# Some notions of smooth ergodic theory

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# Main references

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# Some notions of smooth ergodic theory

Let $M$ be a $C^\infty$ Riemannian compact connected manifold, with $\partial M = \varnothing$
Let $f\colon M \to M$ be of class $C^k$, $k \geq 1$, preserving orientation, such that

$$\forall\, x \in M, \quad m(Df(x)) := \inf_{v \in T_xM \setminus \{0\}} \frac{\|Df(x)v\|}{\|v\|} > 0$$

$\leadsto$ $f$ local diffeomorphism of degree $d \geq 1$

**Goal:** describe the statistical behavior of orbits

$$\mathcal{O}_f^+(x) := \{x, f(x), f^2(x) = f \circ f(x), \ldots\}$$

$\leadsto$ Do empirical measures $\dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1} \delta_{f^k(x)}$ converge to a measure $\mu = \mu_{(x)}$ ?

$$\text{i.e., } \frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ f^k(x) \to_{n\to+\infty} \int_M \varphi \, d\mu, \quad \forall \varphi \in C^0(M, \mathbb{R})$$

*Temporal* averages (Birkhoff averages)

*Spatial* averages

- if so, which properties does the measure $\mu$ have?
  - ergodicity: $\forall A \in \mathscr{B}, f^{-1}(A) = A \implies \mu(A) = 0$ or $1$?
  - mixing: $\forall A, B \in \mathscr{B}$, $\mu(A \cap f^{-n}(B)) \to_{n\to+\infty} \mu(A)\mu(B)$?
- conversely, given $\mu \in \mathscr{M}_f := \{\nu$ Borel probability measure $| f_*\nu = \nu\}$ ($\mathscr{M}_f \neq \emptyset$), which $x \in M$ do follow the statistics of $\mu$?

in other words, $x \in \mathscr{B}(\mu)$, i.e., $\frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k(x)} \to \mu$

**Theorem:** (Birkhoff, 32) $\forall \varphi \in L^1(\mu)$, $\exists \varphi^* \in \mathscr{L}^1(\mu)$ such that

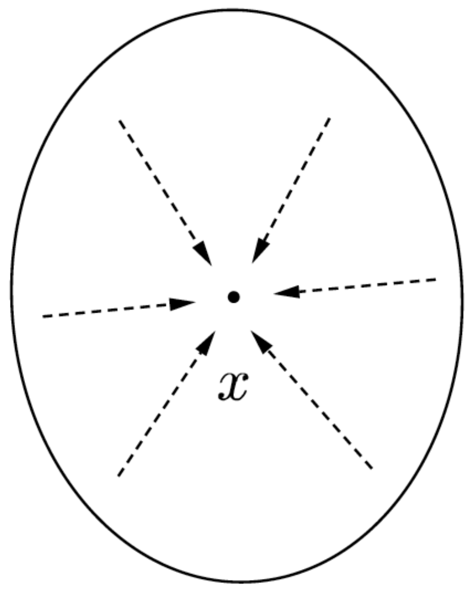
$$\frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ f^k \to \varphi^* \ \mu\text{-a.e.}$$

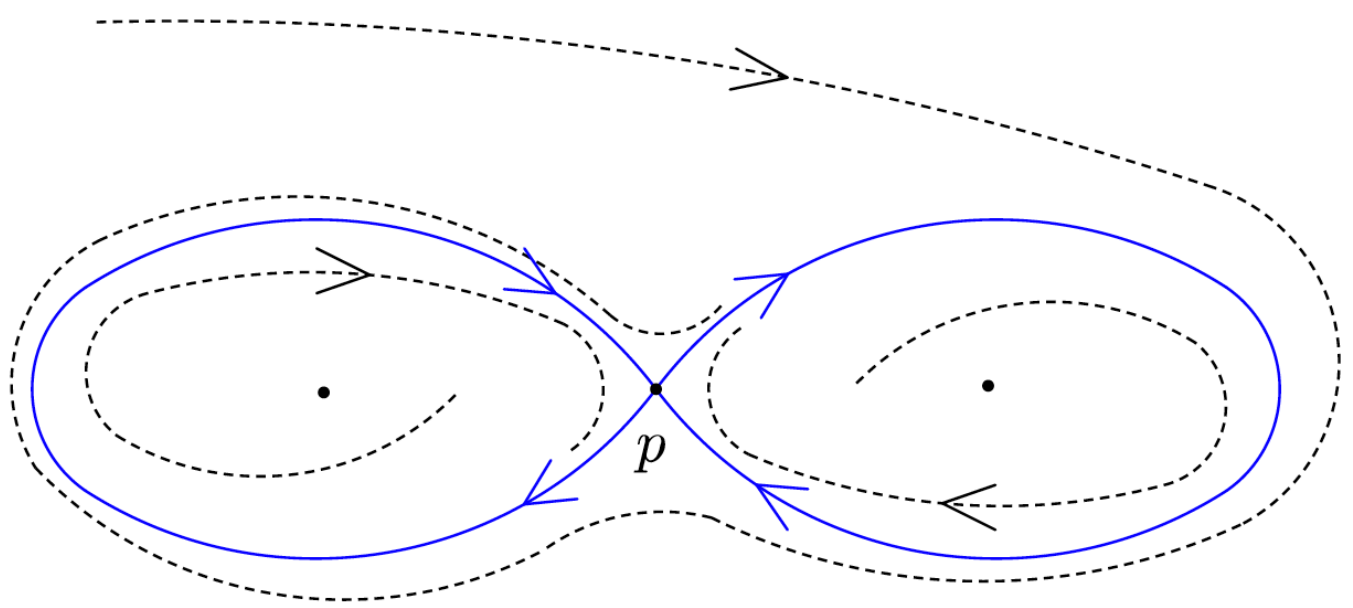
with $\varphi^* \circ f = \varphi^*$ $\mu$-a.e., and $\int_m \varphi^* \, d\mu = \int_M \varphi \, d\mu$

$\rightsquigarrow$ if $(f, \mu)$ ergodic, $\varphi^* = \int_M \varphi \, d\mu$ $\mu$-a.e., and then $\mu$-a.e. $x \in M$ follows the statistics of $\mu$ ($\mu(\mathscr{B}(\mu)) = 1$)

**Question:** does there exist $U \supset \mathrm{supp}(\mu)$ such that
$m(\mathscr{B}(\mu) \cap U) = m(U) > 0$, $m =$ Riemannian volume? (physical measure)

Examples:
- when $\mu$ is ergodic a.c. wrt to $m$ with density $> 0$ on an open set $U$ (Birkhoff)
- when $\mu = \delta_x$ for a sink $x$ / $\mu = \delta_p$ for the figure-eight attractor



# A brief history, conservative/dissipative systems

- **1930s**: Hadamard and Hopf $\rightsquigarrow$ progress on ergodic properties of smooth systems (geodesic flow on negatively curved surfaces)
- **1950s**: Kolmogorov, Arnold and Moser $\rightsquigarrow$ obstructions to ergodicity for Hamiltonian systems (KAM theory)
- **1960s**: work of Anosov and Sinai on hyperbolic systems
- **1970s:** study of broader classes of systems with some hyperbolicity initiated by Brin, Pesin, Hirsch, Pugh, Shub…
  $\rightsquigarrow$ partially hyperbolic systems / non-uniformly hyperbolic systems

## Conservative/dissipative systems

Each map $f$ as above preserves the class of the measure $m$ induced by the Riemannian metric:

$$\frac{df_* m}{dm}(x) = \sum_{y \in f^{-1}(x)} \det Df(y)\, dm(y)$$

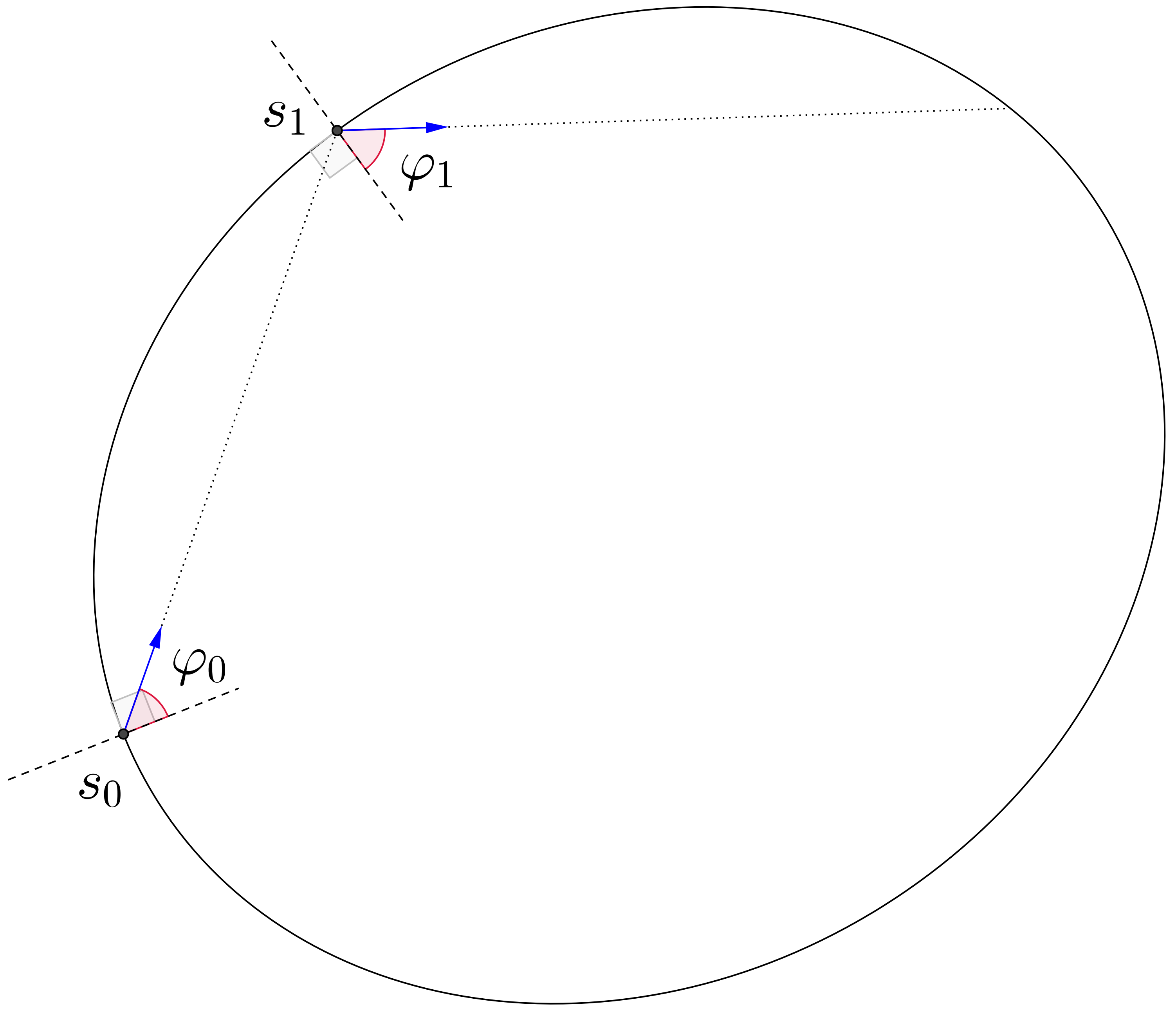
- conservative systems: if $f$ has an invariant measure a.c. with respect to $m$
- dissipative systems: each $f$-invariant measure with full support has a part which is singular with respect to $m$

# Example of conservative systems

Hamiltonian dynamics

(examples: billiards, geodesic flows, $n$-body problem…)

$M = \mathbb{R}^{2n} = \{(x_1, \cdots, x_n, y_1, \cdots, y_n)\}$, endowed with $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$




Let $H\colon U \to \mathbb{R}$ smooth, for an open set $U \subset M$,

$\rightsquigarrow (X^t)_t$ flow induced by the vector field $F$ such that $\omega(F, \cdot\,) = dH$

$$\frac{\partial x_j}{\partial t} = \frac{\partial H}{\partial y_j}, \qquad \frac{\partial y_j}{\partial t} = -\frac{\partial H}{\partial x_j}$$

The flow $(X^t)_t$ preserves the levels $H^{-1}(c)$ and $\omega$, hence the volume $m = \prod_{i=1}^{n} dx_i dy_i$ (Liouville measure)

# Example of dissipative system

$f\colon \mathbb{S}^1 \to \mathbb{S}^1$ with North pole - South pole dynamics between two fixed points $N, S$,
$N$ repelling ($f'(N) > 1$) and $S$ attracting ($0 < f'(S) < 1$)

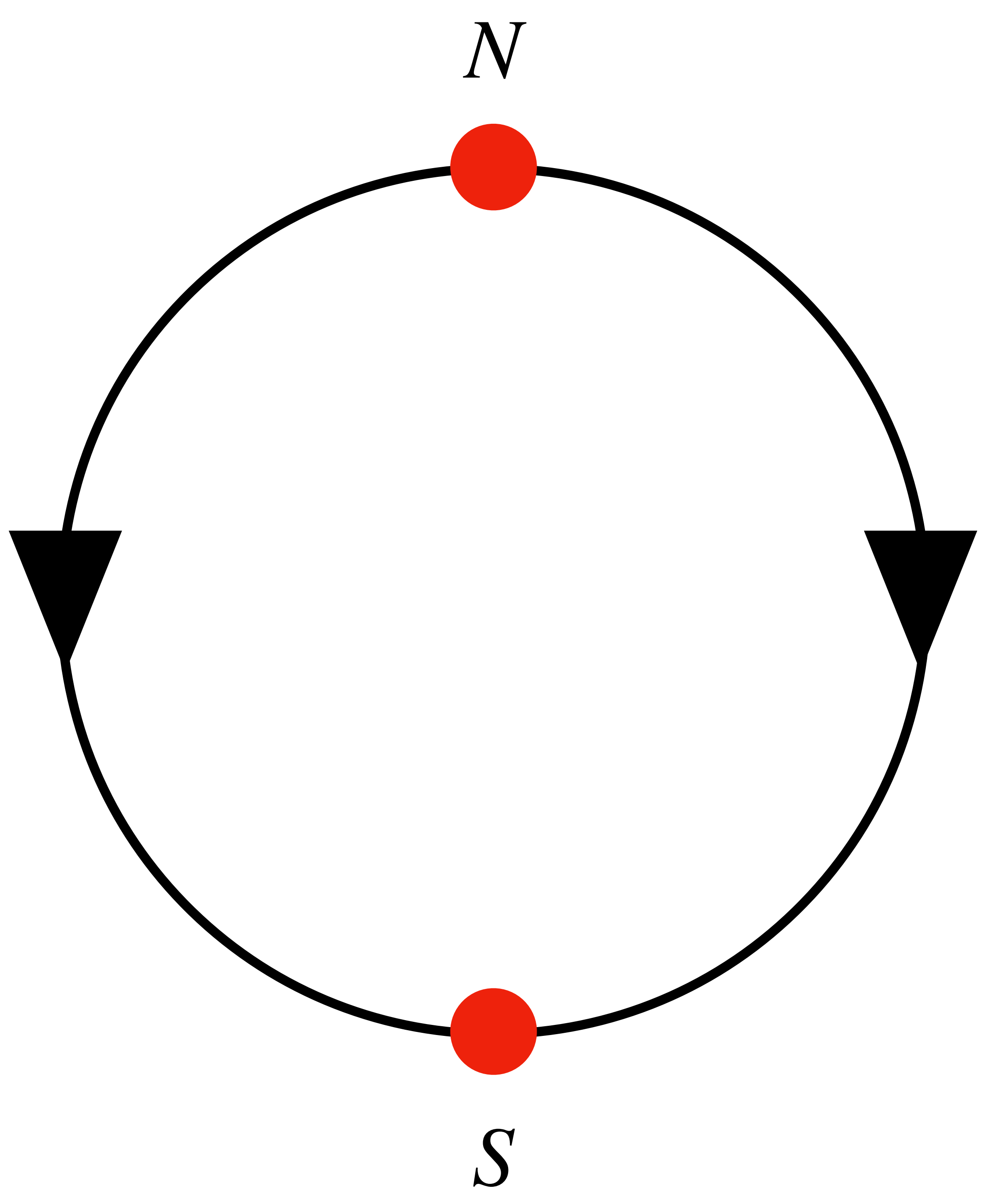
$\rightsquigarrow$ $f$ is dissipative (and $\delta_S$ is a physical measure)

**Proof:** let $\mu$ be an $f$-invariant measure with $\mu(\{S\}) = 0$,
and let $I \subset \mathbb{S}^1 \backslash \{S\}$ open neighborhood of $N$, with $\mu(I) > 0$

$\cap_{k=1}^{+\infty} f^{-k}(I) = \{N\}$ but for all $n \geq 1$,

$\mu(\cap_{k=1}^{n} f^{-k}(I)) = \mu(f^{-n}(I)) = \mu(I)$

$\implies \mu(\{N\}) = \mu(\cap_{k=1}^{+\infty} f^{-k}(I)) = \mu(I) > 0$ ∎

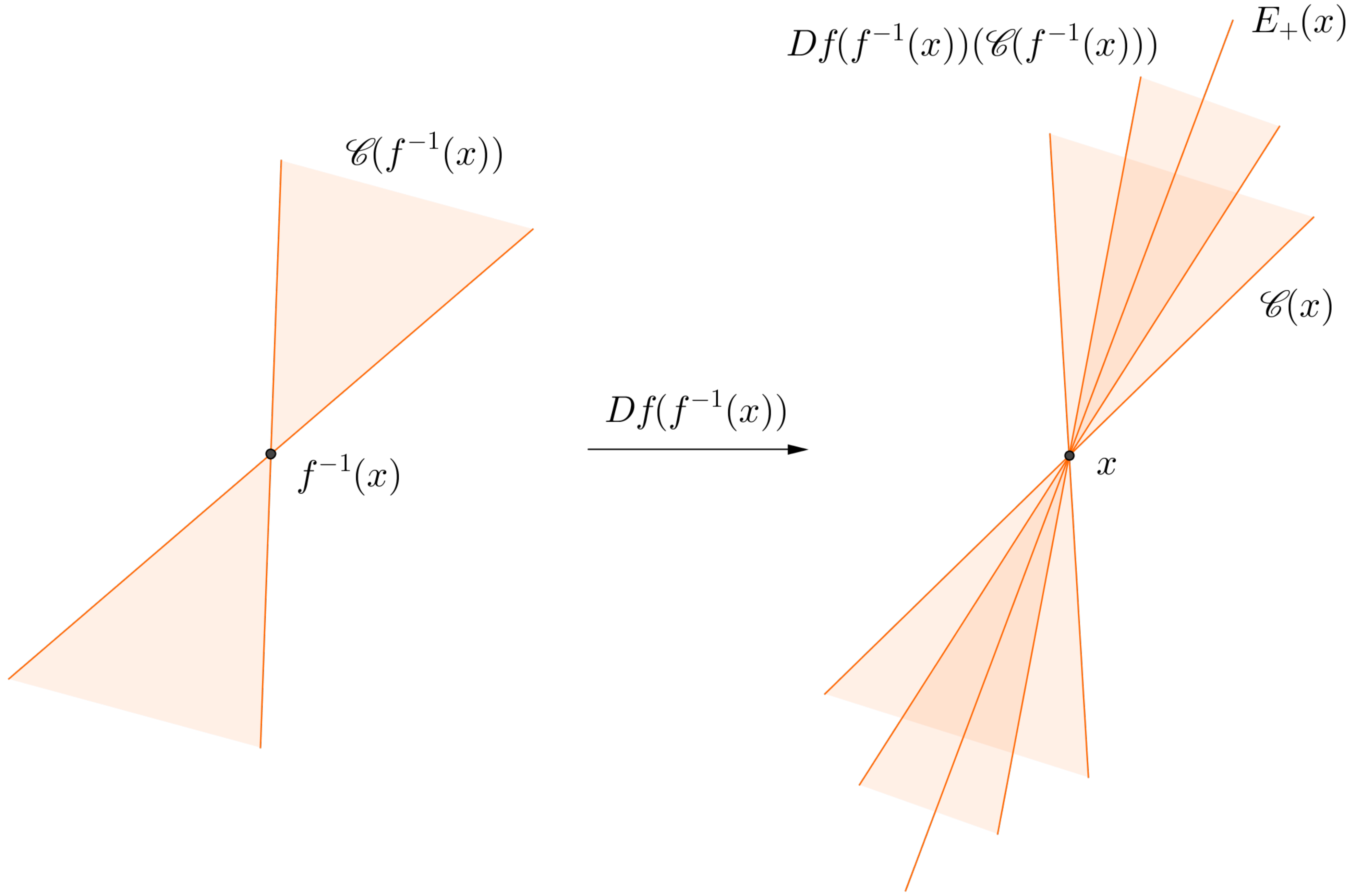


# Dominated splittings and cone-fields

**Definition:** an $f$-invariant set $\Lambda \subset M$ has a dominated splitting if $T_\Lambda M = E_1 \oplus \cdots \oplus E_k$, with

- $\dim(E_i(x))$ independent of $x$, for $i \in \{1, \cdots, k\}$
- (invariance) $\forall\, x \in \Lambda$, $Df(x)E_i(x) = E_i(f(x))$, for $i \in \{1, \cdots, k\}$
- (domination) $\exists\, c > 0, \lambda \in (0,1)$ such that $\forall\, x \in \Lambda$, $\forall i \in \{1, \cdots, k-1\}$,

$$\|Df^n(x)|_{E_i}\| < c\lambda^n m(Df^n(x)|_{E_{i+1}})$$




**Definition:** a cone-field $\mathscr{C}$ on an invariant set $\Lambda \subset M$ is a map $\Lambda \ni x \mapsto \mathscr{C}(x)$ where $\mathscr{C}(x) = \{v \in T_xM : Q_x(v) \geq 0\}$ is a cone, such that in local charts, the quadratic form $\{Q_x\}_{x \in \Lambda}$ defining them can be chosen continuously and have the same signature $(d_+, d_-)$

$\rightsquigarrow$ a cone-field $\mathscr{C}$ on $\Lambda$ is contracted if there exists $N \geq 1$ s.t. for any $x \in \cap_{i=0}^N f^{-k}(\Lambda)$, we have $Df^N(x)\mathscr{C}(x) \subset \mathrm{int}(\mathscr{C}(f^N(x)))$

**Theorem:** (cone-field criterion) let $f \in \mathrm{Diff}^2(M)$, $\Lambda \subset M$ an invariant compact set, and fix $d_+ \geq 1$; then $\Lambda$ has a contracted cone-field $\mathscr{C}$ of dimension $d_+$ if and only if there exists a dominated splitting $T_\Lambda M = E_- \oplus E_+$ with $\dim(E_+) = d_+$

# Foliations, absolute continuity

Let $M$ be a smooth manifold of dimension $n \geq 1$

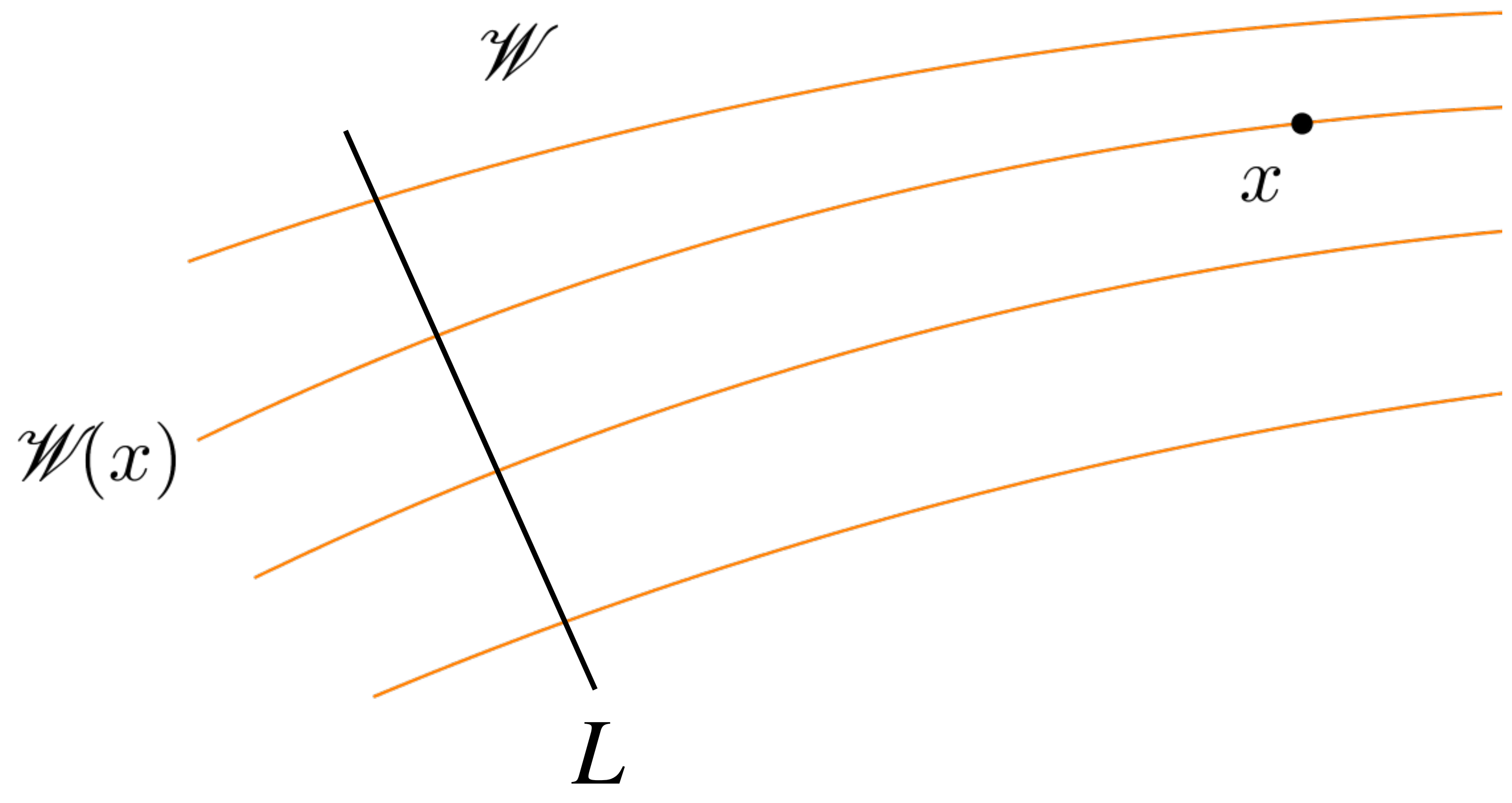
**Definition:** for $1 \leq k \leq n$, a continuous $k$-dimensional foliation $\mathscr{W}$ with $C^1$ leaves of $M$ is a partition of $M$ into $C^1$ submanifolds $\mathscr{W}(x) \ni x$ which locally depend continuously on $x$ in the $C^1$ topology

Let $m$ be the Riemannian volume in $M$
For any submanifold $N \subset M$, let $m_N$ be the induced Riemannian volume in $N$

**Definition:** let $\mathscr{W}$ be a foliation, let $(U, h)$ be a foliation coordinate chart, let $L = h(\{y\} \times B^{n-k})$ be a $C^1$ local transversal
We say that $\mathscr{W}$ is absolutely continuous if for any such $L$ and $U$,
$\exists$ measurable family of positive measurable function
$f_x \colon \mathscr{W}(x) \cap U =: \mathscr{W}_U(x) \to \mathbb{R}$ (conditional densities) s.t. $\forall$ meas. $A \subset U$,

$$m(A) = \int_L \int_{\mathscr{W}_U(x)} \mathbf{1}_A(x, y) f_x(y)\ dm_{\mathscr{W}(x)}(y) dm_L(x)$$

$\rightsquigarrow$ in particular, conditional densities are automatically integrable



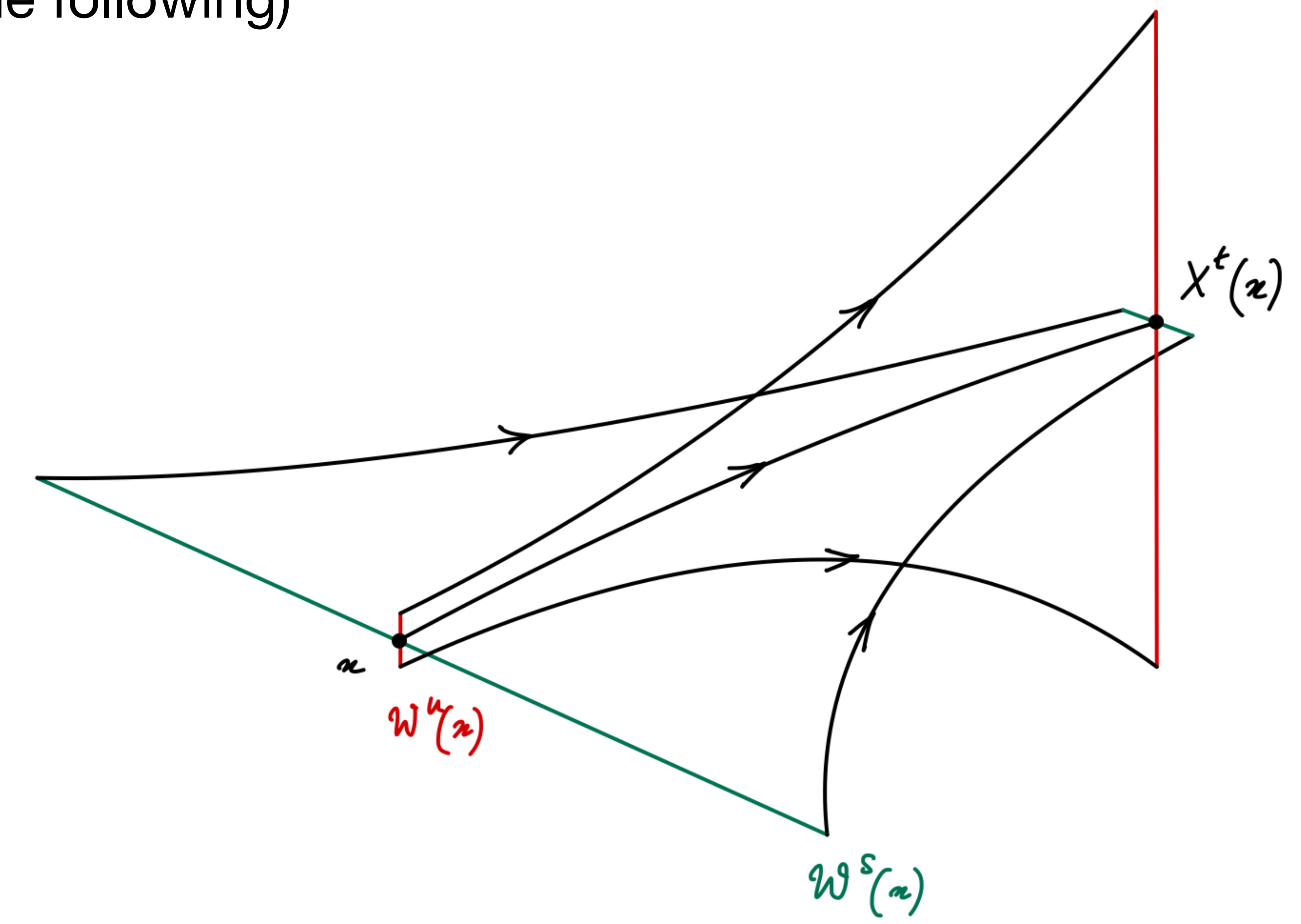
# Hyperbolic Systems

- Let $M$ be a smooth compact Riemannian manifold
- let $f\colon M \to M$ be a $C^k$ (local) diffeomorphism, $k \geq 1$ ($k \geq 2$ in most of the following) or $(X^t\colon M \to M)_{t\in\mathbb{R}}$ (semi-)flow $C^k$, $k \geq 1$, with generator $X$

**Definition:** $\Lambda$ compact $f$-invariant set is hyperbolic if it has a dominated splitting $TM|_\Lambda = E^s \oplus E^u$, where stable/unstable bundle $E^s/E^u$ are uniformly contracted/expanded, i.e., $\exists\, C > 0, 0 < \lambda < 1 < \mu$ such that $\forall x \in \Lambda, n \geq 0$:

$$\begin{aligned} \|Df^n(x)v^s\| &\leq c\lambda^n\|v^s\|, & v^s &\in E^s(x) \\ \|Df^n(x)v^u\| &\geq c^{-1}\mu^n\|v^u\|, & v^u &\in E^u(x) \end{aligned}$$

$\rightsquigarrow TM|_\Lambda = E^s \oplus \mathbb{R}X \oplus E^u$ in the case of a flow




- $E^s, E^u$ integrate uniquely into $f$-invariant foliations $\mathscr{W}^s$ (stable), $\mathscr{W}^u$ (unstable), Hölder continuous, absolutely continuous (a.c.) when $k \geq 2$
- diffeomorphism/flow with $\Lambda = M$: Anosov system

# Partially Hyperbolic Systems

- $M$ compact Riemannian manifold
- $f\colon M \to M$ $C^k$ diffeomorphism, $k \geq 1$

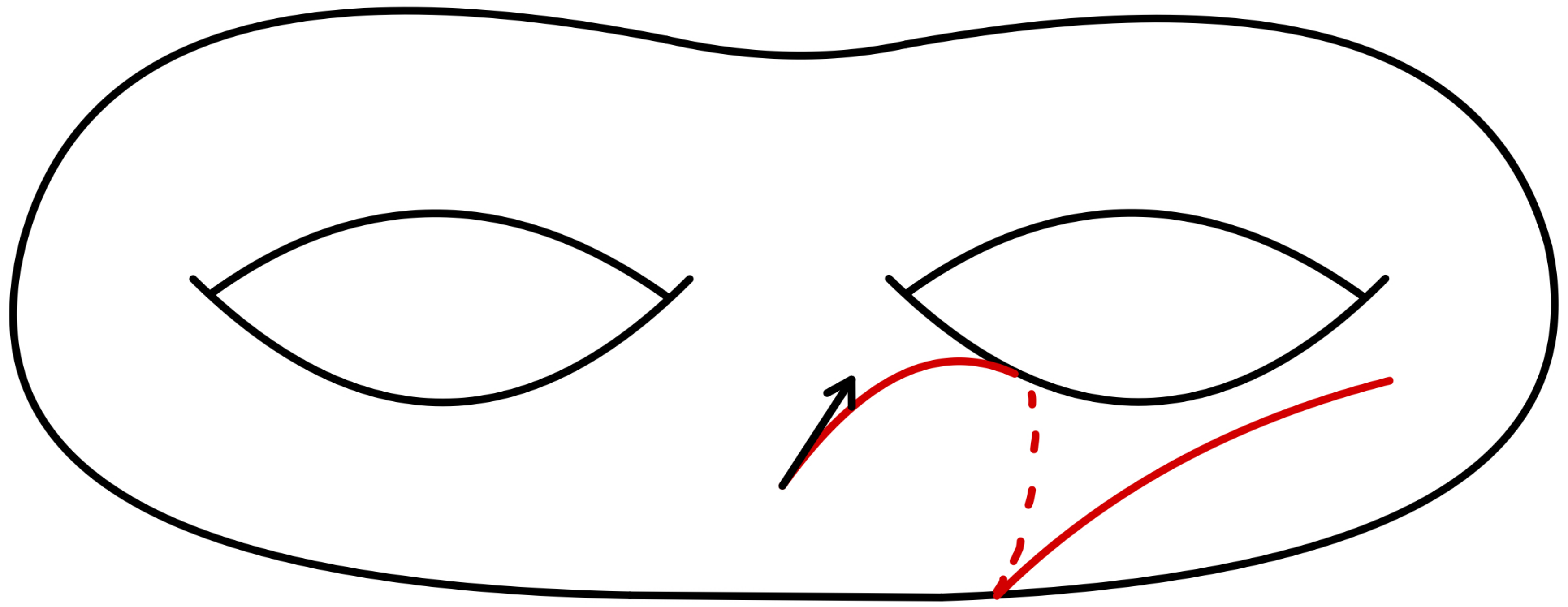
**Definition:** $f$ is partially hyperbolic if there exists a dominated splitting $TM = E^s \oplus E^c \oplus E^u$ s.t. for $c > 0, \lambda_s < \mu_c \leq \lambda_c < \mu_u$, with $\lambda_s < 1 < \mu_u$, we have $\forall\, x \in \Lambda, n \geq 0$:

$$
\begin{aligned}
\|Df^n(x)v^s\| &\leq c\lambda_s^n\|v^s\|, && v^s \in E^s(x)\\
c^{-1}\mu_c^n\|v^c\| \leq \|Df^n(x)v^c\| &\leq c\lambda_c^n\|v^c\|, && v^c \in E^c(x)\\
c^{-1}\mu_u^n\|v^u\| &\leq \|Df^n(x)v^u\|, && v^u \in E^u(x)
\end{aligned}
$$

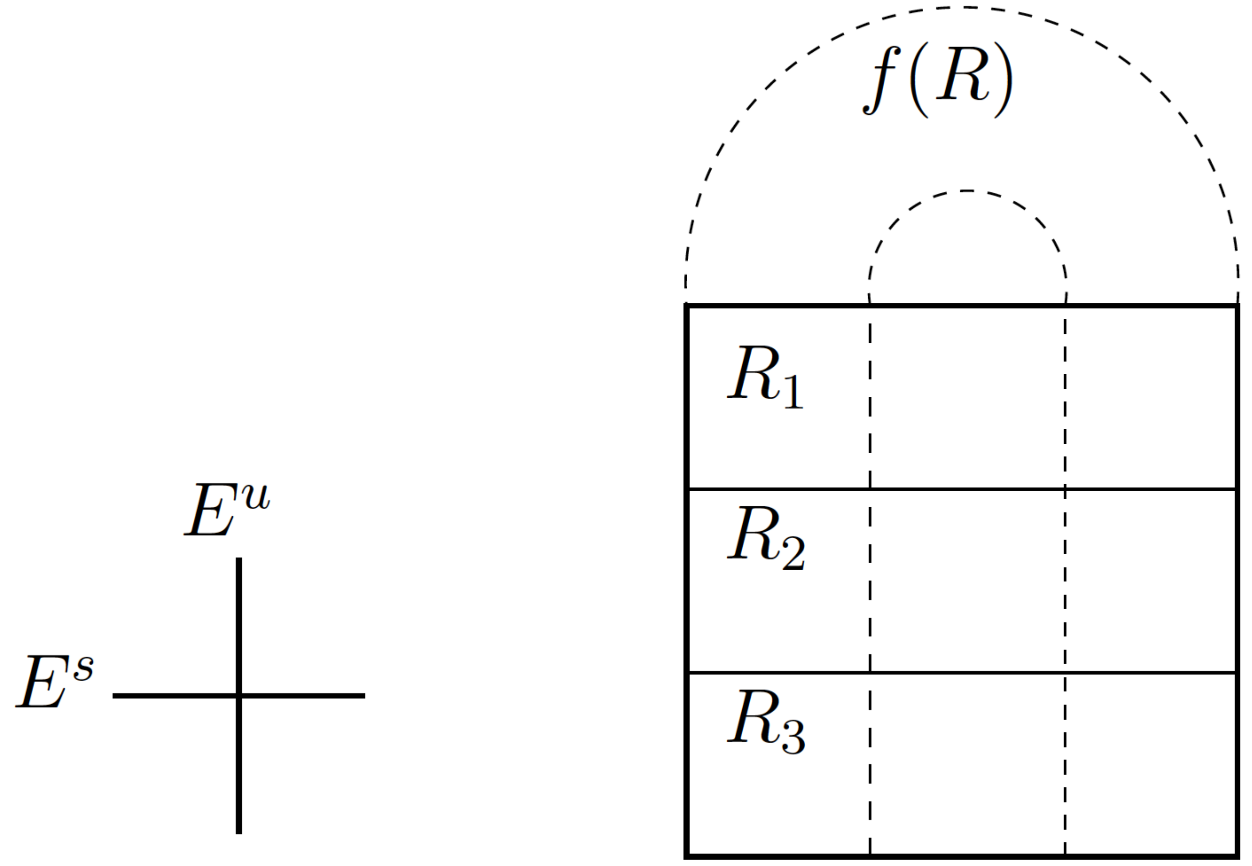
- $E^s, E^u$ integrate uniquely to $f$-invariant foliations $\mathscr{W}^s$ (stable), $\mathscr{W}^u$ (unstable), not necessarily $E^c$
  $\mathscr{W}^s$, $\mathscr{W}^u$ are Hölder continuous, absolutely continuous when $k \geq 2$

# Some (partially) hyperbolic systems

- Anosov flows: geodesic flow on negatively curved surfaces



- hyperbolic set: Smale's horseshoe




- Anosov diffeomorphisms: Arnold's *cat map* on $\mathbb{T}^2$

- partially hyperbolic diffeomorphisms:
    - time-one map $X^1$ of an Anosov flow $(X^t)_t$
    - $A \times \mathrm{Id}$, where $A$ is hyperbolic

# Basic sets, attractors

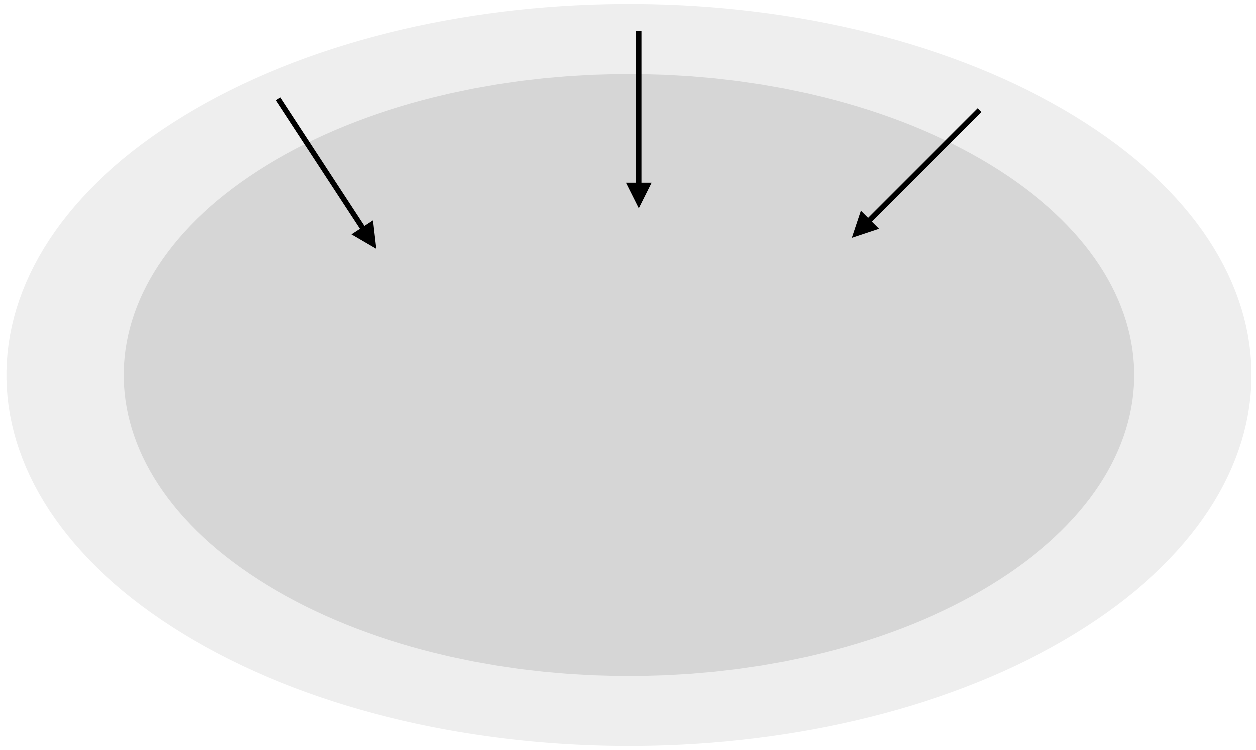
- a hyperbolic set $\Lambda$ is called basic if it is transitive and locally maximal:

$$\Lambda = \cap_{n \in \mathbb{Z}} f^{-n}(\mathscr{U}), \text{ for a neighborhood } \mathscr{U} \text{ of } \Lambda$$

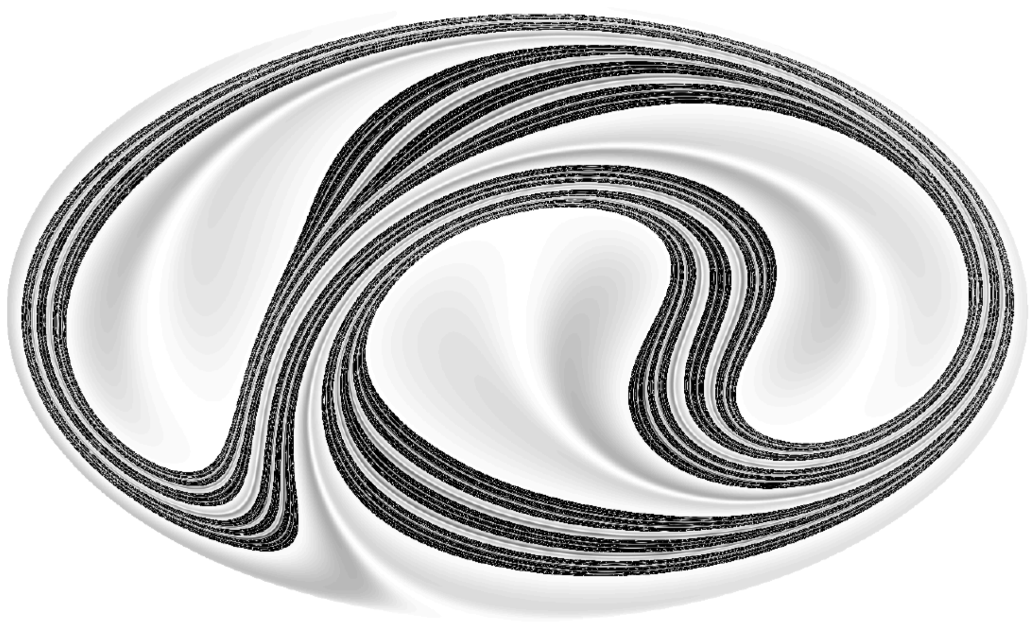
- $\Lambda$ is an attractor if there exists a neighborhood $\mathscr{U}$ of $\Lambda$ such that

$$f(\overline{\mathscr{U}}) \subset \mathscr{U} \text{ and } \Lambda = \cap_{n \in \mathbb{N}} f^{n}(\mathscr{U})$$

$\mathscr{U}$



$\Lambda$



(Plykin attractor)

Basic properties:

- local product structure
- $\Lambda = \cup_{x \in \Lambda} \mathscr{W}^u(x)$
- $\mathscr{U} \subset \cup_{x \in \Lambda} \mathscr{W}^s(x)$

# Basic properties of hyperbolic systems

- sensitivity to initial conditions (« chaotic » systems)

- « good » understanding of statistical properties:
  - ergodicity for $C^2$ conservative systems
  - existence of SRB measures for hyperbolic attractors

- structural stability (Anosov '67) :

If $\Lambda$ is hyperbolic for $f \in \mathrm{Diff}^1(M)$, then there exists a neighborhood $\mathscr{U} \subset \mathrm{Diff}^1(M)$ of $f$ and $h \colon \mathscr{U} \to C^0(\Lambda, M)$ such that for all $g \in \mathscr{U}$,

$$\Lambda_g := h_g(\Lambda) \text{ is hyperbolic for } g,$$

$$h_g \circ f|_\Lambda = g \circ h_g|_\Lambda \quad \text{(topological conjugacy)}$$

In the case of a hyperbolic flow $(X^t)_t$, if $(Y^t)_t$ is $C^1$-close to $(X^t)_t$, there exists an orbit equivalence

- density of periodic orbits for basic sets

# Use of foliations in smooth ergodic theory

1. Statistical properties

- stable ergodicity for $C^2$ conservative diffeomorphisms
- SRB measures: capture the statistical behavior of « many » orbits

$\rightleftarrows$

2. Properties of invariant foliations $\mathscr{W}^{s/u}$

- absolute continuity/Hölder regularity of the foliations $\mathscr{W}^s$ and $\mathscr{W}^u$
- transitivity of the pair $(\mathscr{W}^s, \mathscr{W}^u)$

# Stable Manifold Theorem

Let $f \in \mathrm{Diff}^r(M)$, $r \geq 1$

Let $\Lambda \subset M$ compact $f$-invariant set with a partially hyperbolic splitting $T_\Lambda M = E^s \oplus E^c$, $\dim(E^s) \geq 1$, $E^s$ uniformly contracted

**Definition:** given $\varepsilon > 0$ small, for each $x \in \Lambda$, define the strong stable set:

$$\mathscr{W}^s(x) := \left\{ y \in M : \exists\, c > 0 \text{ s.t. } \forall n \geq 0, d(f^n(x), f^n(y)) < c e^{-\varepsilon n} \min\{m(Df^n|_{E^c(x)}), 1\} \right\}$$

**Theorem:** (Stable Manifold Theorem, Hirsch-Pugh-Shub)

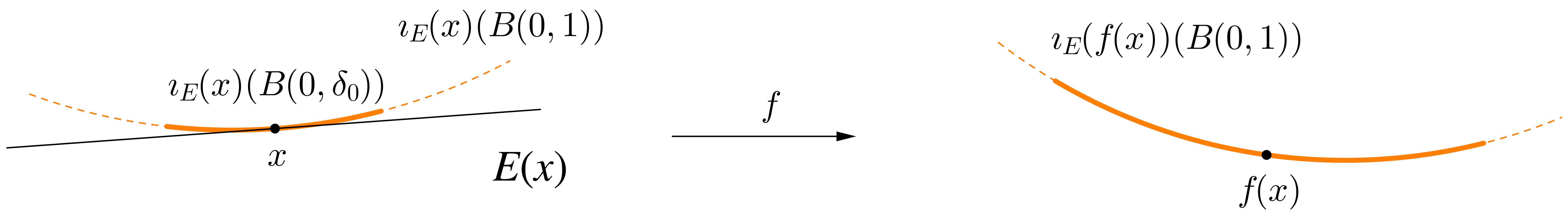
- for any $x \in \Lambda$, the strong stable set $\mathscr{W}^s(x)$ is an injectively immersed $C^r$-submanifold diffeomorphic to $\mathbb{R}^{\dim(E^s)}$, tangent to $E^s(x)$ at $x$
- the strong stable set does not depend on $\varepsilon$ as long as it is small enough
- for any $x, y \in \Lambda$, the strong stable sets $\mathscr{W}^s(x)$, $\mathscr{W}^s(y)$ are either disjoint or coincide
- for $\eta > 0$ small, the ball $\mathscr{W}^s(x, \eta)$ in $\mathscr{W}^s(x)$ of center $x$ and radius $\eta$ depends continuously on $x$ and $f$ for the $C^r$-topology

# Proof of Stable Manifold Theorem: 1) Plaque families

**Theorem:** (Plaque Families, Hirsch-Pugh-Shub)

Let $f \in \mathrm{Diff}^r(M)$, $r \geq 1$, let $\Lambda \subset M$ compact $f$-invariant set with a dominated splitting $T_\Lambda M = E \oplus F$. Then, for every $x \in \Lambda$, there exists a $C^1$ embedding $\imath_E(x) : E(x) \supset B(0,1) \to M$ such that:

- (tangency) for any $x \in \Lambda$, $\imath_E(x)(0) = x$, and $\imath_E(x)(B(0,1))$ is tangent to $E(x)$ at $x$
- (continuity) the embeddings $\{\imath_E(x)\}_x$ depend continuously on $x \in \Lambda$ in the $C^1$-topology
- (local invariance) there exists $\delta_0 \in (0,1)$ such that for $x \in \Lambda$, it holds $f\big(\imath_E(x)(B(0,\delta_0))\big) \subset \imath_E(f(x))(B(0,1))$




**Proof:**

a. Lift $f$ to a $C^r$ local diffeomorphism $f_x := \exp_{f(x)}^{-1} \circ f \circ \exp_x$ from $B(0,\alpha/2) \subset T_xM$ to a neighborhood of $0$ in $T_{f(x)}M$ and glue it with $Df(x)$ on the complement of $B(0,\alpha)$ by a bump function to get a diffeomorphism $\hat{f}_x : T_xM \to T_{f(x)}M$

**Lemma:** for any $\varepsilon > 0$, there exists $\alpha > 0$ such that $d_{C^1}(\hat{f}_x, Df(x)) < \varepsilon$

# 1) Plaque families

b. Let $\mathscr{C}_F$ be a cone-field along $F$ that is contracted by $f$
Let $\mathscr{C}_E$ be a cone-field along $E$ that is contracted by $f^{-1}$
$\rightsquigarrow$ on each tangent space $T_xM$, one obtains a constant cone field which coincides with $\mathscr{C}_E(x)$
$\rightsquigarrow$ $\hat{f}_x^{-1}$ contracts $\mathscr{C}_E(f(x))$ into $\mathscr{C}_E(x)$

Let $L_x$ be the family of Lipschitz graphs tangent to $\mathscr{C}_E(x)$ containing $0$, i.e., the graphs of Lipschitz functions $\psi: E(x) \to F(x)$ such that $\psi(0) = 0$ and $(u, \psi(u)) - (v, \psi(v)) \in \mathscr{C}_E(x), \forall\, u, v \in E(x)$

$\rightsquigarrow$ $L_x$ is complete for the distance $d(\psi_1, \psi_2) := \max_{u \in E(x)} \dfrac{d(\psi_1(u), \psi_2(u))}{\|u\|}$
(distance is bounded because graphs are uniformly Lipschitz)

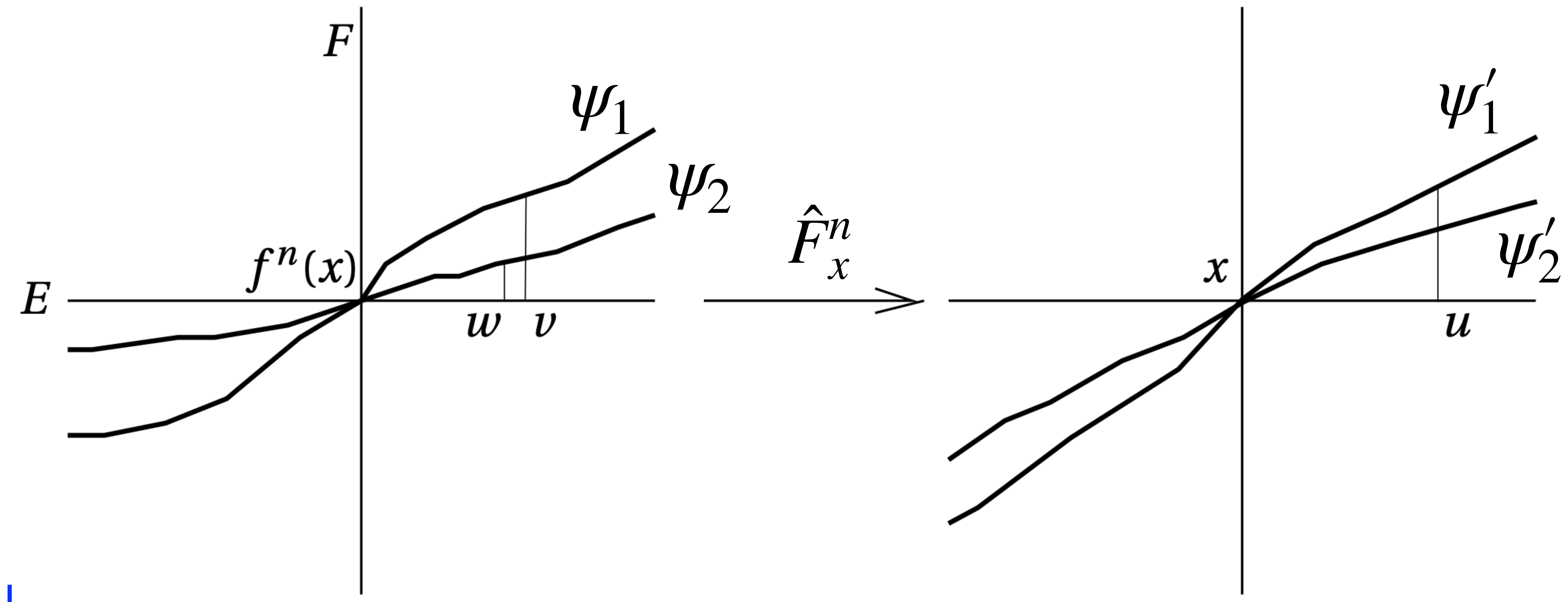
**Lemma:** $\hat{f}_x^{-1}(L_{f(x)}) \subset L_x$
(projection on $E(x)$ is injective on the image of the graph)

# 1) Plaque families

c. **Lemma: (contraction)** for $n \in \mathbb{N}$ large enough, $\hat{F}^n_x := (\hat{f}_{f^{n-1}(x)} \circ \cdots \circ \hat{f}_x)^{-1} : L_{f^n(x)} \to L_x$ is a contraction

**Proof:** let $\psi'_1, \psi'_2$ be the images by $\hat{F}^n_x$ of $\psi_1, \psi_2 \in L_{f^n(x)}$, and fix $u \in E(x)$

$\rightsquigarrow (u, \psi'_1(u)) = \hat{F}^n_x(v, \psi_1(v))$ & $(u, \psi'_2(u)) = \hat{F}^n_x(w, \psi_2(w))$, for $v, w \in E(f^n(x))$




Let us assume that $v = w$ for simplicity, i.e., $(u, \psi'_i(u)) = \hat{F}^n_x(v, \psi_i(v))$, $i = 1,2$

- $(v,0) \in \mathscr{C}_E(f^n(x))$ and $(u,0) \in \mathscr{C}_E(x)$, where $\mathscr{C}_E(f^n(x))$ contracted by $\hat{F}^n_x$, and $\hat{F}^n_x$ close to $(Df^n(x))^{-1}$, hence

$$\|v\| \le \|Df^n(x)|_E\| e^{\varepsilon n} \|u\|$$

- $(0,\psi_1(v) - \psi_2(v)) = (v, \psi_1(v)) - (v, \psi_2(v)) \in \mathscr{C}_F(f^n(x))$,
  $(0,\psi'_1(u) - \psi'_2(u)) = (u, \psi'_1(u)) - (u, \psi'_2(u)) \in \mathscr{C}_F(x)$, where $\mathscr{C}_F(x)$ contracted by $(\hat{F}^n_x)^{-1}$, hence

$$d(\psi_1(v), \psi_2(v)) \ge m(Df^n(x)|_F) e^{-\varepsilon n} d(\psi'_1(u), \psi'_2(u))$$

thus

$$\frac{d(\psi'_1(u), \psi'_2(u))}{\|u\|} \le e^{2\varepsilon n} \frac{\|Df^n(x)|_E\|}{m(Df^n(x)|_F)} \frac{d(\psi_1(v), \psi_2(v))}{\|v\|}$$

$\rightsquigarrow$ $d(\psi'_1, \psi'_2) \le c\lambda^n e^{2\varepsilon n} d(\psi_1, \psi_2)$, by domination, hence uniform contraction for $n$ large enough

# 1) Plaque families

d. Let $\mathscr{L}_x := \prod_{k\in\mathbb{Z}} L_{f^k(x)}$, endowed with the distance given by supremum distance on each $L_{f^k(x)}$

$\rightsquigarrow$ product map $(\hat f^{-1}_{f^k(x)})_{k\in\mathbb{Z}} \colon (\psi_{f^k(x)})_{k\in\mathbb{Z}} \mapsto (\hat f^{-1}_{f^k(x)}(\psi_{f^{k+1}(x)}))_{k\in\mathbb{Z}}$ acts on $\mathscr{L}_x$

and (after iteration) it is a contraction

$\rightsquigarrow$ there exists a fixed point $(\psi_{f^k(x)})_{k\in\mathbb{Z}}$

$\rightsquigarrow$ let then define the embedding $\imath_E(x) \colon E(x) \to M,\ u \mapsto \exp_x(u, \psi_x(u))$

As $\hat f_x|_{B(0,\alpha/2)} \equiv f_x := \exp^{-1}_{f(x)} \circ f \circ \exp_x|_{B(0,\alpha/2)}$ and $\hat f_x$ sends $\psi_x$ to $\psi_{f(x)}$, for $\delta_0 > 0$ small enough,

$$f\big(\imath_E(x)(B(0,\delta_0))\big) = \exp_{f(x)} \hat f_x\big(\mathrm{graph}(\psi_x)|_{B(0,\delta_0)}\big) \subset \exp_{f(x)}\big(\mathrm{graph}(\psi_{f(x)})|_{B(0,1)}\big), \text{ i.e.,}$$

$$f\big(\imath_E(x)(B(0,\delta_0))\big) \subset \imath_E(f(x))(B(0,1))$$

# 1) Plaque families

e. By the cone-field criterion for the maps $\hat{f}_x$, at each $u \in T_xM$ there exists a splitting $\hat{E}(u) \oplus \hat{F}(u)$ s.t.

$\hat{E}(u)$ := vectors $(v_k)_{k\in\mathbb{Z}}$ tgt at $u$ whose iterates under $(\hat{f}_{f^k(x)})_{k\in\mathbb{Z}}$ remain in the cones $(\mathscr{C}_E(f^k(x)))_{k\in\mathbb{Z}}$
$\hat{F}(u)$ := vectors $(v_k)_{k\in\mathbb{Z}}$ tgt at $u$ whose iterates under $(\hat{f}_{f^k(x)})_{k\in\mathbb{Z}}$ remain in the cones $(\mathscr{C}_F(f^k(x)))_{k\in\mathbb{Z}}$

Since $\psi_x$ is Lipschitz, it is differentiable at almost every $u \in T_xM$, hence has a tangent space whose iterates remain in the cones $(\mathscr{C}_E(f^k(x)))_{k\in\mathbb{Z}}$
$\Longrightarrow$ tangent space in $\hat{E}(u)$
$\rightsquigarrow$ Since $\hat{E}(u)$ depends continuously on $u$, $\psi_x$ is $C^1$ and tangent to $\hat{E}(u)$ everywhere
$\rightsquigarrow$ $\iota_E(x)(B(0,1))$ is tangent to $E(x)$ at $0$

f. By construction, $\psi_x$ is close to $\hat{F}^n_x(\psi')$ for $\psi' \in \mathscr{L}_{f^n(x)}$ arbitrary, where $\hat{F}^n_x := (\hat{f}_{f^{n-1}(x)} \circ \cdots \circ \hat{f}_x)^{-1}$
Fixing $n$ and considering $\psi' := \psi_{f^{-n}(x')}$ for $x'$ close to $x$, it implies that $\psi_x$ and $\psi_{x'}$ are close on $B(0,1)$ ∎

# Proof of Stable Manifold Theorem: 2) Coherence argument

**Corollary:** let $f \in \mathrm{Diff}^r(M)$, $r \geq 1$, let $\Lambda \subset M$ compact $f$-invariant set with a partially hyperbolic splitting $T_\Lambda M = E^s \oplus E^c$
Let $\iota_{E^s}(x)$ be the embedding of $E^s(x)$ given by the Plaque families Theorem. Fix $N \in \mathbb{N}$ large, $\varepsilon, \delta_0 > 0$ small, and let

$$\mathscr{W}^s_{\mathrm{loc}}(x) := \iota_{E^s}(x)(B(0,\delta_0))$$

A. For any $x \in \Lambda$, $n \geq 0$, $\mathrm{diam}(f^n(\mathscr{W}^s_{\mathrm{loc}}(x))) \leq e^{n\varepsilon} \prod_{k=0}^{[n/N]} \|Df^{kN}(x)|_{E^s}\|$

B. For any $x \in \Lambda$, $n \geq N$, $f^n(\mathscr{W}^s_{\mathrm{loc}}(x)) \subset \mathscr{W}^s_{\mathrm{loc}}(f^n(x))$

C. For any $x \in \Lambda$, $\mathscr{W}^s(x) = \cup_{k\geq 0} f^{-k}(\mathscr{W}^s_{\mathrm{loc}}(f^k(x)))$

**Proof:** for $\delta_0 > 0$ small enough, the disc $\mathscr{W}^s_{\mathrm{loc}}(x)$ is almost linear, and the action of $f^n$ is close to that of $Df^n|_{E^s(x)}$
A. is proved inductively, checking by the local invariance that $f^n(\mathscr{W}^s_{\mathrm{loc}}(x)) \subset \iota_{E^s}(f^n(x))(B(0,1))$
B. follows from A.
C. by A. and the domination, we deduce that $\mathscr{W}^s(x) \supset \cup_{k\geq 0} f^{-k}(\mathscr{W}^s_{\mathrm{loc}}(f^k(x)))$. Let us show the other inclusion:

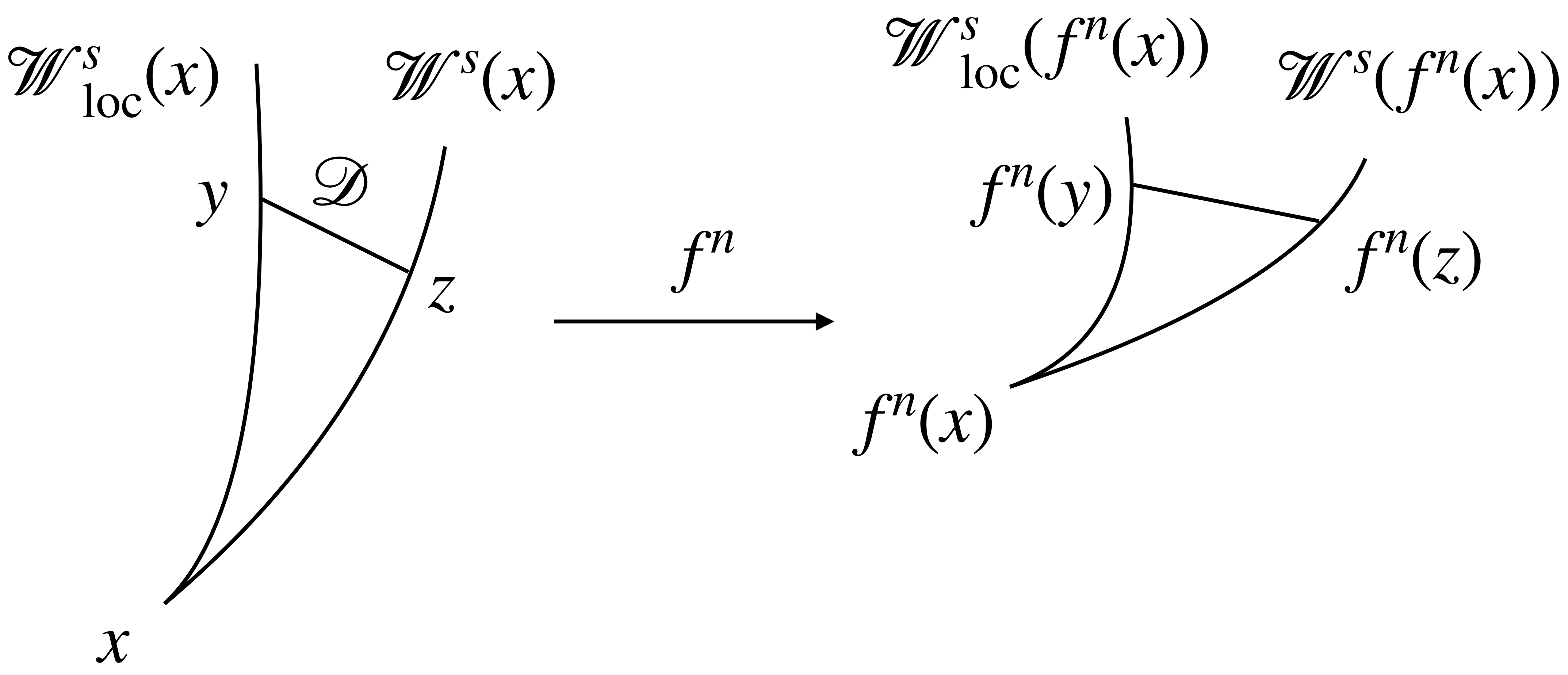
# Proof of Stable Manifold Theorem: 2) Coherence argument

**Coherence argument:** let $z \in \mathscr{W}^s(x)$; up to iteration, assume that their forward iterates remain at distance $\ll \delta_0$

Let $\mathscr{D}$ be a small disk containing $z$ and $y \in \mathscr{W}^s_{\text{loc}}(x)$ and tangent to a contracted cone field $\mathscr{C}_{E^c}$
$\rightsquigarrow$ forward iterates of $\mathscr{D}$ remain tangent to $\mathscr{C}_{E^c}$

For $\varepsilon > 0$ small, and $n \in \mathbb{N}$ large, let us estimate the distance between the points $f^n(x), f^n(y), f^n(z)$:

- $d(f^n(x), f^n(y)) < c_1 e^{n\varepsilon} \| Df^n|_{E^s(x)} \|$
- $d(f^n(x), f^n(z)) < c_2 e^{-\varepsilon n} m(Df^n|_{E^c(x)})$
- $d(f^n(y), f^n(z)) > c_3 e^{-\frac{\varepsilon}{2} n} m(Df^n|_{E^c(x)})$



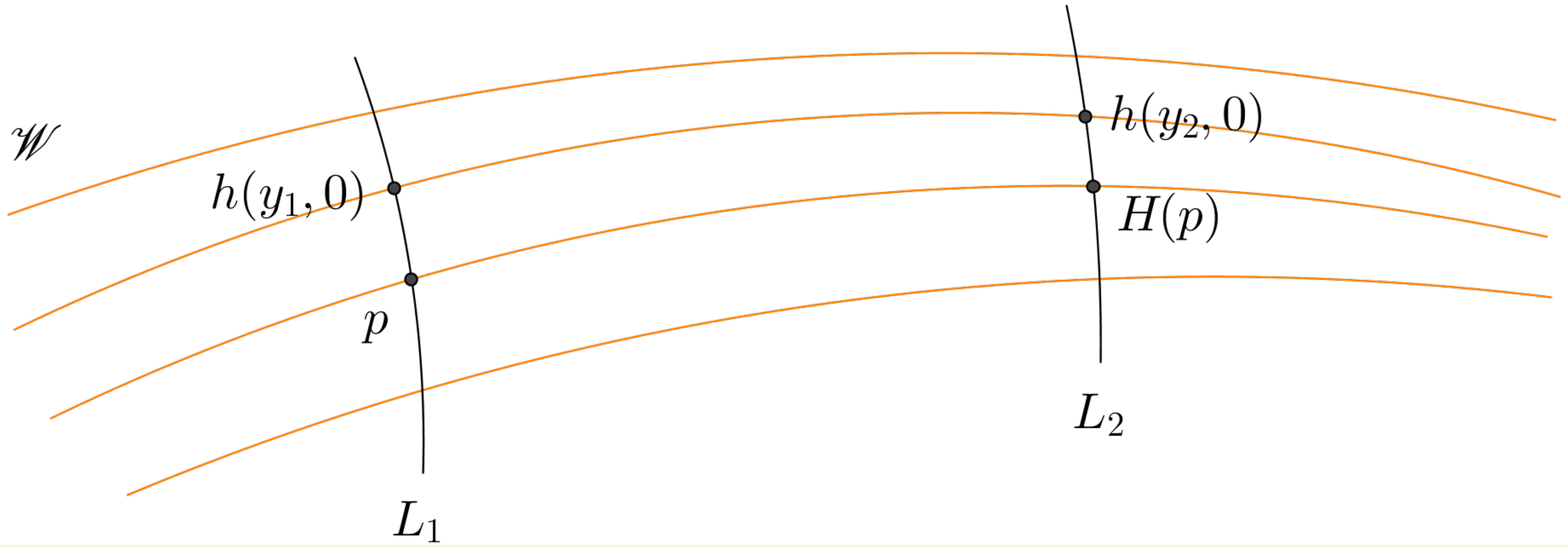

By the domination, and the triangle inequality, we get a contradiction ∎

# Absolute continuity of $\mathscr{W}^s, \mathscr{W}^u$

Let $M$ be a smooth manifold of dimension $n \geq 1$

Let $\mathscr{W}$ be a foliation of $M$, let $(U, h)$ be a foliation coordinate chart, and let $L_i = h(\{y_i\} \times B^{n-k})$ be two $C^1$ local transversals, $i = 1,2$

**Definition:** the holonomy map $H = H_{L_1,L_2} \colon L_1 \to L_2$ is the homeomorphism $h(y_1, z) \mapsto h(y_2, z)$, for $z \in B^{n-k}$



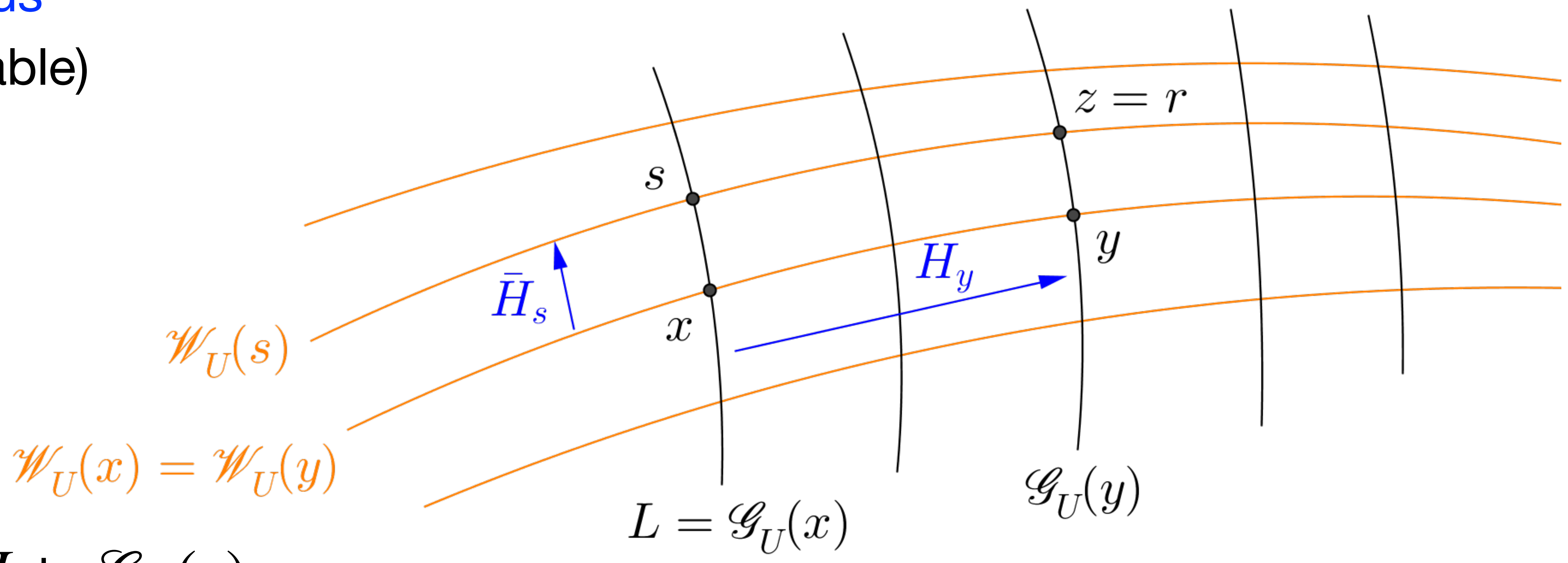

**Definition:** the foliation $\mathscr{W}$ is transversely absolutely continuous if the holonomy map $H$ is absolutely continuous for any foliation chart and any transversals $L_1, L_2$, i.e., there exists a positive measurable function $J \colon L_1 \to \mathbb{R}$ (the Jacobian of $H$) such that for any measurable subset $A \subset L_1$,

$$m_{L_2}(H(A)) = \int_{L_1} \mathbf{1}_A J(z)\, dm_{L_1}(z)$$

# Transverse absolute continuity

**Proposition:** if a foliation $\mathscr{W}$ is transversely absolutely continuous, then it is absolutely continuous

**Proof:** let $(U, h)$ be a foliation coordinate chart on $M$, and let $L = h(\{y\} \times B^{n-k})$ be a $C^1$ local transversal
Let $\mathscr{G}$ be an $n-k$ dimensional $C^1$-foliation such that $L = \mathscr{G}_U(x) := \mathscr{G}(x) \cap U$ and $U = \cup_{y \in \mathscr{W}_U(x)} \mathscr{G}_U(y)$
$\rightsquigarrow$ $\mathscr{G}$ is absolutely continuous and transversally absolutely continuous
Denote by $\{g_y(\,\cdot\,)\}_y$ the densities for $\mathscr{G}$ (continuous, hence measurable)



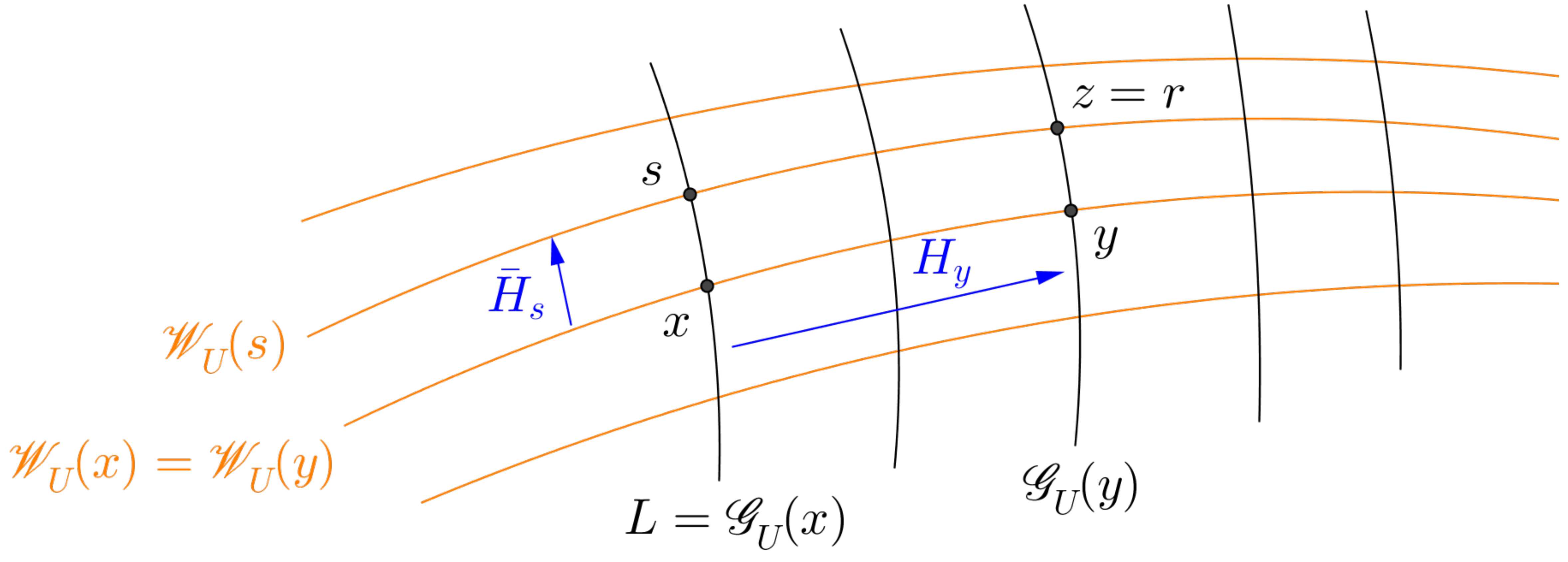

For any measurable set $A \subset U$, by Fubini,

$$m(A) = \int_{\mathscr{W}_U(x)} \int_{\mathscr{G}_U(y)} \mathbf{1}_A(y,z) g_y(z)\ dm_{\mathscr{G}(y)}(z) dm_{\mathscr{W}(x)}(y)$$

Let $H_y$ be the holonomy map along the leaves of $\mathscr{W}$ from $\mathscr{G}_U(x) = L$ to $\mathscr{G}_U(y)$
Let $J_y(\,\cdot\,)$ be the Jacobian of $H_y$

$$\rightsquigarrow \int_{\mathscr{G}_U(y)} \mathbf{1}_A(y,z) g_y(z)\ dm_{\mathscr{G}(y)}(z) = \int_L \mathbf{1}_A(H_y(s)) J_y(s) g_y(H_y(s))\ dm_L(s)$$

# Transverse absolute continuity




Change order of integration in

$$m(A) = \int_{\mathscr{W}_U(x)} \int_{\mathscr{G}_U(y)} \mathbf{1}_A(y, z) g_y(z)\ dm_{\mathscr{G}(y)}(z) dm_{\mathscr{W}(x)}(y)$$

$$\rightsquigarrow m(A) = \int_L \int_{\mathscr{W}_U(x)} \mathbf{1}_A(H_y(s)) J_y(s) g_y(H_y(s))\ dm_{\mathscr{W}(x)}(y) dm_L(s)$$

Let $\bar{H}_s$ be the holonomy map along the leaves of $\mathscr{G}$ from $\mathscr{W}_U(x)$ to $\mathscr{W}_U(s)$, $s \in L$, and let $\bar{J}_s(\cdot)$ be the Jacobian of $\bar{H}_s$

Using a change of variables $r = H_y(s), y = \bar{H}_s^{-1}(r)$, transform integral over $\mathscr{W}_U(x)$ into integral over $\mathscr{W}_U(s)$:

$$\int_{\mathscr{W}_U(x)} \mathbf{1}_A(H_y(s)) J_y(s) g_y(H_y(s))\ dm_{\mathscr{W}(x)}(y) = \int_{\mathscr{W}_U(s)} \mathbf{1}_A(r) J_y(s) g_y(r) \bar{J}_s^{-1}(r)\ dm_{\mathscr{W}(s)}(r)$$

$$\rightsquigarrow m(A) = \int_L \int_{\mathscr{W}_U(s)} \mathbf{1}_A(r) J_y(s) g_y(r) \bar{J}_s^{-1}(r)\ dm_{\mathscr{W}(s)}(r) dm_L(s) \ \blacksquare$$

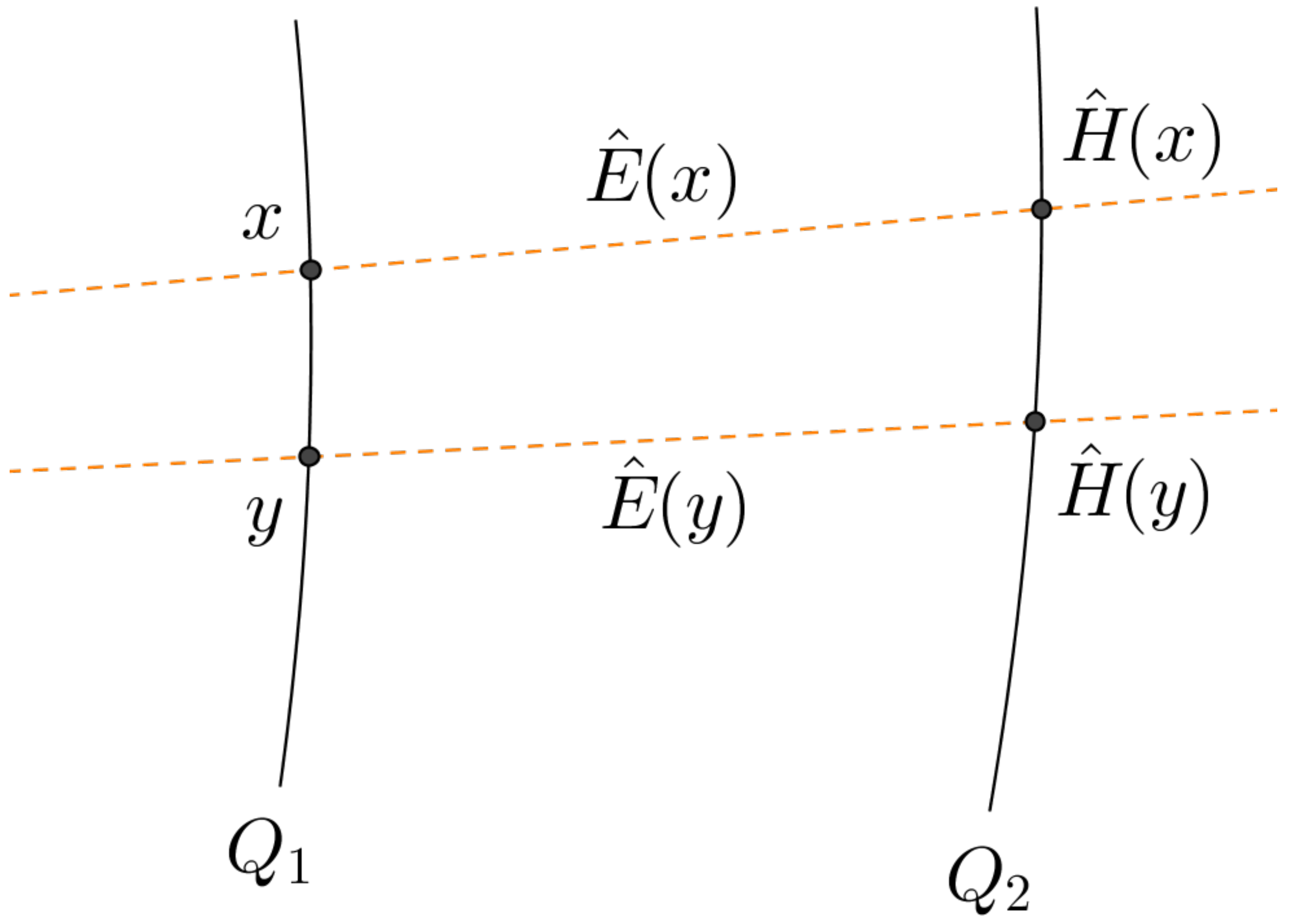
# Proof of absolute continuity of $\mathscr{W}^s, \mathscr{W}^u$

For subspaces $A, B \subset \mathbb{R}^N$, let $\Theta(A,B) := \min\{\|v-w\| : v \in A, \|v = 1\|, w \in B, \|w\| = 1\}$ ($A, B$ are $\theta$-transverse if $\Theta(A,B) \geq \theta > 0$)

**Lemma:** let $\hat{E}$ be a smooth $k$-dimensional distribution on a compact subset of $\mathbb{R}^N$
$\forall\, \xi > 0, \varepsilon > 0$, there exists $\delta > 0$ s.t. if $Q_1, Q_2 \subset \mathbb{R}^N$ are two $N-k$-dim. $C^1$ submanifolds with a smooth holonomy map $\hat{H}\colon Q_1 \to Q_2$ along $\hat{E}$ s.t. for all $x \in Q_1$,

$$\Theta(T_xQ_1, \hat{E}(x)) \geq \xi, \qquad \Theta(T_{\hat{H}(x)}Q_2, \hat{E}(x)) \geq \xi,$$
$$\mathrm{dist}(T_xQ_1, T_{\hat{H}(x)}Q_2) \leq \delta, \qquad \|\hat{H}(x) - x\| \leq \delta,$$

then the Jacobian of $\hat{H}$ is smaller than $1 + \varepsilon$




**Proof:** only the first derivatives of $Q_1, Q_2$ affect the Jacobian of $\hat{H}$
$\rightsquigarrow$ it is equal to the Jacobian of the holonomy map $\bar{H}\colon T_xQ_1 \to T_{\hat{H}(x)}Q_2$ along $\hat{E}$
After linear change of coordinates (depending only on $\xi$), we may assume that in the new coordinates $(u, v) \in \mathbb{R}^N$,

$$x = (0,0), \quad T_{(0,0)}Q_1 = \{v = 0\}, \quad \hat{H}(x) = (0,v_0), \|v_0\| = \|\hat{H}(x) - x\|, \quad T_{(0,v_0)}Q_2 = \{v = v_0 + Bu\}$$

for some $k \times (N-k)$ matrix $B$ whose norm depends only on $\delta$,

$$\hat{E}(0,0) = \{u = 0\}, \quad \hat{E}(w,0) = \{u = w + A(w)v\}$$

for some $(N-k) \times k$ matrix $A(w)$ which is $C^1$ in $w$, $A(0) = 0$

# Proof of absolute continuity of $\mathscr{W}^s, \mathscr{W}^u$

$$x = (0,0), \quad T_{(0,0)}Q_1 = \{v = 0\}, \quad \hat{H}(x) = (0,v_0), \|v_0\| = \|\hat{H}(x) - x\|, \quad T_{(0,v_0)}Q_2 = \{v = v_0 + Bu\}$$

$$\hat{E}(0,0) = \{u = 0\}, \quad \hat{E}(w,0) = \{u = w + A(w)v\}$$

$\rightsquigarrow$ Image of $(w,0)$ under $\hat{H}$ is $\{u = w + A(w)v\} \cap \{v = v_0 + Bu\}$

Norm of $B$ bounded from above in terms of $\xi \rightsquigarrow$ enough to estimate $\dfrac{\partial u}{\partial w}$ at $w = 0$

$$u = w + A(w)v_0 + A(w)Bu$$

$$\frac{\partial u}{\partial w} = I + \frac{\partial A(w)}{\partial w}v_0 + \frac{\partial A(w)}{\partial w}Bu + A(w)B\frac{\partial u}{\partial w}$$

and then for $w = 0$, (recall $u(0) = 0, A(0) = 0$)

$$\frac{\partial u}{\partial w}\Big|_{w=0} = I + \frac{\partial A(w)}{\partial w}\Big|_{w=0} v_0 \quad \blacksquare$$

# Proof of absolute continuity of $\mathscr{W}^s, \mathscr{W}^u$

**Theorem:** the stable and unstable foliations $\mathscr{W}^s, \mathscr{W}^u$ of a $C^2$ Anosov diffeomorphism are transversely absolutely continuous

**Proof:** let $f\colon M \to M$ be a $C^2$ Anosov diffeomorphism with stable and unstable distributions $E^s$, $E^u$, $c > 0$, $0 < \lambda < 1 < \mu$ s.t. $\forall x \in \Lambda, n \geq 0$:

$$\|Df^n(x)v^s\| \leq c\lambda^n\|v^s\|, \qquad v^s \in E^s(x)$$
$$\|Df^n(x)v^u\| \geq c^{-1}\mu^n\|v^u\|, \qquad v^u \in E^u(x)$$

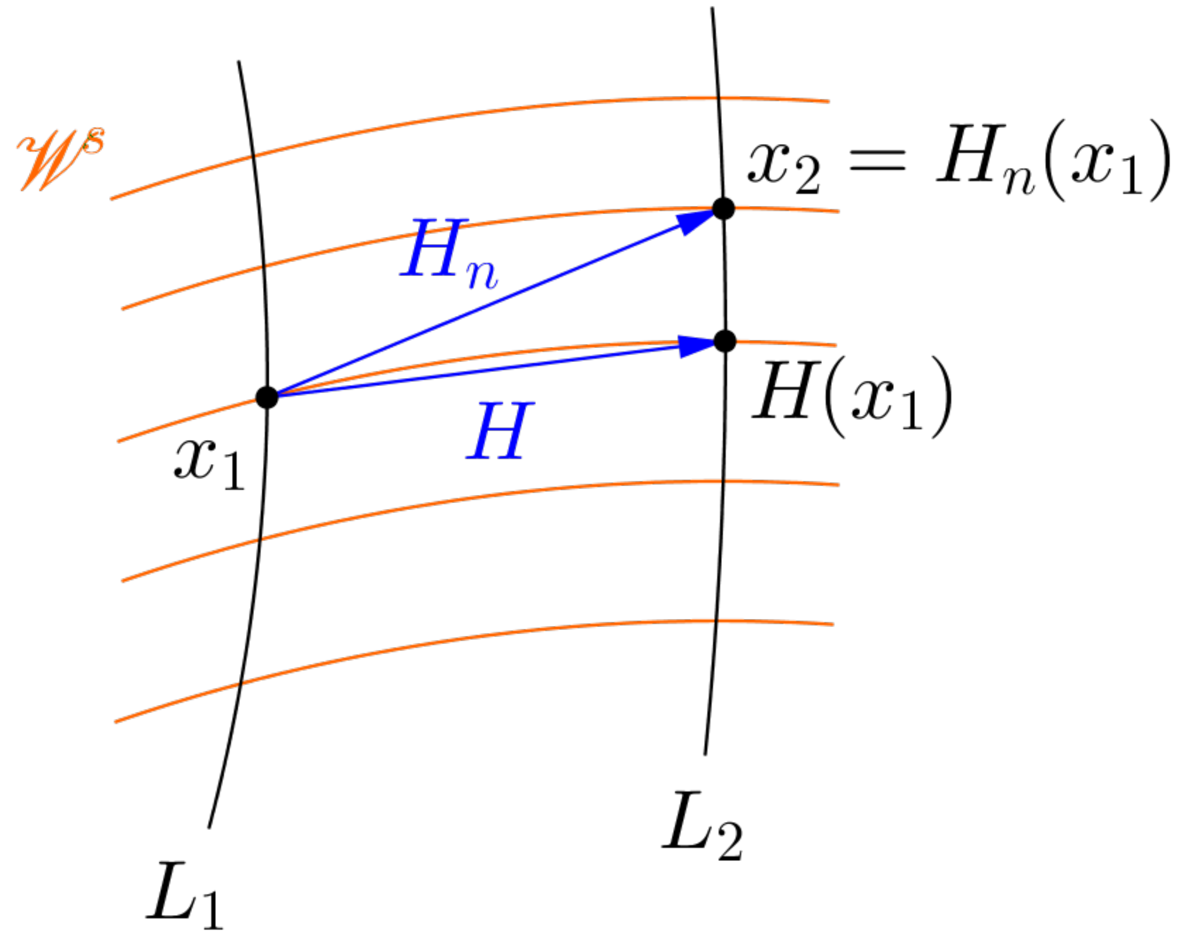
We will focus on $\mathscr{W}^s$ $\rightsquigarrow$ idea: uniformly approximate holonomy maps by continuous maps with uniformly bounded Jacobians
$\rightsquigarrow$ let $\hat{E}^s$ be a smooth distribution that approximate the continuous distribution $E^s$

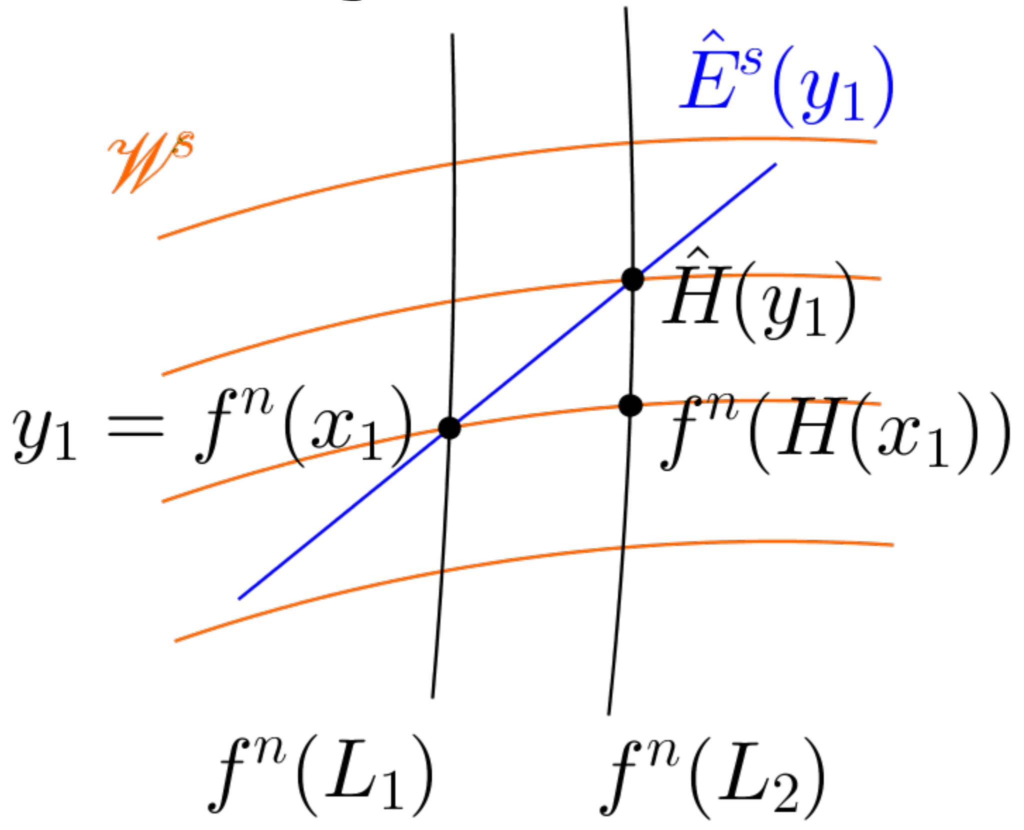
We assume that $M \subset \mathbb{R}^N$. By compactness, for some $\theta_0 > 0$, $\Theta(E^s(x), E^u(x)) \geq \theta_0$, for all $x \in M$,
and there exist foliation charts $(U_i, h_i)_{i=1}^l$ of $\mathscr{W}^s$ $\rightsquigarrow$ $\exists\, \epsilon, \delta > 0$ s.t. $\forall\, y \in U_j$, $\forall\, L \subset U_j$ compact submanifold of $U_j$ s.t.

- $L$ intersects transversely each local stable leaf of $U_j$
- $\Theta(T_zL, E^s) > \dfrac{\theta_0}{3}$, for all $z \in L$
- $\operatorname{dist}(y, L) < \delta$

then for any subspace $E \subset \mathbb{R}^N$ with $\operatorname{dist}(E, E^s(y)) < \epsilon$, $y + E \pitchfork L = \{\pi(y)\}$, with $\|y - \pi(y)\| < \dfrac{6\delta}{\theta_0}$

# Proof of absolute continuity of $\mathscr{W}^s, \mathscr{W}^u$




Let $(U, h)$ be a foliation coordinate chart, $L_1, L_2$ local transversals in $U$ with holonomy map $H\colon L_1 \to L_2$ (along the leaves of $\mathscr{W}^s$), let $\hat{H}\colon f^n(L_1) \to f^n(L_2)$ and $H_n\colon L_1 \to L_2$ be the maps given by

$$\hat{H}\colon f^n(x) \mapsto (f^n(x) + \hat{E}^s(f^n(x))) \pitchfork f^n(L_2), \qquad H_n\colon x \mapsto f^{-n}(\hat{H}(f^n(x)))$$

For $x_1 \in L_1$, $x_2 = H_n(x_1)$, $y_i = f^n(x_i)$, $i = 1,2$; it holds $d(f^k(x_1), f^k(x_2)) \leq c\lambda^k d(x_1, x_2)$, $\forall\, k \geq 0$

If $\hat{E}^s$ is close enough to $E^s$, it is uniformly transverse to $f^n(L_1), f^n(L_2)$, hence $d(\hat{H}(f^n(x_1)), f^n(H(x_1))) \leq c_1 d(f^n(x_1, f^n(H(x_1))) \leq c_1 c\lambda^n d(x_1, H(x_1))$

The angle $\angle(x_2 - x_1, H(x_1) - x_1) \lesssim \left(\dfrac{\lambda}{\mu}\right)^n$, hence

$$d(H_n(x_1), H(x_1)) \leq c_2 \left(\frac{\lambda}{\mu}\right)^n d(x_1, H(x_1)),$$ i.e. $H_n$ converges uniformly to $H$ as $n \to +\infty$

# Proof of absolute continuity of $\mathscr{W}^s, \mathscr{W}^u$

**Lemma:** the Jacobians of $H_n$ are uniformly bounded

**Proof:** by previous estimates, $d(f^k(x_1), f^k(x_2)) \le c_3\lambda^k$, for $k \ge 0$

Let $J(f^k(x_i))$ be the Jacobian of $f$ in the direction of $T_{f^k(x_i)}f^k(L_i)$, for $i = 1,2$, $k \ge 0$

Let $J_n$ be the Jacobian of $H_n$,

Let $\hat{J}$ the Jacobian of $\hat{H}$ (it is uniformly bounded by the previous lemma)

$$\rightsquigarrow J_n(x_1) = \prod_{k=0}^{n-1} (J(f^k(x_2)))^{-1} \hat{J}(f^n(x_1)) \prod_{k=0}^{n-1} J(f^k(x_1))$$

$\rightsquigarrow$ it remains to bound $\displaystyle\prod_{k=0}^{n-1} \frac{J(f^k(x_1))}{J(f^k(x_2))}$ from above:

follows from $\mathrm{dist}(T_{f^k(x_1)}f^k(L_1), T_{f^k(x_2)}f^k(L_2)) \le c_4\lambda^{\alpha k}$, and Lipschitz continuity of $J$ ∎

**End of the proof of a.c.:** by the previous lemma, $\exists J > 0$ s.t. for any measurable set $A \subset L_1$, $m_{L_2}(H_n(A)) \le J m_{L_1}(A)$

Enough to work with balls: let $B(x, r) \subset L_1$ be a ball in $L_1$; for $\delta > 0$ small, $n \ge 0$ large, $H(B(x, r - \delta)) \subset H_n(B(x, r))$,

$$\implies m_{L_2}(H(B(x, r-\delta))) \le m_{L_2}(H_n(B(x, r))) \le J m_{L_1}(B(x, r))$$
$$\implies m_{L_2}(H(B(x, r))) = \lim_{\delta \to 0} m_{L_2}(H(B(x, r-\delta))) \le J m_{L_1}(B(x, r)) \blacksquare$$

# Ergodicity of conservative Anosov diffeomorphisms

**Theorem:** (Anosov, Sinai '67) if $f\colon M \to M$ is a $C^2$ conservative Anosov diffeomorphism ($f_* m = m$, $m$ volume) on a compact connected Riemannian manifold $M$, then $f$ is ergodic

**Proof:** (Hopf argument) $\forall\, \varphi \in L^2(M, \mathbb{R})$, let

$$\bar{\varphi}_f := \limsup_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k$$

Birkhoff's Theorem $\Longrightarrow \bar{\varphi}_f = \pi_f(\varphi)$, where $\pi_f$ is the projection

$$\pi_f\colon L^2(M, \mathbb{R}) \to L^2_f(M, \mathbb{R}) := \{\psi \in L^2(M, \mathbb{R}) : \psi \circ f = \psi\}$$

**Claim:** to show that $f$ is ergodic, enough to show that $\forall\, \varphi \in C^0(M, \mathbb{R})$, $\bar{\varphi}_f = \mathrm{cst}$ a.e. ($= \int_M \varphi \, dm$)

**Proof:** since $\pi_f$ is continuous, and $C^0(M, \mathbb{R})$ is dense in $L^2(M, \mathbb{R})$, we thus have $\pi_f(L^2(M, \mathbb{R})) = L^2_f(M, \mathbb{R}) = \{\mathrm{cst\ fcts}\}$ ∎

# Ergodicity of conservative Anosov diffeomorphisms

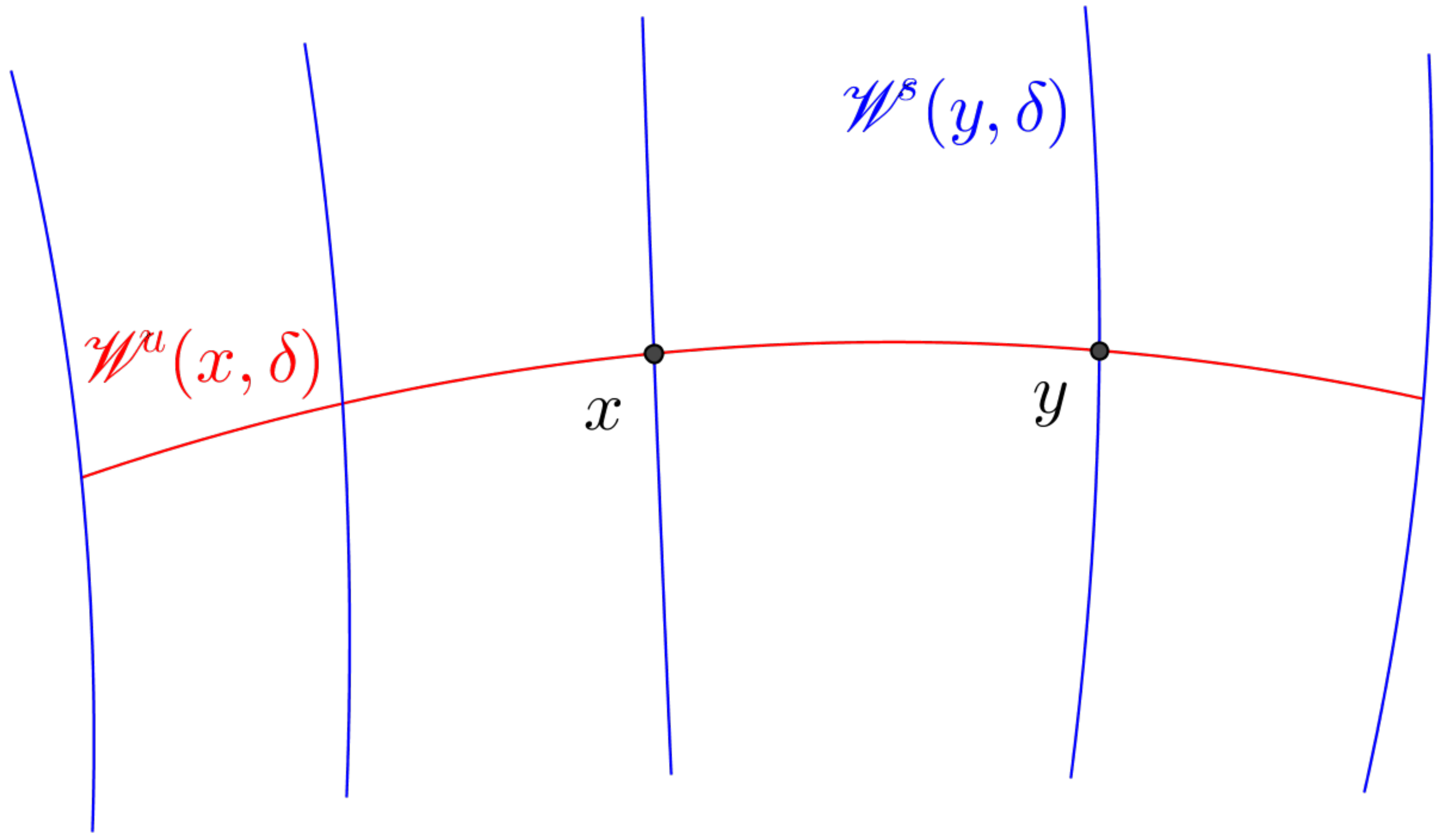
1. Local ergodicity: fix $\varphi \in C^0(M, \mathbb{R})$; then $\bar{\varphi}_f$ is locally constant

Key remarks:

i. $\bar{\varphi}_f$ is constant along the leaves of $\mathscr{W}^s$ ( $\leftarrow \lim_{k\to+\infty} d(f^k(x), f^k(y)) = 0$ if $y \in \mathscr{W}^s(x)$ + Cesàro)

ii. $\bar{\varphi}_f = \bar{\varphi}_{f^{-1}}$ a.e. ($f$-invariance $\Leftrightarrow f^{-1}$-invariance)

iii. $\bar{\varphi}_{f^{-1}}$ is constant along the leaves of $\mathscr{W}^u$




$\rightsquigarrow$ let $A := \{\bar{\varphi}_f = \bar{\varphi}_{f^{-1}}\} \subset M$ ( $\rightsquigarrow m(A^c) = 0$)

$\rightsquigarrow$ let $\delta > 0$ be small s.t. $U = U_x(\delta) := \cup_{y\in\mathscr{W}^u(x,\delta)} \mathscr{W}^s(y,\delta)$ is the homeomorphic image of $[-\delta,\delta]^{\dim(E^u)} \times [-\delta,\delta]^{\dim(E^s)}$ (local product structure)

Absolute continuity of $\mathscr{W}^u \Longrightarrow$ for a.e. $x \in M$, $m_{\mathscr{W}^u(x,\delta)}(A^c \cap \mathscr{W}^u(x,\delta)) = 0$

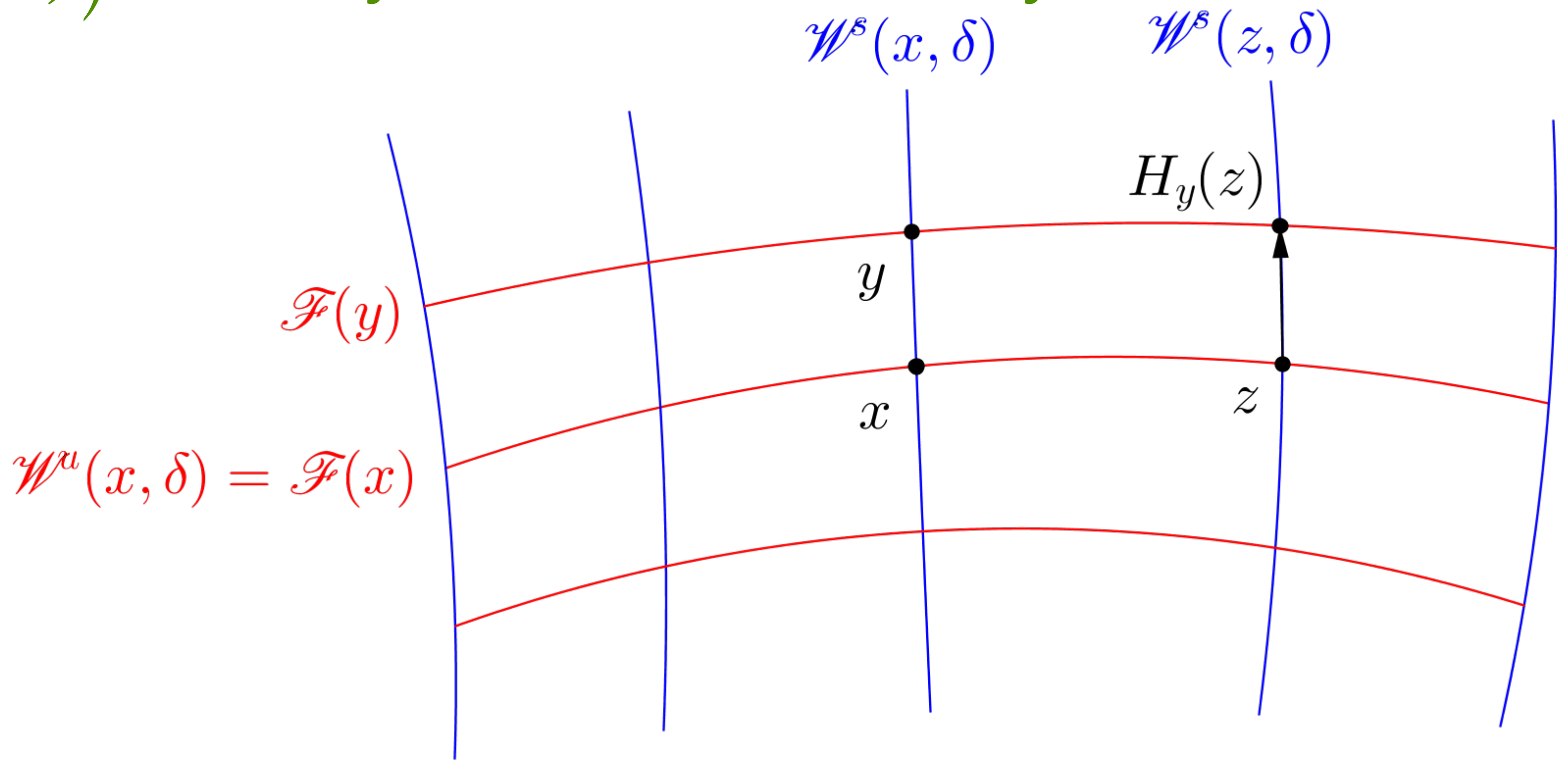
Transverse absolute continuity of $\mathscr{W}^s \Longrightarrow m\Big(\cup_{y\in A^c\cap\mathscr{W}^u(x,\delta)} \mathscr{W}^s(y,\delta)\Big) = 0$, i.e., $m\Big(\cup_{y\in A\cap\mathscr{W}^u(x,\delta)} \mathscr{W}^s(y,\delta)\Big) = m(U)$

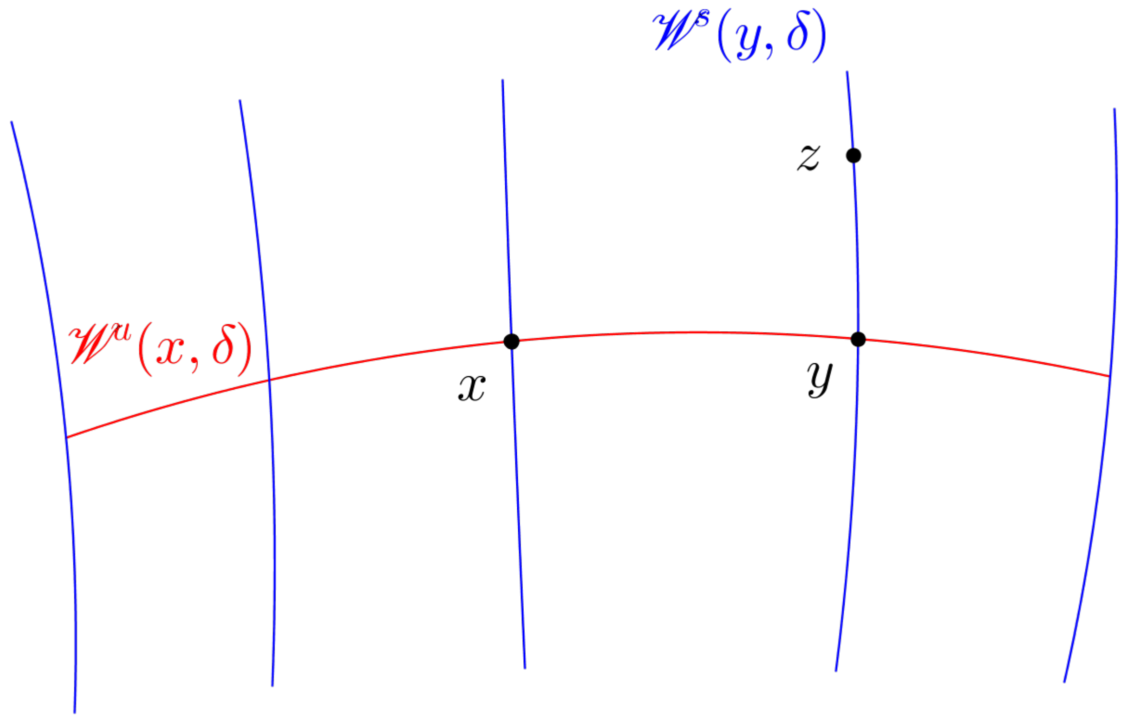
(see below for more details)

# Ergodicity of conservative Anosov diffeomorphisms

Indeed, consider an absolutely continuous foliation $\mathscr{F}$ on $U$ transverse to $\mathscr{W}^s$ (e.g. $\mathscr{W}^u$ or a smooth non-dynamical foliation) with $\mathscr{F}(x) = \mathscr{W}^u(x,\delta)$, and for $y \in U$, consider the holonomy map $H_y\colon \mathscr{W}^s(x,\delta) \to \mathscr{F}(y)$ along the leaves of $\mathscr{W}^s$

$\rightsquigarrow$ for $m$-a.e. $y \in U$, $m_{\mathscr{F}(y)}\Big(\bigcup_{y\in A^c\cap\mathscr{W}^u(x,\delta)} \mathscr{W}^s(y,\delta)\Big) = m_{\mathscr{F}(y)}(H_y(A^c \cap \mathscr{W}^u(x,\delta))) = 0$, and then $m\Big(\bigcup_{y\in A^c\cap\mathscr{W}^u(x,\delta)} \mathscr{W}^s(y,\delta)\Big) = 0$, by absolute continuity of $\mathscr{F}$




Let now $z \in \bigcup_{y\in A\cap\mathscr{W}^u(x,\delta)} \mathscr{W}^s(y,\delta)$, i.e., $z \in \mathscr{W}^s(y,\delta), y \in A \cap \mathscr{W}^u(x,\delta)$:

$\rightsquigarrow$ $\bar\varphi_f(z) \overset{\text{i.}}{=} \bar\varphi_f(y) \overset{y\in A}{=} \bar\varphi_{f^{-1}}(y) \overset{\text{iii.}}{=} \bar\varphi_{f^{-1}}(x)$, i.e., $\bar\varphi_f$ cst $m$-a.e. on $U$ (local ergodicity)

2. Global ergodicity: we have seen that for any $x \in M$, there exists a neighborhood $U_x \subset M$ of $x$ where $\bar\varphi_f$ is $m$-a.e. constant

   By connectedness, we conclude that $\bar\varphi_f$ is constant $m$-a.e. ∎

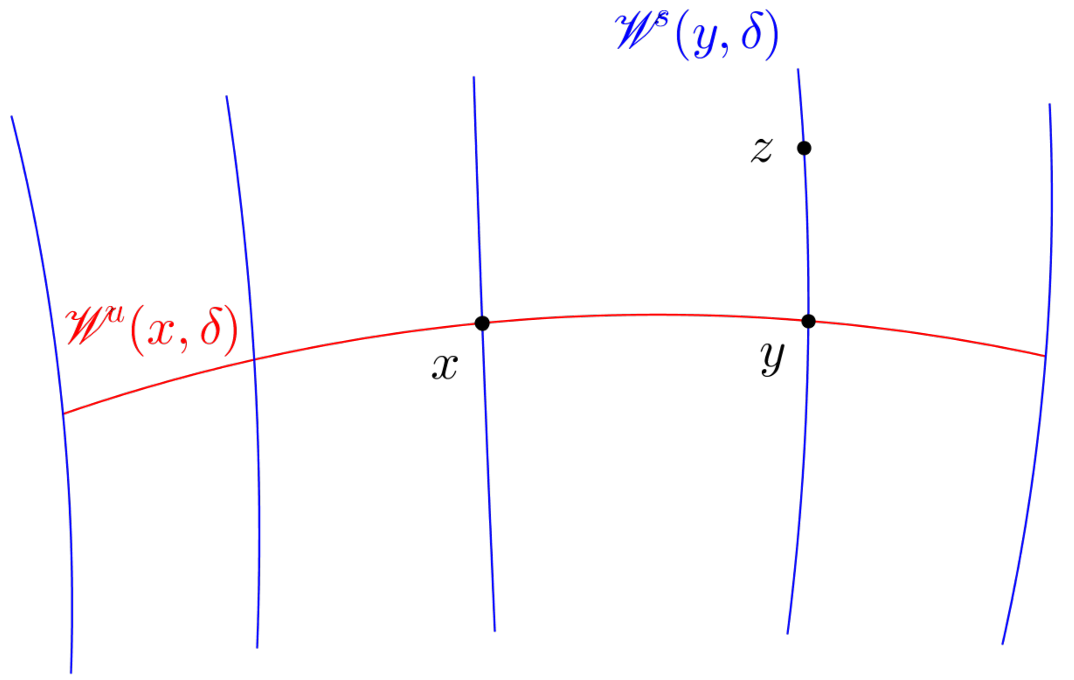
# Sinai-Ruelle-Bowen (SRB) measures

What about the dissipative case? Still a way to describe the statistics of a « large » (for Lebesgue) set of orbits

**Definition:** (SRB measure) let $\mu$ be an $f$-invariant Borel probability measure. We say it is SRB if for every measurable partition $\xi$ subordinate to $\mathscr{W}^u$, the conditional measures $\{\mu_x^\xi\}_x$ are a.c. wrt $m_{\mathscr{W}^u(x)}$ for $\mu$-a.e. $x \in M$

**Proposition:** any ergodic SRB measure $\mu$ is physical (Birkhoff + a.c. of $\mathscr{W}^s, \mathscr{W}^u$)

**Proof:** fix a measurable partition $\xi$ subordinate to $\mathscr{W}^u$, with conditional measures $\{\mu_x^\xi\}_x$
$\mu$ ergodic $\rightsquigarrow \mu(\mathscr{B}(\mu)) = 1$ (by Birkhoff), hence for $\mu$-a.e. $x$, $\mu_x^\xi(\mathscr{B}(\mu)) = 1$
as $\mu$ is SRB, for such $x$ we have $m_{\mathscr{W}^u(x,\delta)}(\mathscr{B}(\mu)) = m_{\mathscr{W}^u(x,\delta)}(\mathscr{W}^u(x,\delta)) \quad (\star)$
Let $U := \bigcup_{y\in\mathscr{W}^u(x,\delta)} \mathscr{W}^s(y,\delta)$; note that for any $z \in \mathscr{W}^s(y,\delta)$, $y \in \mathscr{W}^u(x,\delta) \cap \mathscr{B}(\mu)$,
we have $z \in \mathscr{B}(\mu)$; by transverse absolute continuity of $\mathscr{W}^s$ and $(\star)$ we conclude that
(as in proof of local ergodicity for conservative Anosov diffeos.) $m$-a.e. $z \in U$ is in $\mathscr{B}(\mu)$ ∎




**Theorem** (Sinai-Ruelle-Bowen): let $\Lambda \subset U$ be a transitive attractor of a $C^2$ diffeomorphism $f$; then there exists a unique $f$-invariant Borel probability measure $\mu$ on $\Lambda$ such that for any $f \in C^0(U,\mathbb{R})$, for $m$-a.e. $x \in U$, it holds

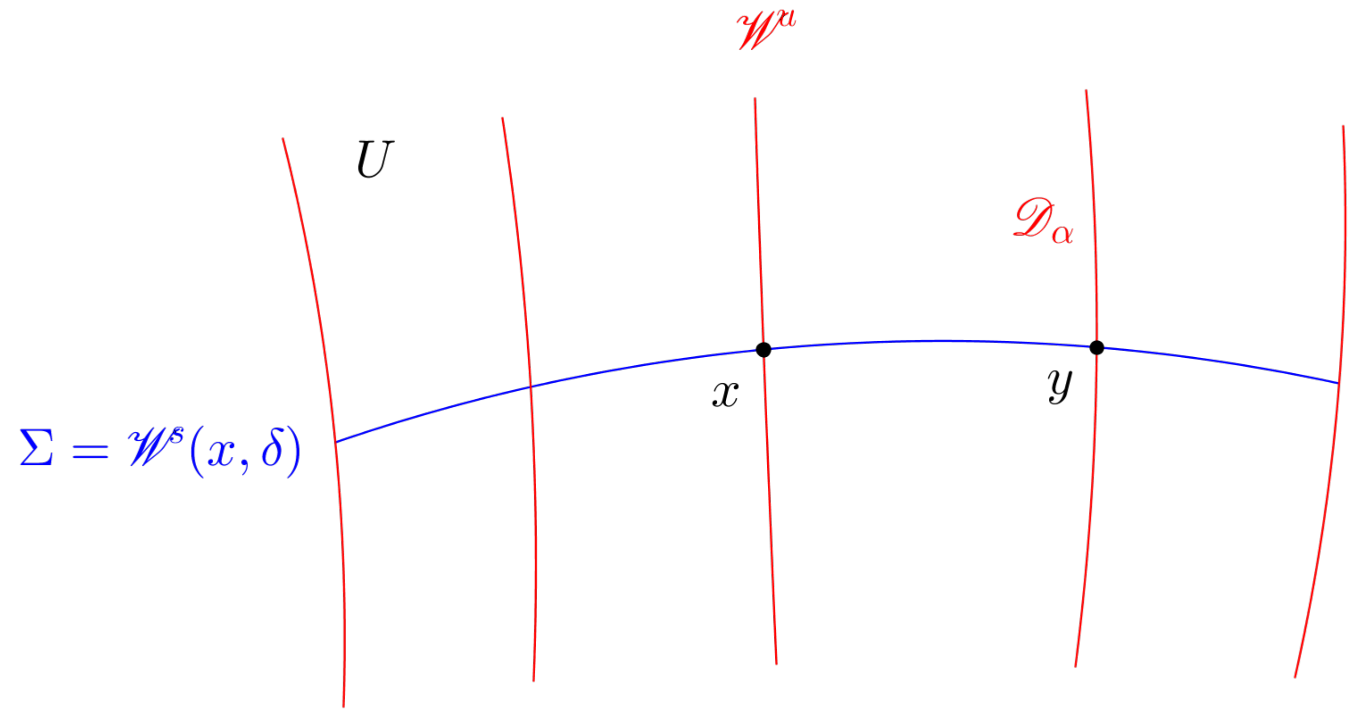
$$\lim_{k\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \varphi(f^k(x)) = \int_U \varphi\, d\mu$$

An equivalent characterization of $\mu$ is that it is SRB. Moreover, $(f,\mu)$ is ergodic

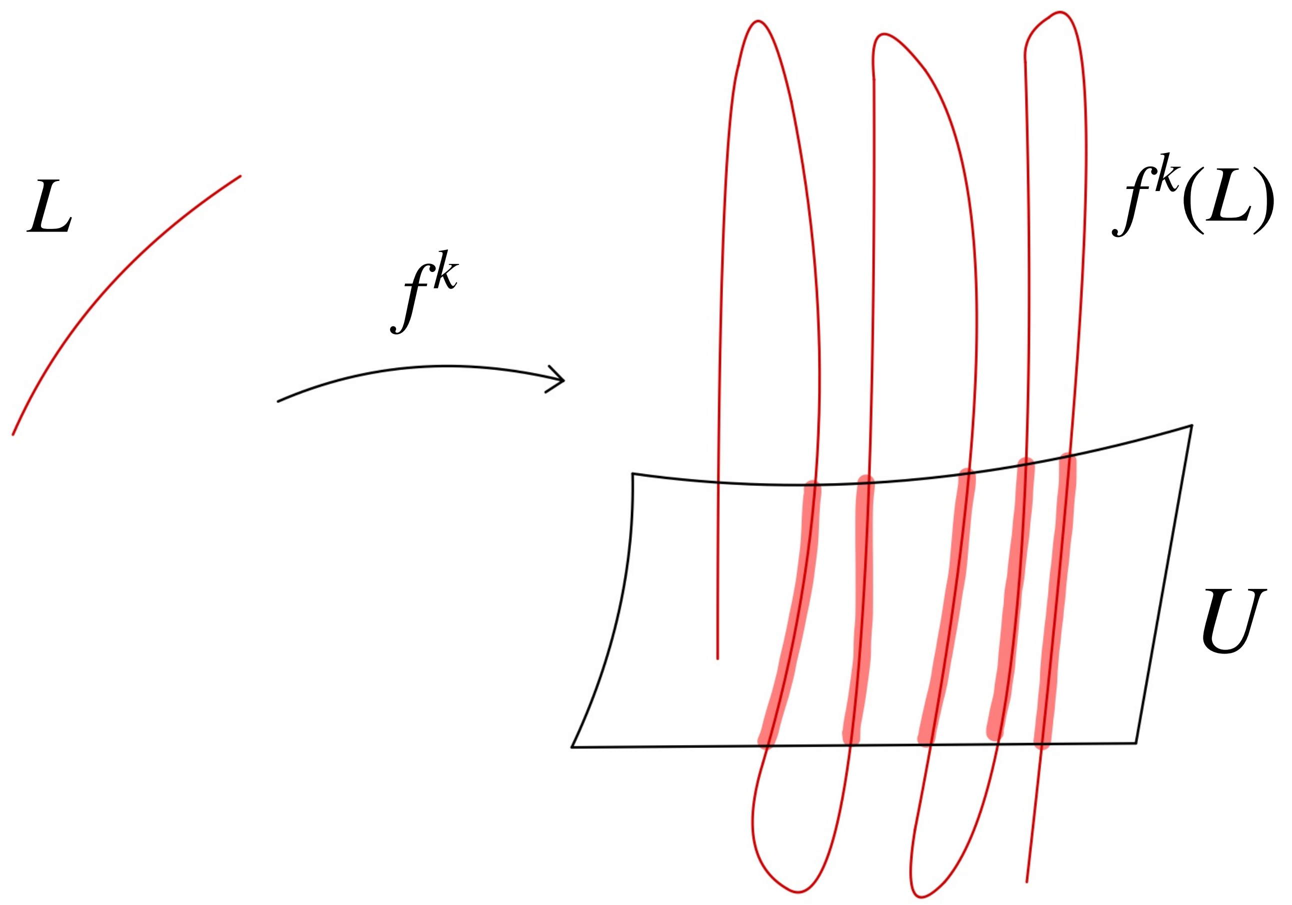
# Proof of Sinai-Ruelle-Bowen result

a. Construction of $\mu$ + SRB property

Let $\Sigma := \mathscr{W}^s(x, \delta)$ be a stable disk, $\delta > 0$ small,
let $U = U(\Sigma, \delta) := \cup_{y \in \Lambda \cap \Sigma} \mathscr{W}^u(s, \delta)$ (topological product for $\delta$ small enough) $\rightsquigarrow$ canonical neighborhood
We write $U = \cup_\alpha \mathscr{D}_\alpha$ the partition of $U$ into local unstable disks




Fix $x_0 \in \Lambda$, let $L := \mathscr{W}^u_{\mathrm{loc}}(x_0)$, and let $\mu_0 := \dfrac{m_L}{m_L(L)}$
For $k \geq 0$, let $\mu_k$ be the measure $(f^k)_* \mu_0$ living on $f^k(L)$
For any canonical neighborhood $U$, let $U_k$ be the union of $\mathscr{D}_\alpha$'s in $U$ completely contained in $f^k(L)$
Let $\hat{\mu}_k = \hat{\mu}_{k,U} := \mathbf{1}_{U_k} \mu_k$; then $\mu_k(U) - \hat{\mu}_k(U) \to 0$ as $k \to +\infty$
(indeed, if $f^k(x) \in \mathscr{D}_\alpha$, with $x \in L$ not too close to $\partial L$, then $\mathscr{D}_\alpha \subset f^k(L)$)




Let $\mu$ be an accumulation point of the averages $\left(\dfrac{1}{n} \sum_{k=0}^{n-1} \mu_k\right)_n \rightsquigarrow \mu$ is $f$-invariant

Moreover, for any canonical neighborhood $U$, $\left(\dfrac{1}{n} \sum_{k=0}^{n-1} \hat{\mu}_{k,U}\right)_n$ also converges to $\mu$

$f|_{\mathscr{W}^u}$ is uniformly expanding, and has uniformly bounded $C^2$ derivatives
$\rightsquigarrow$ by a distortion argument, $\exists \alpha, \beta > 0$ s.t. $\forall k \geq 0$, $\forall \mathscr{D}_\alpha \subset U_k$,

$$\alpha \leq \frac{d\hat{\mu}_{k,U}}{dm_{\mathscr{D}_\alpha}} \leq \beta$$

This bound also works for $\left(\dfrac{1}{n} \sum_{k=0}^{n-1} \hat{\mu}_{k,U}\right)_n$, thus for the accumulation point $\mu \rightsquigarrow \mu$ is SRB ∎

# Proof of Sinai-Ruelle-Bowen result

**Definition:** given $\nu \in \mathcal{M}_f$, we say that a point $x \in M$ is future generic wrt $\nu$ if it is in the basin $\mathcal{B}(\nu)$, i.e.

$$\forall\, \varphi \in C^0(M, \mathbb{R}), \qquad \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int_M \varphi \, d\nu$$

we say it is past generic if it is future generic for $f^{-1}$, generic if it is future + past generic

**Proposition:** (ergodic decomposition of invariant measures) let $\mathcal{E}_f$ be the set of ergodic $f$-invariant Borel probability measures of $f$

Then for any $f$-invariant Borel probability $\nu \in \mathcal{M}_f$, there is a Borel probability measure $\tau_\nu$ on $\mathcal{E}_f$ such that $\nu = \displaystyle\int_{\mathcal{E}_f} \nu' \, d\tau_\nu(\nu')$

$\rightsquigarrow$ $\nu$-a.e. $x \in M$ is generic wrt some ergodic measure $\nu_{(x)} \in \mathcal{E}_f$ (by Birkhoff); denote this set by $\mathrm{G}(\nu)$

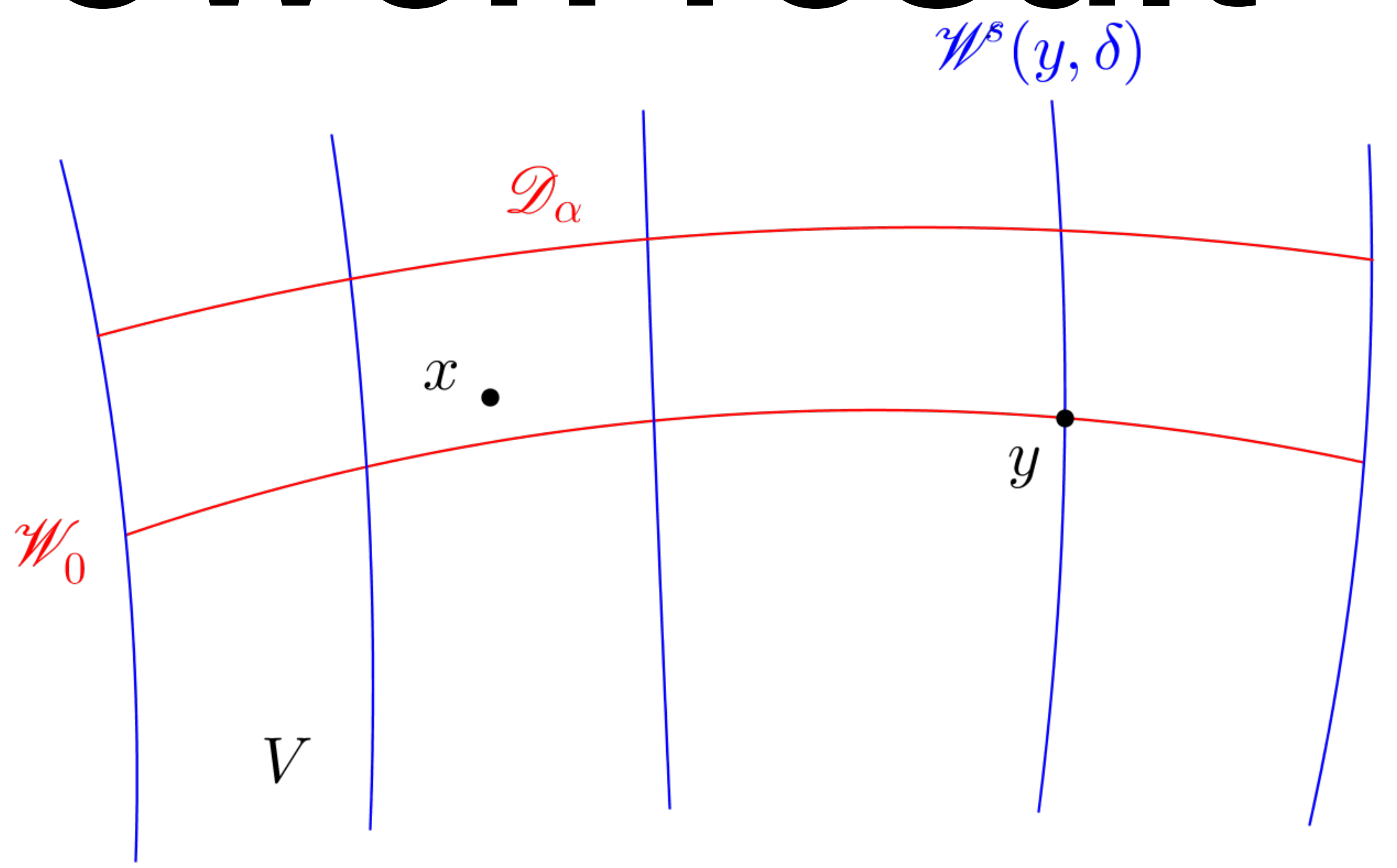
b. Local ergodicity of $(f, \mu)$: (genericity wrt same measure locally)

For any density point $x$ of $\mu$, $\exists\, V$ neighborhood of $x$, $\nu = \nu(x) \in \mathcal{E}_f$ s.t.

i. $\mu$-a.e. $z \in V$ is future generic wrt to $\nu$

ii. $m$-a.e. $z \in V$ is future generic wrt $\nu$

# Proof of Sinai-Ruelle-Bowen result




**Proof:** let $U = \cup_\alpha \mathscr{D}_\alpha$ be a canonical neighborhood centered at $x$
Disintegrate $\mu$ wrt to $\{\mu^u_{\mathscr{D}_\alpha}\}_\alpha$ wrt disks in $U$, and let $\mathscr{W}_0$ be one of these disks s.t.

- $x \in V := \cup_{y \in \mathscr{W}_0} \mathscr{W}^s(y,\delta)$
- $m_{\mathscr{W}_0}$-a.e. $y \in \mathscr{W}_0$ is generic wrt to some $\mu_{(y)}$
  (follows from SRB property and the previous fact that $\mu(\mathrm{G}(\mu)) = 1$,
  i.e., $\mu$-a.e. $y \in M$ is generic wrt to some $\mu_{(y)}$)

But $\forall\, y, y' \in \mathscr{W}_0 \cap \mathrm{G}(\mu)$, $\lim_{n\to+\infty} d(f^{-k}(y), f^{-k}(y')) = 0 \Longrightarrow$ past generic wrt same measure, i.e., $\mu_{(y)} = \mu_{(y')} =: \nu$
Similarly, any $z \in V(\nu) := \cup_{y \in \mathscr{W}_0 \cap \mathrm{G}(\mu)} \mathscr{W}^s(y,\delta)$ is future generic wrt $\nu$
Moreover, as $m_{\mathscr{W}_0}(\mathscr{W}_0 \cap \mathrm{G}(\mu)) = 1$ and $\mathscr{W}^s$ transversally a.c. $\Longrightarrow \forall\, \mathscr{D}_\alpha \subset U$, $m_{\mathscr{D}_\alpha}(\mathscr{D}_\alpha \cap V(\nu)) = 1$
+ SRB property $\Longrightarrow$ for $\mu$-a.e. $\mathscr{D}_\alpha$, $\mu^u_{\mathscr{D}_\alpha}(\mathscr{D}_\alpha \cap V(\nu)) = 1$, hence $\mu(V(\nu)) = \mu(V)$, and $m(V(\nu)) = m(V)$ ∎

c. Global ergodicity: local ergodicity + topological transitivity of the attractor
Indeed, any density point $x$ has a neighborhood $V_x$ where $\mu$-a.e. point is future generic wrt some $\nu(x)$
For any $x, x'$, there exists $n \geq 0$ s.t. $V_x \cap f^{-n}(V_{x'}) \neq \emptyset$ open set where a.e. point is future generic to both $\nu(x)$, $\nu(x')$ $\rightsquigarrow \nu(x) = \nu(x') =: \nu$
Density points have full measure: $\mu$-a.e. point is generic wrt $\nu$, i.e., $(f, \mu)$ ergodic

d. The measure $\mu$ is physical: follows from previous result for ergodic SRB ∎