number is badly approximated by rational numbers no Diophantine Condition
(avoid problems coming from "almost periodic" points)
- local C^α -linearization (conjugation to rotation) obtained by Arnold ('61)
- global results in the C^∞ category : results by Herman ('79), Yoccoz ('84)
- our purpose : **hyperbolic systems**
 - structural stability vs C^0 conjugacy for \mathcal{L} hyperbolic systems which are close abundance of periodic orbits vs many obstructions ! **Sufficient condition ?**
(when periodic obstructions vanish, are the systems smoothly conjugate?)
 - Specific tools : use of **invariant foliations** to study the regularity ...
 - periodic information may be redundant (periods, Lyapunov exponents, Birkhoff invariants...)

- Rigidity of Geometric flows (geodesic flow in negative curvature, billiards...)

How much information on the geometry is given by periodic data?

(Length Spectrum: set of lengths of periodic orbits...)

- Celebrated results by Guel and Croke ('90) for negatively curved surfaces
- for dispersing billiards with analytic boundary: De Simoi - Kaloshin - L. ('19)

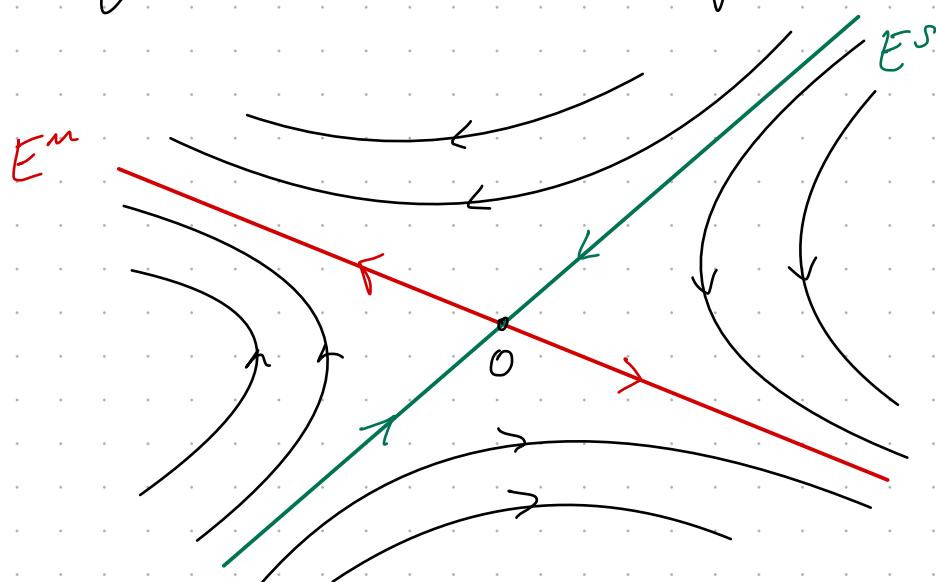


Some notions of hyperbolic dynamics

Example : ① A linear invertible map acting on \mathbb{R}^n with eigenvalues of moduli $\neq 1$.

$$E^s := \bigoplus_{|\lambda| < 1} E_\lambda, \quad E^u := \bigoplus_{|\lambda| > 1} E_\lambda \quad (E_\lambda: \text{eigenspace of some eigenvalue } \lambda)$$

- • 0 is fixed
- $\forall v \in E^s \setminus \{0\}$, $A^n v \xrightarrow[n \rightarrow \infty]{} 0$ and $A^{-n} v$ escapes to ∞ in the past
- $\forall v \in E^u \setminus \{0\}$, $A^n v \xrightarrow[n \rightarrow -\infty]{} 0$ and $A^{-n} v$ escapes to ∞ in the future
- any other orbit escapes in the future and in the past



+ picture is stable by perturbation

(Same is true for linear maps
B close to A)

→ A is called hyperbolic

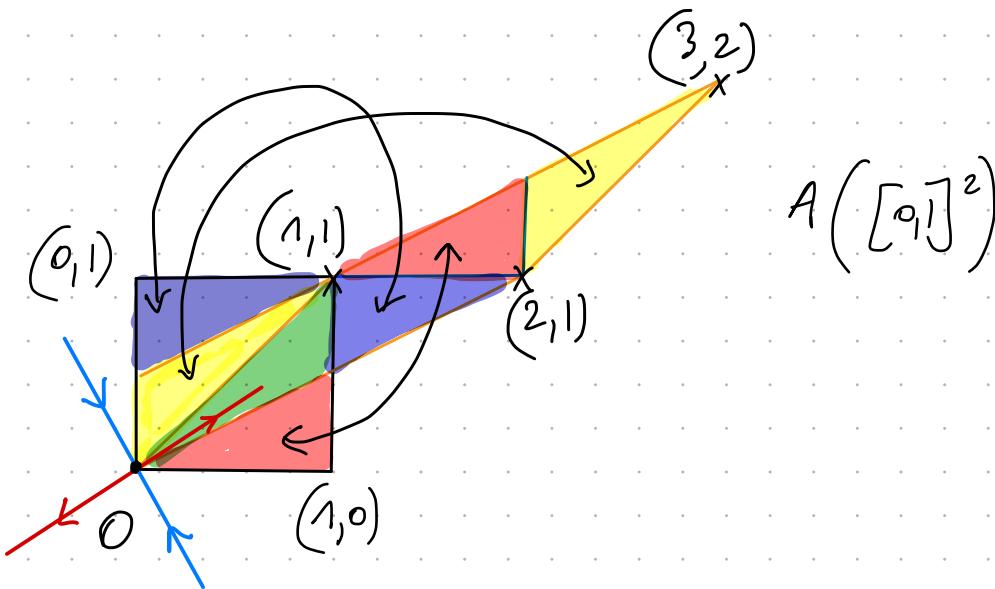
② Hyperbolic toral automorphisms

$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in SL(2, \mathbb{R})$ induce a diffeomorphism of $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$,

$$\Rightarrow f_A : T^2 \rightarrow T^2, (x, y) \mapsto (2x + y \bmod 1, x + y \bmod 1)$$

eigenvalues $0 < \lambda < 1 < \lambda^{-1} = \frac{3+\sqrt{5}}{2}$

eigenvectors $v_{\lambda^{-1}} = \left(\frac{1+\sqrt{5}}{2}, 1 \right), v_\lambda = \left(\frac{1-\sqrt{5}}{2}, 1 \right)$

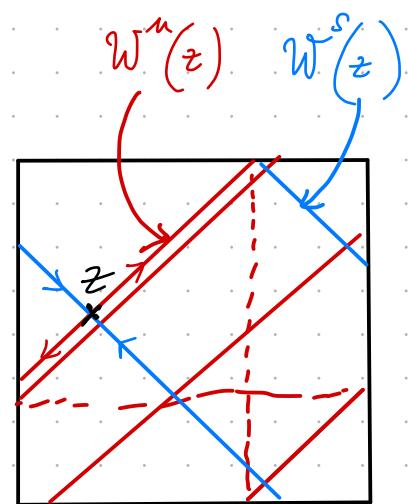


- periodic points for $f_A = \{ \text{points with rational coordinates} \} \quad (\text{dense in } \mathbb{T}^2)$
- Stable / unstable foliation:

lines in \mathbb{R}^2 parallel to v_λ project to a family W^s of parallel lines on \mathbb{T}^2
 $v_{\lambda^{-1}}$ project to a family W^u of parallel lines on \mathbb{T}^2

$W^s(z) = \text{stable manifold of } z \in \mathbb{T}^2$

$W^u(z) = \text{unstable manifold of } z \in \mathbb{T}^2$



f_A contracts / expands stable / unstable manifolds by λ / λ^{-1}

$(W^{s/u}(z) = \{ z' \in \mathbb{T}^2 : d(f_A^n(z), f_A^n(z')) \rightarrow 0 \text{ as } n \rightarrow \pm\infty\})$

+ each stable / unstable manifold is dense in \mathbb{T}^2 .

• W^u / W^s stable / unstable foliation: collection of all stable / unstable manifolds

invariance: $f_A(W^*(z)) = W^*(f_A(z))$, $\forall z \in \mathbb{T}^2$, $* = s/u$

③ Horseshoe (Smale)

In the 2-Sphere S^2 ,

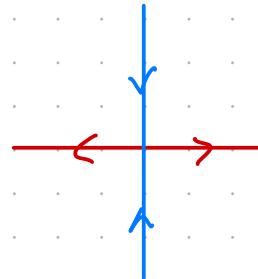
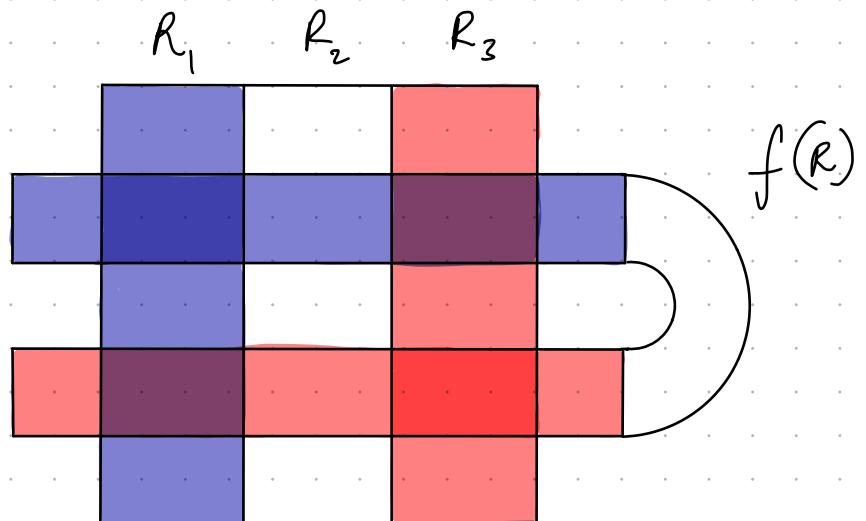
let R be a rectangle diffeomorphic to $[0,1]^2$

$$R = R_1 \cup R_2 \cup R_3$$

$$R_1 = \left[0, \frac{1}{3}\right] \times [0,1]$$

$$R_2 = \left[\frac{1}{3}, \frac{2}{3}\right] \times [0,1]$$

$$R_3 = \left[\frac{2}{3}, 1\right] \times [0,1]$$



horizontal dim. expanded
vertical dim. contracted

Let $f: S^2 \rightarrow S^2$ be a diffeomorphism s.t.

$$f|_{R_1}: (x,y) \mapsto \left(5x, \frac{1}{5}y\right) + c_1$$

$$\text{with } \pi_1(f(R_1)) \supset [0,1] \quad (\pi_1(x,y) = x)$$

- $f(R_2) \cap R = \emptyset$
- $f|_{R_3}: (x,y) \mapsto \left(5x, \frac{1}{5}y\right) + c_3$
with $\pi_1(f(R_3)) \supset [0,1]$

Maximal invariant set $\Lambda = \bigcap_{k \in \mathbb{Z}} f^k(\mathcal{R})$ is the product of 2 Cantor sets

It is invariant and locally maximal: \exists open set $U \supset \Lambda$ s.t. any f -invariant subset of U containing Λ is equal to Λ .

Moreover, Λ is hyperbolic: $\forall x \in \Lambda$, E_x^S = vertical space is contracted wif. by df_x
 E_x^U = horizontal space is expanded wif. by df_x^{-1}

Λ can be coded symbolically: the word $w = (w_j)_{j \in \mathbb{Z}} \in \{1, 3\}^{\mathbb{Z}}$,

\exists ! point $z \in \Lambda$ s.t. $f^h(z) \in R_{w_n}$, $\forall n \in \mathbb{Z}$

and if $h: z \mapsto w(z)$ conjugates $f|_{\Lambda}$ to the full 2-sided shift

$\tau: (w_j)_{j \in \mathbb{Z}} \mapsto (w_{j+1})_{j \in \mathbb{Z}}$

$h \circ f = \tau \circ h$ on Λ

Definition : (hyperbolic set) M manifold with a Riemannian metric

$f : M \rightarrow M$ C^1 diffeomorphism on M .

A set $\Lambda \subset M$ is a *hyperbolic set* for f if it is a f -invariant compact set s.t.

$$TM|_{\Lambda} = E^s \oplus E^u, \text{ with}$$

- E^s, E^u invariant : $\forall x \in \Lambda, df_x(E_x^{s/u}) = E_{f(x)}^{s/u}$
- E^s is uniformly contracted / expanded : $\exists C > 0, \lambda \in (0, 1)$ s.t. $\forall x \in \Lambda, \forall v^s \in E_x^s, \|df^n(x) \cdot v^s\| \leq C\lambda^n \|v^s\|$

$$\begin{aligned} \forall n \geq 0, \|df^n(x) \cdot v^s\| &\leq C\lambda^n \|v^s\| && \left(\text{It is possible to find a metric such that } C=1 \right) \\ \|df^n(x) \cdot v^u\| &\geq C^{-1}\lambda^{-n} \|v^u\| && \left(\text{w.r.t. adapted norm} \right) \end{aligned}$$

Remarks :

- $\forall x \in \Lambda$, the stable/unstable space is unique

- $\forall n \neq 0, \Lambda$ is hyperbolic for $f \Leftrightarrow \Lambda$ is hyperbolic for f^n

- the definition is independent of the choice of a Riemannian metric

Endomorphisms

A C^r -endomorphism of a manifold M is a differentiable map of class C^r on M which is not necessarily injective nor surjective.

A compact set $\Lambda \subset M$ is **invariant** for an endomorphism f of M if $f^{-1}(\Lambda) = \Lambda$.

A compact set $\Lambda \subset M$ is **stable** for an endomorphism f of M if $f(\Lambda) = \Lambda$.

Definition: (expanding map)

Let f be a C^1 -endomorphism of a manifold M . An invariant stable compact

set $\Lambda \subset M$ is **expanded** by f if $\exists n \geq 1$ s.t. $\forall x \in \Lambda$, $d_x f^n$ is invertible

and with contracted inverse.

When $\Lambda = M$ we say that f is **expanding**.

The cone field criterion

Let $f: M \rightarrow M$ be a C^1 diffeomorphism. A set $U \subset M$ satisfies the cone field criterion if

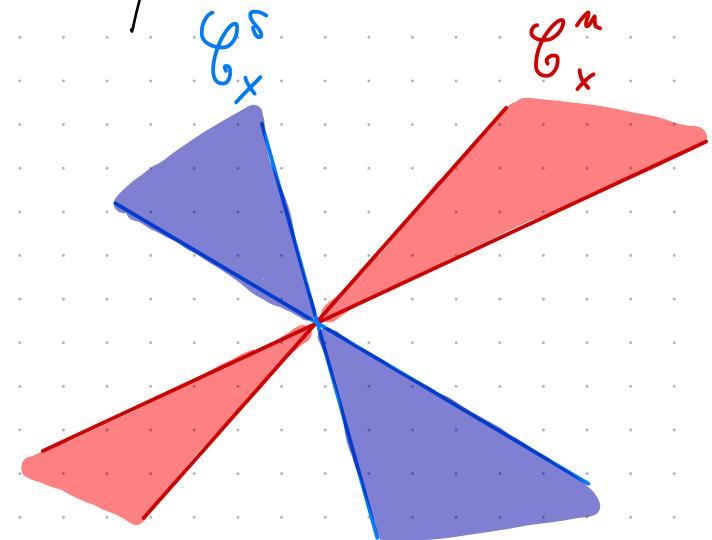
- \exists tangent splitting $TM|_U = \hat{E}^s \oplus \hat{E}^u$ into (not nec. invariant) subbundles \hat{E}^s, \hat{E}^u
- a continuous family of norms $\|\cdot\|$ defined on $TM|_U$
- Constants $\lambda, C \in (0, 1)$, $C > 0$ s.t. $\forall x \in U$, the cones

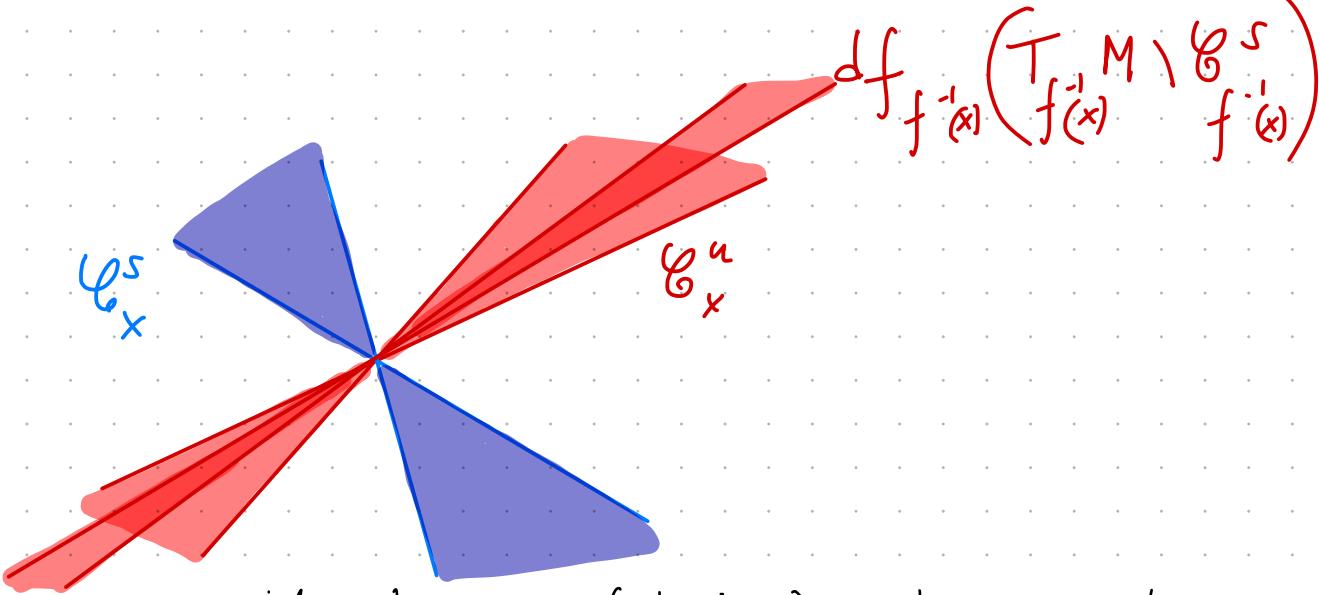
$$\mathcal{C}_x^u = \left\{ v = v^s + v^u \in T_x M \mid e^{-\lambda} \|v^u\| \geq \|v^s\| \right\}$$

$$\mathcal{C}_x^s = \left\{ v = v^s + v^u \in T_x M \mid e^{-\lambda} \|v^s\| \geq \|v^u\| \right\}$$

satisfy: $\forall n \geq 1$, $\forall z \in \bigcap_{k=0}^n f^{-k}(U)$,

- $df_z^n(T_z M \setminus \mathcal{C}_z^s) \subset \mathcal{C}_{f^n(z)}^u$
- $\forall v \in \mathcal{C}_z^u$, $\|df_z^n \cdot v\| \geq C \lambda^{-n} \|v\|$
- $\forall v \in \mathcal{C}_{f^n(z)}^s$, $\|df_{f^n(z)}^{-n} \cdot v\| \geq C \lambda^{-n} \|v\|$





Proposition: If Λ CM satisfy the cone-field criterion then it is hyperbolic.

Conversely, if Λ CM is hyperbolic and $\|\cdot\|_o$ is an adapted metric, then

the cone field criterion is satisfied by the splitting $TM|_\Lambda = E^s \oplus E^u$ and the metric

$$\|v\| = \sqrt{\|v^s\|_o^2 + \|v^u\|_o^2}$$

In this case, we have : $\forall x \in \Lambda$, $E_x^u = \bigcap_{n=0}^{+\infty} df_{f^{-n}(x)}^n E_{f^{-n}(x)}^u$

↓
Same dim. as E_x^u

and same thing for E_x^s taking f^{-1} instead of f .

Corollary: Let Λ be a hyperbolic set of a diffeomorphism f .

Then $\exists U \supset \Lambda$ neighborhood of Λ and $\exists \mathcal{U} \ni f$ neighborhood of f in $\text{Diff}'(M)$ s.t. any invariant compact set $\Lambda' \subset U$ for any diffeo $g \in \mathcal{U}$ is also hyperbolic for g .

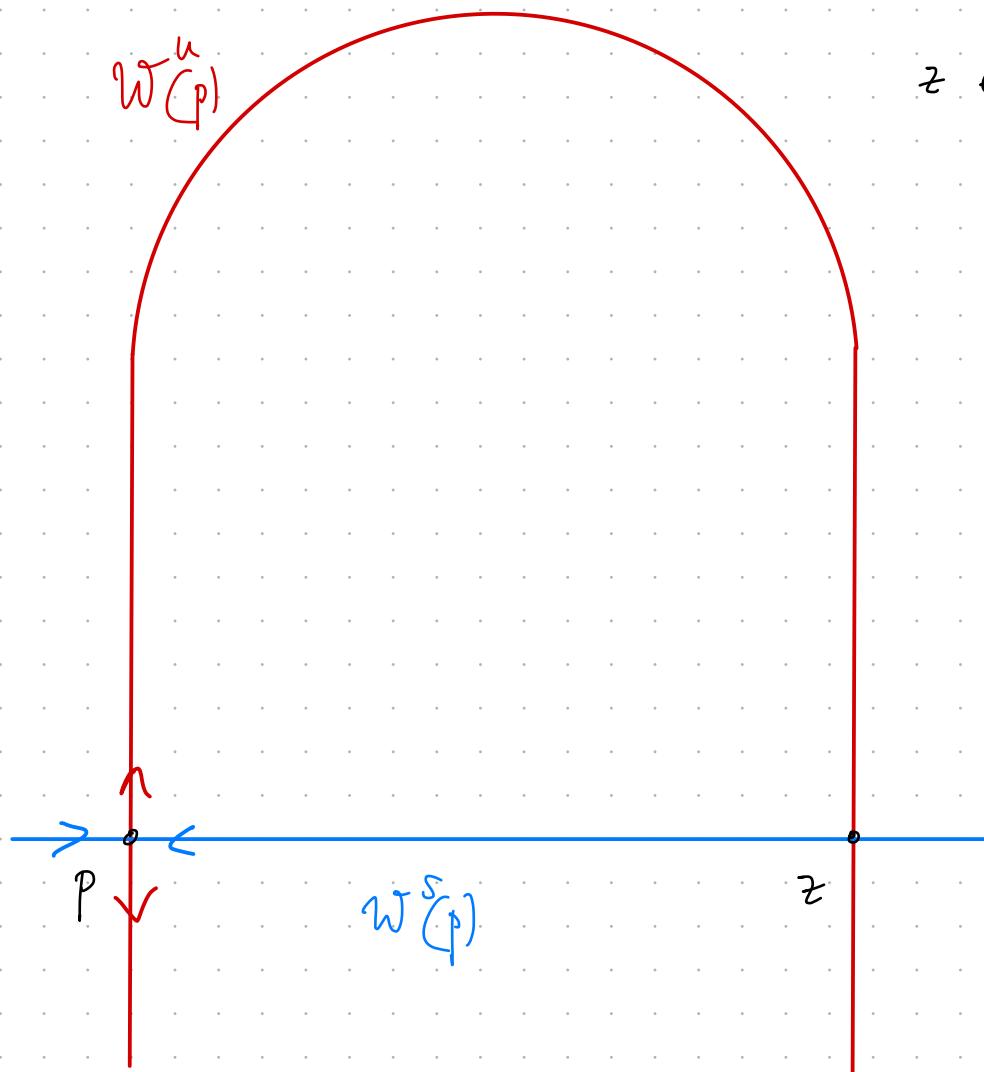
Examples:

- a periodic orbit $K = \{x, f(x), \dots, f^{q-1}(x), f^q(x) = x\}$ is hyperbolic if and only if the differential $df^q(x)$ is hyperbolic
(note that $\forall k \in \mathbb{Z}$, $df^q(f^k(x))$ are conjugated)
 - a) sink if $E_K^u = \{0\}$
 - b) source if $E_K^s = \{0\}$
 - c) saddle otherwise
- the whole torus T^2 is hyperbolic in the case of the linear automorphism f_A in example 2 above.
- the horseshoe Λ in example 3 is a hyperbolic set

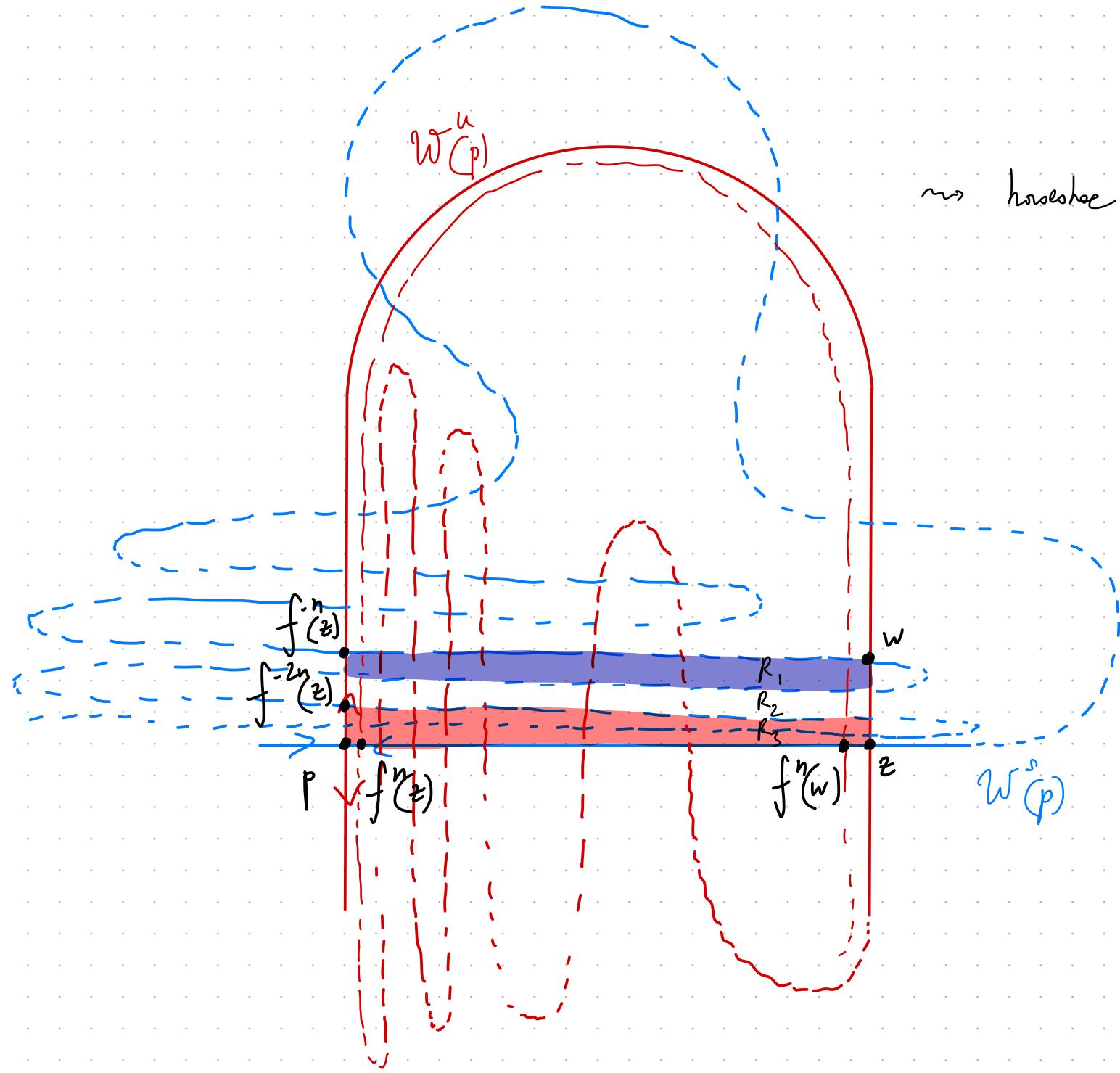
Homoclinic intersections and horseshoes

$f : S \rightarrow S$ diffeo. with a hyperbolic fixed point p

$$W^{s/m}(p) = \{q \in S \mid d(f^n(q), f^n(p)) \xrightarrow[n \rightarrow \pm\infty]{} 0\}$$



$z \in S$ point of transverse intersection
between $W^s(p)$ and $W^u(p)$

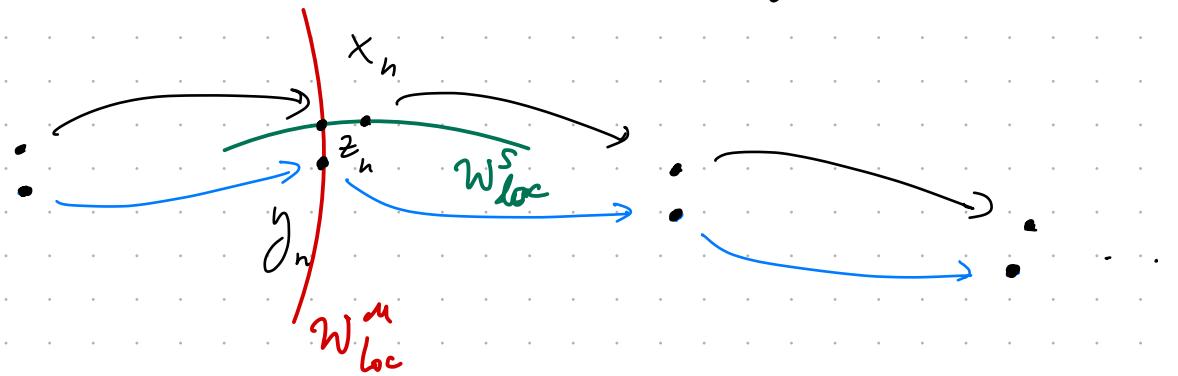


Expansiveness

Lemma : every hyperbolic compact set N for a diffeomorphism f is expansive:

$\exists \varepsilon > 0$ s.t. if $(x_n = f^n(x_0))_{n \in \mathbb{Z}}$ and $(y_n = f^n(y_0))_{n \in \mathbb{Z}}$ are ε -close, then $x_0 = y_0$.

Proof:



For $\varepsilon > 0$ sufficiently small, $W^s_{\varepsilon}(x_n)$ and $W^u_{2\varepsilon}(y_n)$ intersect at a unique point

$$z_n := W^s_{\varepsilon}(x_n) \cap W^u_{2\varepsilon}(y_n)$$

But $W^{s/u}(x_{n+1}) = W^{s/u}(f(x_n)) = f(W^{s/u}(x_n))$ and similarly for y_n

$\Rightarrow (z_n)_{n \in \mathbb{Z}}$ is an f -orbit. But $z_n \in W^u_{2\varepsilon}(y_n)$ and $f|_{W^u}$ is expanding

$$\Rightarrow y_n = z_n, \forall n \geq 0.$$

Same argument for $f^{-1}|_{z_n}, x_n \Rightarrow x_n = z_n, \forall n \leq 0$. Conclusion: $x_0 = y_0$. ■

Structural stability

Definition: Let $f: M \rightarrow M$ be a C^r diffeomorphism of some manifold M , $r \geq 1$.

Let $\Lambda \subset M$ be an invariant set for f . We say that $f|_{\Lambda}$ is **structurally stable** if

$\forall C^r$ perturbation g of f which is sufficiently close to f , $\exists h = i_g: \Lambda \rightarrow M$

continuous injection such that $i_g \circ f|_{\Lambda} = g \circ i_g|_{\Lambda}$.

Theorem: (Anosov '67, Moser '69)

A hyperbolic compact set Λ for a C^1 diffeomorphism is structurally stable.

Proof: we want to solve the equation

$$g \circ h \circ f^{-1} = h$$

for g C^1 -close to f and h C^0 -close to the canonical inclusion $i: \Lambda \hookrightarrow M$.

\rightsquigarrow We will use the implicit function theorem for the map

$$\underline{\Phi} : C^0(1, M) \times C^1(M, M) \ni (h, g) \mapsto g \circ h \circ f^{-1} \in C^0(1, M)$$

We see that $\underline{\Phi}$ is a C^1 -differentiable map of Banach manifolds. Besides

$$\underline{\Phi}(i, f) = i.$$

To apply the inverse function theorem, it is enough to verify that $\text{id} - \partial_h \underline{\Phi}(i, f)$ is an isomorphism.

The tangent space of the Banach manifold $C^0(1, M)$ at the inclusion i is the Banach space $\Gamma = \{g \in C^0(1, TM) : \forall x \in 1, g(x) \in T_x M\}$.

The partial derivative of $\underline{\Phi}$ at (i, f) is :

$$\underline{\mathcal{T}} := \partial_h \underline{\Phi}(i, f) : \Gamma \ni \sigma \mapsto Df \circ \sigma \circ f^{-1} \in \Gamma.$$

Is $(\text{id} - \underline{\mathcal{T}})$ invertible?

We split Γ into two \mathbb{F} -invariant subspaces $\Gamma = \Gamma^m \oplus \Gamma^s$ with

$$\Gamma^m = \left\{ \gamma \in C^0(\mathbb{I}, TM) : \forall x \in \mathbb{I}, \gamma(x) \in E_x^m \right\}$$

$$\Gamma^s = \left\{ \gamma \in C^0(\mathbb{I}, TM) : \forall x \in \mathbb{I}, \gamma(x) \in E_x^s \right\}.$$

The norm of $\mathbb{F}|_{\Gamma^s}$ is less than 1, hence the map $(id - \mathbb{F})|_{\Gamma^s}$ is invertible, with inverse

$$(id - \mathbb{F}|_{\Gamma^s})^{-1} = \sum_{k=0}^{+\infty} (\mathbb{F}|_{\Gamma^s})^k$$

Similarly, $\mathbb{F}|_{\Gamma^m}$ is invertible with contracting inverse, then $(id - \mathbb{F})|_{\Gamma^m}$ is invertible, with inverse

$$- (\mathbb{F}|_{\Gamma^m}) \circ (id - (\mathbb{F}|_{\Gamma^m})^{-1}) = - \sum_{k=1}^{+\infty} (\mathbb{F}|_{\Gamma^m})^{-k}$$

Therefore, by the implicit function theorem, $\exists g$ C' -close to f , \exists a continuous map

h that is C^0 -close to i that semi-conjugates the dynamics:

$$g \circ h = h \circ f$$

Since i is injective and close to h , if $h(x) = h(y)$, then x and y are close.

By semi-conjugacy, $h \circ f^n(x) = g^n \circ h(x) = g^n \circ h(y) = h \circ f^n(y)$, $\forall n \in \mathbb{Z}$
 $\Rightarrow f^n(x)$ is close to $f^n(y)$, for every $n \in \mathbb{Z}$.

By expansiveness, we conclude that $x = y$, i.e. h is injective. \blacksquare

Shadowing

$f : M \rightarrow M$ C^1 diffeomorphism.

Definition : given $\varepsilon > 0$, $(x_n)_{n \in \mathbb{Z}}$ is a ε -pseudo-orbit if

$$\forall n \in \mathbb{Z}, d(x_{n+1}, f(x_n)) < \varepsilon.$$

For hyperbolic sets, pseudo-orbits can be approximated by orbits of the system.

Theorem : (Shadowing lemma)

Let $f : M \rightarrow M$ be a C^1 diffeo. and $H \subset M$ be a hyperbolic set for f .

Then $\exists U \supset H$ neighborhood of H and $\mathcal{U} \ni f|_H$ C^1 -neighborhood of $f|_H$, $\theta, \varepsilon_0 > 0$ s.t.

If $g \in \mathcal{U}$, \mathcal{U} -pseudo-orbit $(x_n)_n$ of g contained in U , with $\varepsilon \in (0, \varepsilon_0)$,

There exists $x \in M$ s.t.

$$d(g^n(x), x_n) < \theta \varepsilon, \quad \forall n \in \mathbb{Z}.$$

Moreover, $(g^n(x))_n$ is the unique g -orbit which ε_0 -shadows $(x_n)_n$ $\left(\begin{array}{l} d(g^n(x), x_n) < \varepsilon_0 \\ \forall n \in \mathbb{Z} \end{array} \right)$

Definition : x is chain-recurrent for f if $\forall \varepsilon > 0$, $\exists k \geq 1$ and $(x_n)_{n \in \mathbb{Z}}$ ε -pseudo-orbit s.t. $x_0 = x_k = x$.

Corollary : (closing lemma)

with the same assumptions as above, if $x \in \Lambda$ is chain-recurrent for $f|_\Lambda$,

then x is accumulated by periodic points whose orbits are contained in arbitrarily small neighborhoods of Λ .

Proof : x chain-recurrent w.r.t. $\varepsilon > 0$: let $(x_n)_n$ be an ε -pseudo-orbit in Λ

and $T \geq 1$ be such that $x_0 = x$ and $x_{n+T} = x_n$, $\forall n \in \mathbb{Z}$.

We deduce that $\exists y_\varepsilon$ whose orbit $(f^n(y_\varepsilon))_{n \in \mathbb{Z}}$ approximates the pseudo-orbit.

Since $(x_n)_n$ is T -periodic, the orbit $(f^{n+T}(y_\varepsilon))_n$ also approximates $(x_n)_n$.

By uniqueness, $f^T(y_\varepsilon) = y_\varepsilon$.

When $\varepsilon \rightarrow 0$, the orbit of y_ε is contained in an arbitrarily small neighborhood of Λ

and $y_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{} x$ as claimed. ■