C^r-PREVALENCE OF STABLE ERGODICITY FOR A CLASS OF PARTIALLY HYPERBOLIC SYSTEMS

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Abstract. We prove that for $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, for any dynamically coherent, center bunched and strongly pinched volume preserving C^r partially hyperbolic diffeomorphism $f\colon X \to X$, if either (1) its center foliation is uniformly compact, or (2) its center-stable and center-unstable foliations are of class C^1, then there exists a C^1-open neighbourhood of f in $\mathrm{Diff}^r(X, \mathrm{Vol})$, in which stable ergodicity is C^r-prevalent in Kolmogorov's sense. In particular, we verify Pugh-Shub's stable ergodicity conjecture in this region. This also provides the first result that verifies the prevalence of stable ergodicity in the measure-theoretical sense. Our theorem applies to a large class of algebraic systems. As applications, we give affirmative answers in the strongly pinched region to: 1. an open question of Pugh-Shub in [32]; 2. a generic version of an open question of Hirsch-Pugh-Shub in [25]; and 3. a generic version of an open question of Pugh-Shub in [25].

Contents

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1. Introduction

Smooth ergodic theory, that is, the study of statistical and geometric properties of measures invariant under a smooth transformation or flow, is a much studied subject in the modern dynamical systems. It has its root in Boltzmann's Ergodic Hypothesis in the study of gas particles back in the 19^{th} century. Ever since Birkhoff's proof of his ergodic theorem, there has been a constant interest in understanding the genericity of ergodic systems. The pioneering work of A. Kolmogorov in the 1950's provided the first obstruction to the genericity of ergodicity for Hamiltonian systems. His idea was later developed into what is now known as the Kolmogorov-Arnold-Moser (KAM) theory. On the other hand, the work of E. Hopf, D. Anosov and Y. Sinai provided open sets of ergodic systems, known as Anosov systems, or uniformly hyperbolic systems.

Definition 1.1 (Anosov diffeomorphisms). Given a compact Riemannian manifold X, a diffeomorphism $f \in \mathrm{Diff}^1(X)$ is called *uniformly hyperbolic* or *Anosov* if there exists a continuous splitting $TX = E_f^s \oplus E_f^u$ of the tangent bundle into Df-invariant subbundles and constants $\bar{\chi}^u, \bar{\chi}^s > 0$ such that for any $x \in X$, we have

$$\|Df(v_1)\| < e^{-\bar{\chi}^s}\|v_1\|, \quad \forall\, v_1 \in E_f^s(x)\backslash\{0\}, \tag{1.1}$$

$$\|Df(v_2)\| > e^{\bar{\chi}^u}\|v_2\|, \quad \forall\, v_2 \in E_f^u(x)\backslash\{0\}. \tag{1.2}$$

For the next nearly thirty years after Anosov-Sinai's work, uniformly hyperbolic systems remained the only examples where ergodicity was known to appear robustly, a property which is also called "stable ergodicity": we say that a C^2 volume preserving diffeomorphism f is stably ergodic if any volume preserving g sufficiently close to f in the C^2 topology is also ergodic. A breakthrough came when M. Grayson, C. Pugh and M. Shub [22] gave the first non-hyperbolic example of a stably ergodic system, i.e., the time-one map of the geodesic flow on the unit tangent bundle of a surface of constant negative curvature. Such systems are special cases of partially hyperbolic systems, which are defined as follows.

Definition 1.2 (Partially hyperbolic diffeomorphisms). Given a smooth Riemannian manifold X, a C^1 diffeomorphism $f\colon X \to X$ is called *partially hyperbolic* if its C^1 norm is uniformly bounded and there exist a nontrivial continuous splitting of the tangent bundle into Df-invariant subbundles, $TX = E_f^s \oplus E_f^c \oplus E_f^u$, and continuous functions $\bar{\chi}^u, \bar{\chi}^s \colon X \to \mathbb{R}_{>0}$, $\bar{\chi}^c, \hat{\chi}^c \colon X \to \mathbb{R}$, such that

$$-\bar{\chi}^s < \bar{\chi}^c \le \hat{\chi}^c < \bar{\chi}^u, \tag{1.3}$$

and for any $x \in X$, any $v_1 \in E_f^s(x)\backslash\{0\}$, $v_2 \in E_f^c(x)\backslash\{0\}$, $v_3 \in E_f^u(x)\backslash\{0\}$, we have

$$\|D_x f(v_1)\| < e^{-\bar{\chi}^s(x)}\|v_1\|, \tag{1.4}$$

$$e^{\bar{\chi}^c(x)}\|v_2\| < \|D_x f(v_2)\| < e^{\hat{\chi}^c(x)}\|v_2\|, \tag{1.5}$$

$$e^{\bar{\chi}^u(x)}\|v_3\| < \|D_x f(v_3)\|. \tag{1.6}$$

We set $E_f^{cs} := E_f^c \oplus E_f^s$ and $E_f^{cu} := E_f^c \oplus E_f^u$.

Partially hyperbolic systems have served as the principal source for finding stably ergodic systems (for other examples, see [9, 12, 41]). They also appear in the study of SRB measures, statistical mechanics, rigidity theory and homogeneous dynamics. Based on [22] and other results, Pugh and Shub formulated the following fundamental conjecture:

Conjecture 1.3 (Pugh-Shub's Stable Ergodicity Conjecture, [32]). *Stable ergodicity is C^r-dense among the set of C^r volume preserving partially hyperbolic diffeomorphisms on a compact connected manifold, for any integer $r \geq 2$.*

Since its introduction, this conjecture and related questions on stable ergodicity have been extensively studied, for instance in the following series of works [33, 15, 20, 39, 38, 37, 14, 4, 5]. We will later elaborate on the connections between them in Subsection 1.1. We mention that Conjecture 1.3 is far from being solved: results directly related to Conjecture 1.3 are only known for $\dim E^c_f = 1$.[1]

Our main result (Theorem E), will be given in Section 2; as we will see, we actually obtained prevalence in Kolmogorov's sense, a notion which is much stronger than density. In the next section, we will state Theorem A, a corollary of Theorem E, to help the reader understand the main features of our result.

1.1. **Stable ergodicity and accessibility.** In [33], the authors proposed a route to prove the Stable Ergodicity Conjecture. They divided the conjecture into two parts, using a geometric notion originating in an argument due to E. Hopf [24].

Let f be C^r partially hyperbolic diffeomorphism of a smooth compact Riemannian manifold X, $r \in \mathbb{N}_{\geq 1} \cup \{\infty\}$. It is well-known (see [25]) that E^s and E^u uniquely integrate to continuous foliations $\mathcal{W}^s_f$ and $\mathcal{W}^u_f$ respectively, called the *stable* and *unstable* foliations. For any $x \in X$ and $* = s, u$, the leaf of $\mathcal{W}^*_f$ through x, denoted by $\mathcal{W}^*_f(x)$, is an immersed C^r-manifold, and $f(\mathcal{W}^*_f(x)) = \mathcal{W}^*_f(f(x))$. If $f \in \mathcal{PH}^2(X)$, the transverse regularity of $\mathcal{W}^s_f$ and $\mathcal{W}^u_f$ is Hölder (see [35]).

Definition 1.4 (Accessibility). An *su-path* of f is a path obtained by concatenating finitely many subpaths, each of which lies entirely in a single leaf of $\mathcal{W}^s_f$ or $\mathcal{W}^u_f$. The map f is said to be *accessible* if any two points in X can be connected by some *su*-path. We say that f is (C^1-)*stably accessible* if there exists $\mathcal{U}$, a C^1-open neighbourhood of f, such that any $g \in \mathcal{U}$ is accessible.

Conjecture 1.5 (Accessibility implies ergodicity). *Essential accessibility implies ergodicity among C^2 volume preserving partially hyperbolic diffeomorphisms.*

Recall that *essential accessibility* is a weakening of the notion of accessibility: it means that for any two measurable sets A, B of positive volume, there exist $a \in A$ and $b \in B$ which can be connected by some *su*-path.

Conjecture 1.6 (Density of accessibility). *For any integer $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, stable accessibility is open and dense among C^r partially hyperbolic diffeomorphisms, volume preserving or not.*

The state of the art on Conjecture 1.5 is the result of K. Burns and A. Wilkinson [14]. They proved Conjecture 1.5 under one mild technical condition called *center bunching*, which asserts, in rather loose terms, that the hyperbolic part dominates nonconformality of the center. Their result improved earlier work of Pugh-Shub in [33], which required two technical conditions: *dynamical coherence* and a stronger form of center bunching. Dynamical coherence is a very commonly used notion, which asserts certain joint integrability of the invariant subspaces E^c, E^s, E^u. We will give the formal definitions in Definitions 1.8-1.9.

[1] The C^1 version of Conjecture 1.3 is proved in [5].

In comparison, there is a paucity of progress towards Conjecture 1.6. When the center dimension is one, Conjecture 1.6 was proved by F. Rodriguez-Hertz, M.A. Rodriguez-Hertz and R. Ures in [38]. It is still open for any $\dim E^c > 1$. To describe the current state of Conjecture 1.6, we mention several related results, which were obtained among certain classes of systems.

- K. Burns and A. Wilkinson [15] proved a version of Conjecture 1.6 for compact group extensions over Anosov systems.
- In a recent paper [26], V. Horita and M. Sambarino obtained some C^r-density result for a class of partially hyperbolic systems with $\dim E^c = 2$ and *uniformly compact* center foliations (see Definition 1.10).
- Another C^r-density result for partially hyperbolic systems with $\dim E^c = 2$ was obtained recently by A. Avila and M. Viana [6] using a very different method.
- Z. Zhang [43] recently proved C^r-density of C^2-stable ergodicity for a class of skew products over Anosov maps, satisfying pinching, bunching conditions with certain type of dominated splitting in the center subspace.

The difficulty of Conjecture 1.6 is mainly due to the C^2-smallness of the perturbation. In fact, the C^1-density of stable accessibility was already proved by D. Dolgopyat and A. Wilkinson [20] in 2003. There was a line of research focused on the C^1 version of Conjecture 1.3. In the case where $\dim E^c = 1, 2$, this was proved in [11] and [37]. Recently, the C^1-version of Conjecture 1.3 was completely solved by A. Avila, S. Crovisier and A. Wilkinson [5].[2]

As the main result of this paper, we will verify C^r-density of stable ergodicity in C^1-neighbourhoods of two classes of partially hyperbolic systems, defined by some technical conditions. Let us first recall some notions needed to state our result.

The following definition is commonly-used in the study of partially hyperbolic systems (see [32, 33, 35, 36]).

Definition 1.7 (Pinching). An Anosov diffeomorphism f with constants $\bar{\chi}^s, \bar{\chi}^u$ as in (1.1), (1.2) is called *θ-pinched* for some $\theta \in (0,1)$ if there exist $\hat{\chi}^u, \hat{\chi}^s > 0$ s.t.

$$e^{-\hat{\chi}^s} < \|Df^{-1}\|^{-1} \leq \|Df\| < e^{\hat{\chi}^u},$$
$$-\bar{\chi}^s + \theta\hat{\chi}^u < 0, \quad \bar{\chi}^u - \theta\hat{\chi}^s > 0.$$

A partially hyperbolic system f with functions $\bar{\chi}^s, \bar{\chi}^u, \bar{\chi}^c, \hat{\chi}^c$ as in (1.3)–(1.6) is called *θ-pinched* for some $\theta \in (0,1)$ if there exist $\hat{\chi}^u, \hat{\chi}^s > 0$ such that

$$e^{-\hat{\chi}^s} < \|Df^{-1}\|^{-1} \leq \|Df\| < e^{\hat{\chi}^u}, \tag{1.7}$$
$$\theta\hat{\chi}^u < \bar{\chi}^s + \bar{\chi}^c, \quad \theta\hat{\chi}^s < \bar{\chi}^u - \hat{\chi}^c.$$

By definition, given any Anosov or partially hyperbolic diffeomorphism f, there exists $\theta \in (0,1)$ such that f is θ-pinched.

A related notion is the following.

Definition 1.8 (Center bunching). A partially hyperbolic system f with functions $\bar{\chi}^s, \bar{\chi}^u, \bar{\chi}^c, \hat{\chi}^c$ as in (1.3)–(1.6) is called *center bunched* (see [14]) if

$$\hat{\chi}^c < \bar{\chi}^s + \bar{\chi}^c, \qquad -\bar{\chi}^c < \bar{\chi}^u - \hat{\chi}^c. \tag{1.8}$$

[2]In [5] the authors showed that stable ergodicity is C^1-dense among C^1 volume preserving partially hyperbolic diffeomorphisms. If Conjecture 1.3 is true, then such statement would be an immediate corollary by [3].

DEFINITION 1.9 (Dynamical coherence, plaque expansiveness). We say that a partially hyperbolic system f is:

- *dynamically coherent* (see [25]) if E_f^{cs}, resp. E_f^{cu}, integrates to a f-invariant foliation $\mathcal{W}_f^{cs}$, resp. $\mathcal{W}_f^{cu}$, called the *center-stable foliation*, resp. the *center-unstable foliation*. In this case, for any $x \in X$, we let $\mathcal{W}_f^c(x) := \mathcal{W}_f^{cs}(x) \cap \mathcal{W}_f^{cu}(x)$; the collection of all such leaves forms a foliation $\mathcal{W}_f^c$, called the *center foliation*, which integrates E_f^c, and subfoliates both $\mathcal{W}_f^{cs}$ and $\mathcal{W}_f^{cu}$ (see [16]);
- *plaque expansive* (see [25, Section 7]) if f is dynamically coherent and there exists $\varepsilon > 0$ with the following property: if $(p_n)_{n \geq 0}$ and $(q_n)_{n \geq 0}$ are ε-pseudo orbits which respect $\mathcal{W}_f^c$ and if $d(p_n, q_n) \leq \varepsilon$ for all $n \geq 0$, then $q_n \in \mathcal{W}_f^c(p_n)$. It is known that plaque expansiveness is a C^1-open condition (see Theorem 7.4 in [25]).

DEFINITION 1.10 (Uniformly compact foliation). A foliation is *uniformly compact* if all the leaves are compact, and the leaf volume of the leaves is uniformly bounded.

We can now state our main result.

THEOREM A. *Let X be a compact smooth Riemannian manifold. Let $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, and assume that $f \in \mathrm{Diff}^r(X)$ is a dynamically coherent, center bunched partially hyperbolic diffeomorphism. Let $c := \dim E_f^c$. If either $c = 1$*[3] *and f is plaque expansive, or $c > 1$ and f satisfies at least one of the following assertions:*

- *f is $(\frac{c-1}{c})^{\frac{1}{7}}$-pinched and has uniformly compact center foliation;*
- *f is $(\frac{c-1}{c})^{\frac{1}{9}}$-pinched and the maps $x \mapsto E_f^{cs}(x), E_f^{cu}(x)$ are of class C^1,*

then there exists a C^1-open neighbourhood of f in $\mathrm{Diff}^r(X)$, denoted by $\mathcal{U}$, such that C^1-stable accessibility is prevalent in the $C^r - J-$Kolmogorov sense in $\mathcal{U}$, for any $J \geq J_0$ (see Subsection 3.1 below for a precise definition). Here J_0 is an integer depending only on $\dim X$.

Moreover, let Vol be a smooth volume form on X, and assume that $f \in \mathrm{Diff}^r(X, \mathrm{Vol})$ satisfies one of the previous conditions. Then the above conclusion is true for $\mathcal{U}_0$, a C^1-open neighbourhood of f in $\mathrm{Diff}^r(X, \mathrm{Vol})$, in place of $\mathcal{U}$. In particular, C^1-stable ergodicity is C^r-dense in $\mathcal{U}_0$.

Theorem A generalizes all the results on stable ergodicity from [15, 26, 6, 43] to arbitrary center dimension in the strongly pinched region. Compared to the previous works, our result has two significant novelties:

1. this is the first time that C^r-density of stable ergodicity is proved for fully nonlinear systems with arbitrary center dimension, for $r \geq 2$[4];
2. this is the first result that shows that stable ergodicity and accessibility are prevalent in the measure-theoretical sense.

To provide motivations for (2), let us mention that there are two ways to approach the question of genericity: topological and metric. These notions are sometimes

[3]Any partially hyperbolic diffeomorphism with center dimension 1 is automatically center bunched.

[4]Among the set of compact group extensions, C^r-density of stable ergodicity was proved in [15]. These systems also have higher center dimension, but they are simpler as the action on the fiber is by group translation, hence is characterised by finitely many parameters.

conflicting[5]. The study of prevalence properties goes back to Kolmogorov [29]. We say that a property is prevalent if it holds for a *typical dynamical system in Kolmogorov's sense* (see Definition 3.3) as in [7, 29, 34] (see [27, 30] for different notions). Even when $\dim E^c_f = 1$, we have strengthened the result of [38] as we show that stable ergodicity is not only C^r-dense but also prevalent among center bunched partially hyperbolic diffeomorphisms with one-dimensional center, assuming plaque expansiveness. In [34, Conjecture 3], the authors conjectured that: for the generic finite dimensional submanifold V in $\mathrm{Diff}^r(M)$ and almost every $f \in V$, the equivalence classes of points in the chain recurrent set of f are open in the chain recurrent set. They also mentioned that the validity of such conjecture would give a "finite spectral decomposition for f where each piece of the decomposition has something akin to the accessibility property". P. Berger [7] constructed a counter-example to this conjecture in the Newhouse domain. Our result strongly suggests that the accessibility property could be prevalent among partially hyperbolic diffeomorphisms.

1.2. **Further illustrations of our result.** Conjecture 1.3 has its origin in several concrete models. For instance, given an integer $n \geq 1$, the linear automorphism of $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ associated to a matrix $A \in \mathrm{SL}(n,\mathbb{Z})$ is defined as the unique diffeomorphism $f_A\colon \mathbb{T}^n \to \mathbb{T}^n$ such that the following diagram commutes, where $\pi\colon \mathbb{R}^n \to \mathbb{T}^n$ denotes the natural projection:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^n \\ \pi\downarrow & & \downarrow\pi \\ \mathbb{T}^n & \xrightarrow{f_A} & \mathbb{T}^n \end{array}$$

Back in the 1970's, Hirsch-Pugh-Shub [25] already asked whether any ergodic linear automorphism of $\mathbb{T}^n$ stably ergodic, for $n \geq 2$.

Positive answer to this question is known when the map is Anosov by [2]. This question was solved by F. Rodriguez-Hertz in [39] for any $n \leq 5$. This in particular answered a special case of the question, asked in [22] for an explicit 4×4 matrix. More generally, in [39], the author investigated the case of pseudo-Anosov maps with 2 eigenvalues of modulus 1. In [32], the authors mentioned that the validity of Conjecture 1.3 would give a positive answer to the following weaker version of the previous question[6], namely, given two integers $n, r \geq 2$, whether the C^r-generic volume preserving perturbation of any ergodic automorphism of $\mathbb{T}^n$ ergodic. To the best of our knowledge, the question remains open for any $\dim E^c > 1$.

Let us now consider $M = \mathrm{SL}(n,\mathbb{R})/\Gamma$ for some uniform discrete subgroup Γ of $\mathrm{SL}(n,\mathbb{R})$. Let $A \in \mathrm{SL}(n,\mathbb{R})$ with at least one eigenvalue of modulus different from 1, and let $L_A\colon M \to M$ be the left translation by A. In [32], the authors ask whether L_A is stably ergodic among C^2 volume preserving diffeomorphisms of M.

Unlike the case of $\mathbb{T}^n$, the topological complexity of the homogeneous manifold M has so far prevented a generalization of [39]. One can also consider the generic version of the previous question, namely, whether the C^∞ generic volume preserving perturbation of L_A is stably ergodic. This is true when the map L_A is Anosov, by

[5]For instance, among circle diffeomorphisms, a topological generic map has rational rotation number, while those with irrational rotation number occupy positive measures in many one-parameter families.

[6]See the remark below [32, Conjecture 1].

[2], or when the center dimension $\dim E^c$ is equal to one, but the question remains open for any $\dim E^c > 1$.

As corollaries of the main result of this paper, we answer the previous questions in any dimension in the strongly pinched region, namely for maps with pinching exponents close to 1.

Theorem B. *Let $n \geq 2$ be some integer. For any $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, any linear partially hyperbolic automorphism $f_A : \mathbb{T}^n \to \mathbb{T}^n$, ergodic or not, that is $(\frac{c-1}{c})^{\frac{1}{5}}$-pinched, where c is the number of eigenvalues of $A \in \mathrm{SL}(n, \mathbb{Z})$ of modulus 1, there exists $\mathcal{U}$, a C^1-open neighbourhood of f_A in $\mathcal{PH}^r(\mathbb{T}^n, \mathrm{Vol})$, such that for some C^r-dense subset $\mathcal{U}'$ of $\mathcal{U}$, any map in $\mathcal{U}'$ is a stably ergodic diffeomorphism.*

Theorem C. *Let Γ be a uniform discrete subgroup of $\mathrm{SL}(n, \mathbb{R})$, let $M := \mathrm{SL}(n, \mathbb{R})/\Gamma$ and let $L_A : M \to M$ be the left translation by $A \in \mathrm{SL}(n, \mathbb{R})$, assuming that A has an eigenvalue with modulus different from 1. Then, the C^∞ generic volume preserving perturbation of L_A is stably ergodic for any θ-pinched L_A, where $\theta \in (0, 1)$ depends only on the integer n.*

The partially hyperbolic splittings for Theorems B and C are the canonical ones: the center spaces are the neutral subspaces of the affine actions. Note that even for linear automorphisms with two-dimensional center, Theorem B partially improves and generalizes the main result in [39], in the following sense: (1) we removed the pseudo-Anosov condition; (2) our result also applies to non-ergodic maps; (3) we weakened the regularity condition (for the perturbations) from C^5 to C^1.

Another open question in [32] is the following.[7] Given two compact Riemannian manifolds M, N, where M supports a C^r volume preserving Anosov diffeomorphism $g : M \to M$, $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, is the C^r-generic volume preserving perturbation of $g \times \mathrm{Id} : M \times N \to M \times N$ ergodic? Again, this question has a positive answer when the map in question satisfies $\dim N = 1$. A recent result of A. Avila and M. Viana [6] also gives an affirmative answer to this question when $\dim N = 2$ (see also [26] for a related result). This question remains open for any N of dimension at least 3. As a corollary of our main result, we can answer this question in any dimension in the strongly pinched region.

Theorem D. *Let M, N be two compact Riemannian manifolds, where M supports a C^r volume preserving Anosov diffeomorphism $g : M \to M$, $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, and let $h : M \to \mathrm{Isom}(M)$ be a C^r map. Assume that g is $\left(\frac{n-1}{n}\right)^{\frac{1}{7}}$-pinched where $n := \dim N$. Then a C^r-generic volume preserving perturbation of the map $(x, y) \mapsto (g(x), h(x, y))$ is stably ergodic.*

Theorems B, C and D are immediate consequences of a more general result, Theorem E, stated in Section 2.

1.3. **Idea of the proof.** We follow closely the method in [43]. In [43], the author studied a class of skew products over Anosov systems, and divided the problem into: I. showing that the property of having a stably open accessibility class is C^r-generic; II. showing that the property of having an open accessibility class with intermediate volume is C^r-meager. In Step I, we prove the existence of open accessibility classes using a quantitative version of a theorem of M. Bonk and B. Kleiner in [10] (the details about this quantitative statement can be found in Section 7).

[7] Actually, the version we give here is stronger than that initially stated in [32].

In our setting, this boils down to destroying common intersections between many different holonomy loops. This is done by a parameter exclusion within a family of random perturbations.

The main new observation in this paper (*à la* Avila) is that: by letting the number of loops be sufficiently large compared to the dimension of the manifold, we are left with enough room to create open accessibility class at every point, due to the fact that the random perturbations we use are not very sensitive to the map and the point. The details about this argument can be found in Section 10. This allows us to bypass Step II in [43] which only works under restrictive assumptions (for a different application of the method in Step II, see [44]). Our construction is also suitable to study the measure-theoretical prevalence. Our method suggests that under a strong pinching condition, the failure of accessibility should be a phenomenon of *infinite codimension*.

On the technical level, in Section 6, we use [20] to construct families of center disks to connect different regions of the space. Small complications arise in the study of prevalence, since we need to organize several families for different maps. For each disk we consider a parametrized family of random perturbations generated by vector fields with disjoint supports, and parametrize a part of the accessibility class of any given point x in a slightly smaller disk by $[0,1]^c$, where c is the center dimension. We then apply Bonk-Kleiner's criterion to show the openness of the accessibility class of x for most parameters in the family. The regularity results in [35, 36] are used to reduce the problem to a finite set of points and loops.

Convention. Given a compact smooth Riemannian manifold X with a smooth volume form Vol, for any $r \in \mathbb{N}_{\geq 1} \cup \{\infty\}$, we denote by $\mathcal{PH}^r(X)$ (resp. $\mathcal{PH}^r(X, \mathrm{Vol})$) the set of all C^r (resp. C^r volume preserving) partially hyperbolic diffeomorphisms on X with bounded C^r norms.

In the course of the paper, we will often use constants depending on a diffeomorphism f (and that may or may not depend on other things). We say that a constant C depending on a C^r diffeomorphism f is C^r*-uniform* if it works for all diffeomorphisms in a C^r-open neighbourhood of f. We introduce several constants related to a diffeomorphism in Notations 3.12, 5.2 and Construction 9.1.

Given $l \geq 0$ and diffeomorphisms $f_1, f_2, \ldots, f_l$, we use the notation $\prod_{i=1}^{l} f_i$ to denote $f_l \circ \cdots \circ f_1$, where by convention $\prod_{i=j+1}^{j} f_i := \mathrm{Id}$ for any $j = 1, \ldots, l-1$.

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2. Main result

In the following, we fix a smooth Riemannian manifold X with a smooth volume form Vol. Let $f : X \to X$ be a partially hyperbolic diffeomorphism with functions

$\bar{\chi}^s, \bar{\chi}^u, \bar{\chi}^c, \hat{\chi}^c$ as in (1.3)–(1.6). For any real number $\varrho \in \mathbb{R}$, we set

$$\beta(f,\varrho) := \min\left(\frac{\bar{\chi}^s + \bar{\chi}^c}{\hat{\chi}^u}, \frac{\bar{\chi}^u - \hat{\chi}^c}{\hat{\chi}^s}\right) \min\left(\frac{\bar{\chi}^s}{\hat{\chi}^s}, \frac{\bar{\chi}^u}{\hat{\chi}^u}\right)^{\varrho}. \tag{2.1}$$

We will focus on the case where f is dynamically coherent (recall Definition 1.9) and satisfies one of the following properties:

(a) the center foliation $\mathcal{W}_f^c$ is uniformly compact (see Definition 1.10);
(b) the maps $x \mapsto E_f^{cs}(x), E_f^{cu}(x)$ are of class C^1.

Let us state the most general version of our result, which contains Theorem A.

Theorem E. *Let $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, let $f \in \mathcal{PH}^r(X)$ be dynamically coherent and center bunched, and let $c := \dim E_f^c$. If either $c = 1$ and f is plaque expansive, or $c > 1$ and f satisfies one of the following conditions:*

(1) condition (a) holds for f, and $\beta(f,3) > \frac{c-1}{c}$;
(2) condition (b) holds for f, and $\beta(f,4) > \frac{c-1}{c}$,

then there exist $\mathcal{U}$, a C^1-open neighbourhood of f in $\mathrm{Diff}^r(X)$, and an integer J_0 only depending on $\dim X$, such that for any $J \geq J_0$, C^1-stable accessibility is prevalent in $\mathcal{U}$ in the $C^r - J-$Kolmogorov sense.

Moreover, let $f \in \mathrm{Diff}^r(X, \mathrm{Vol})$ satisfy the above condition. Then the above conclusion is true for $\mathcal{U}_0$, a C^1-open neighbourhood of f in $\mathrm{Diff}^r(X, \mathrm{Vol})$, in place of $\mathcal{U}$. In particular, C^1-stable ergodicity is C^r-dense in $\mathcal{U}_0$.

Remark 2.1. *If f is θ-pinched for some $\theta \in (0,1)$ and satisfies (a), resp. (b), then we can see that $\beta(f,3) > \theta^7$, resp. $\beta(f,4) > \theta^9$. Therefore, Theorem A is a consequence of Theorem E. Similarly, Theorems B, C and D follow by making appropriate choice of functions $\bar{\chi}^s, \bar{\chi}^u, \bar{\chi}^c, \hat{\chi}^c$. We omit these straightforward computations.*

3. Preliminaries

In the following section, we recall some general notions about parameter families, prevalence and partially hyperbolic diffeomorphisms.

3.1. **Prevalence.** Let X be a smooth Riemannian manifold with a smooth volume form Vol. We refer the reader to [21, Chapter II, §3] and [23] for more details about the notions recalled in the following.

Definition 3.1 (C^r topology). Let $m, n \geq 1$ be integers. Given $k \in \mathbb{N}$ and $f \in C^k(\mathbb{R}^m, \mathbb{R}^n)$, we set $\|f\|_{C^k} := \sup_{0 \leq i \leq k,\ x \in \mathbb{R}^m} \|\partial_x^i f\|$. Here, $\partial_x^i f$ is a multi-linear map from $(T_x\mathbb{R}^m)^i$ to $T_{f(x)}\mathbb{R}^n$, and $\|\partial_x^i f\|$ denotes the norm of this linear map. We let d_{C^k} be the distance induced by $\|\cdot\|_{C^k}$. Given $f, g \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$, we set $d_{C^\infty}(f,g) := \sum_{k=0}^\infty 2^{-k} \frac{\|f-g\|_{C^k}}{\|f-g\|_{C^k}+1}$. For $r \in \mathbb{N} \cup \{\infty\}$, the C^r topology on $C^r(\mathbb{R}^m, \mathbb{R}^n)$ is the topology induced by d_{C^r}. Given smooth Riemannian manifolds M, N, we define accordingly the C^r topology for maps between M and N, and we denote by $C_b^r(M,N)$ the subset of maps in $C^r(M,N)$ with bounded distance to the constant maps.

Definition 3.2 (Parameter family). Given integers $r, m, n, J \geq 1$, we define the space $C^r([0,1]^J, C_b^r(\mathbb{R}^m, \mathbb{R}^n))$ as the set of families $\{f_\omega\}_{\omega \in [0,1]^J}$ of maps $f_\omega \in$

$C^r(\mathbb{R}^m,\mathbb{R}^n)$ such that for every $0 \le i,j \le r$ the derivative $\partial_\omega^i \partial_x^j f_\omega(x)$ is a multilinear map which depends continuously on $(\omega,x) \in [0,1]^J \times \mathbb{R}^m$. The C^r topology on the space $C^r([0,1]^J, C_b^r(\mathbb{R}^m,\mathbb{R}^n))$ is the topology induced by the norm $\|\cdot\|_{C^r}$:

$$\|\{f_\omega\}_\omega\|_{C^r} := \sup_{\substack{0\le i,j\le r,\\ (\omega,x)\in[0,1]^J\times\mathbb{R}^m}} \|\partial_\omega^i \partial_x^j f_\omega(x)\|.$$

The C^∞ topology on $C^\infty([0,1]^J, C_b^\infty(\mathbb{R}^m,\mathbb{R}^n))$ is defined by analogy.

Let $r \in \mathbb{N}\cup\{\infty\}$. Given smooth Riemannian manifolds M, N and $\mathcal{U} \subset C_b^r(M,N)$, a *$C^r - J-$family in $\mathcal{U}$* is an element $\{f_\omega\}_\omega \in C^r([0,1]^J,\mathcal{U})$. We define the C^r topology on $C^r([0,1]^J,\mathcal{U})$ analogously and denote by d_{C^r} the associated metric.

Definition 3.3 (Prevalence). Let $\mathcal{U}$ be an open subset in $\mathrm{Diff}^r(X)$ or $\mathrm{Diff}^r(X,\mathrm{Vol})$, $r \in \mathbb{N}\cup\{\infty\}$. A property $\mathscr{P}$ for maps in $\mathrm{Diff}^r(X)$ or $\mathrm{Diff}^r(X,\mathrm{Vol})$ is said to be *prevalent in the $C^r - J-$Kolmogorov sense* in $\mathcal{U}$, if for a C^r-generic $C^r - J-$family $\{f_\omega\}_\omega$ in $\mathcal{U}$ and for almost every $\omega \in [0,1]^J$, f_ω satisfies $\mathscr{P}$.

We introduce the following notion for technical reasons.

Definition 3.4 (Good family). Let $r \in \mathbb{N}\cup\{\infty\}$. Given an integer $J \ge 1$, a $C^r - J-$family $\{f_\omega\}_\omega \in C^r([0,1]^J, \mathrm{Diff}^r(X))$ is said to be *good* if the fixed points of f_ω^k are isolated for all integer $k \ge 1$ and almost every $\omega \in [0,1]^J$.

Proposition 3.5. *There exists $J_0 = J_0(\dim X) > 0$, which we fix in the rest of this paper, with the following property. For any $r \in \mathbb{N}_{\ge 2}\cup\{\infty\}$ and any integer $J \ge J_0$, the set of good $C^r - J-$families is dense in the set of $C^r - J-$families with respect to the C^r topology.*

Proof. This is essentially contained in the proof of [28, Theorem 2.2]. In [28], the author showed the prevalence of Kupka-Smale diffeomorphisms in $\mathrm{Diff}^r(X)$. In contrast to the Kupka-Smale property, our notion of good family is only a transversality condition on the level of 0-jets. Next lemma suffices for our purpose.

lemma 3.6. *For any integers $p,q \ge 3$, any $r \in \mathbb{N}_{\ge 2}\cup\{\infty\}$, for any C^r map $f\colon (-1,1)^p \to \mathbb{R}^q$, there exist an integer $L \ge 1$ and C^∞ divergence-free vector fields $V_1,\dots,V_L$ on $\mathbb{R}^q$, supported in $(-1,1)^q$, such that the following is true. Let us denote by $\mathscr{F}_{V_i}^{b_i}\colon \mathbb{R}^q \to \mathbb{R}^q$ the time-b_i map of the flow generated by V_i, and let $F\colon (-1,1)^p \times (-1,1)^L \to \mathbb{R}^q$ be defined by $F\colon (x,b) \mapsto \mathscr{F}_{V_L}^{b_L}\circ\cdots\circ\mathscr{F}_{V_1}^{b_1}(f(x))$, for all $b = (b_1,\dots,b_L)$. Then the map*

$$\mathscr{G}\colon (-1,1)^{p+L} \to \mathbb{R}^p\times\mathbb{R}^q, \quad (x,b)\mapsto (x,F(x,b))$$

is a submersion for any (x,b) such that $F(x,b) \in (-\frac{1}{2},\frac{1}{2})^q$.

Lemma 3.6 is proved via a direct construction. Using Lemma 3.6 in place of [28, Lemma 1.5], the proof of Proposition 3.5 proceeds as that of [28, Theorem 2.2]. □

3.2. **Partially hyperbolic diffeomorphisms.** Fix an integer $d \ge 1$. We let X be a smooth d-dimensional Riemannian manifold with a smooth volume form Vol.

Let $f\colon X \to X$ be a partially hyperbolic diffeomorphism or an Anosov map.

In the following, we will call a leaf of $\mathcal{W}_f^c, \mathcal{W}_f^{cu}$, etc. a center leaf, center-unstable leaf, etc.

Definition 3.7 (Holonomies). Let $f \in \mathcal{PH}^1(X)$ be dynamically coherent.

- Let $x_1 \in X$ and $x_2 \in \mathcal{W}_f^s(x_1)$. By transversality, for $i = 1, 2$, there exists a neighbourhood $\mathcal{C}_i$ of x_i within $\mathcal{W}^{cu}_{f,\mathrm{loc}}(x_i)$ such that for any $x \in \mathcal{C}_1$, the local s-leaf through x intersects $\mathcal{C}_2$ at a unique point, denoted by $H^s_{f,x_1,x_2}(x) = H^s_{f,\mathcal{C}_1,\mathcal{C}_2}(x)$. We thus get a well-defined local homeomorphic embedding $H^s_{f,\mathcal{C}_1,\mathcal{C}_2}\colon \mathcal{C}_1 \to \mathcal{C}_2$, called the (local) *stable holonomy map* between $\mathcal{C}_1$ and $\mathcal{C}_2$. For $i = 1, 2$, set $\tilde{\mathcal{C}}_i := \mathcal{C}_i \cap \mathcal{W}^c_{f,\mathrm{loc}}(x_i)$; by restriction, $H^s_{f,\mathcal{C}_1,\mathcal{C}_2}$ induces a local homeomorphism $H^s_{f,\tilde{\mathcal{C}}_1,\tilde{\mathcal{C}}_2}\colon \tilde{\mathcal{C}}_1 \to \tilde{\mathcal{C}}_2$. Unstable holonomies are defined accordingly.
- Let $x_1 \in X$ and $x_2 \in \mathcal{W}^c_f(x_1)$ be two sufficiently close points in the same center leaf. Let $* \in \{u, s\}$. Then, the (local) *center holonomy map* H^c_{f,x_1,x_2} along local leaves in $\mathcal{W}^c_f$ is a well-defined local homeomorphism from a neighbourhood of x_1 in $\mathcal{W}^*_f$ to $\mathcal{W}^*_f(x_2)$.

The following result [35, Theorem A] relates the pinching condition in Definition 1.7 with the regularity of u, s-holonomy maps.

Proposition 3.8. *If $f \in \mathcal{PH}^1(X)$ is θ-pinched for some $\theta \in (0, 1)$, then the local unstable and stable holonomy maps are uniformly θ-Hölder.*

The following result is contained in the proof of [35, Theorem B]. It relates the center bunching condition in Definition 1.8 to the regularity of u, s-holonomies.

Proposition 3.9. *If $f \in \mathcal{PH}^2(X)$ is dynamically coherent and center bunched, then local stable/unstable holonomy maps between center leaves are C^1 when restricted to some center-stable/center-unstable leaf and have uniformly continuous derivatives.*

In Proposition 3.9, uniformity of the continuity is a simple consequence of the invariant section theorem and the uniform C^2 bound.

3.3. **On leaf conjugacy.** Later on, we will focus on dynamically coherent systems satisfying one of the conditions (a) or (b) in Section 2. The following result is due to Hirsch-Pugh-Shub.

Proposition 3.10 (Theorem 7.1, [25], see also Theorem 1 in [36]). *Let f be a dynamically coherent partially hyperbolic diffeomorphism satisfying (a) or (b). Then any $g \in \mathcal{PH}^1(X)$ which is sufficiently C^1-close to f is also dynamically coherent. Moreover, there exists a homeomorphism $\mathfrak{h} = \mathfrak{h}_g\colon X \to X$, called a leaf conjugacy, such that: (1) $\mathfrak{h}$ maps a f-center leaf to a g-center leaf; (2) both h and h^{-1} tend to* Id *in the uniform norm as $d_{C^1}(f, g)$ tends to* 0.

Proof. It suffices to see that any f in the proposition is plaque expansive (recall Definition 1.9). The plaque expansiveness is proved in [17] (see also [36, Proposition 13]) under (a), respectively in [25] under (b). □

The following result, due to Pugh-Shub-Wilkinson [36], ensures that the leaf conjugacy $\mathfrak{h}$ in Theorem 3.10 has Hölder regularity under (a) or (b).

Proposition 3.11 (Theorems A-B, [36]). *Let $f \in \mathcal{PH}^1(X)$ be dynamically coherent, satisfying (a) (resp. (b)), let $\bar{\chi}^s, \bar{\chi}^u, \hat{\chi}^s, \hat{\chi}^u$ be as in* (1.3)–(1.7)*, and let θ be a constant such that*

$$0 < \theta < \min\left(\frac{\bar{\chi}^s}{\hat{\chi}^s}, \frac{\bar{\chi}^u}{\hat{\chi}^u}\right) \leq 1. \tag{3.1}$$

Then for some C^1-open neighbourhood $\mathcal{U}_0 = \mathcal{U}_0(f,\theta)$ of f, for any $g \in \mathcal{U}_0$, the leaf conjugacy $\mathfrak{h}_g$ in Proposition 3.10 exists and can be made uniformly θ-Hölder, and local center holonomies between sufficiently close leaves are uniformly θ-Hölder (resp. θ^2-Hölder).

We will later use Propositions 3.8, 3.9 and 3.11 while keeping track of the uniformity of various quantities. We summarise these statements as follows.

Notation 3.12. Let X be a d-dimensional compact Riemannian manifold ($d \geq 3$) with metric $d(\cdot,\cdot)$, and $f \in \mathcal{PH}^1(X)$ be dynamically coherent and plaque expansive. We denote by $d_{\mathcal{W}_f^s}, d_{\mathcal{W}_f^u}, d_{\mathcal{W}_f^c}, d_{\mathcal{W}_f^{cs}}, d_{\mathcal{W}_f^{cu}}$ the associated leafwise distances. For any $x \in X$, $\sigma > 0$, and $* = s,u,c,cs,cu$, we set $\mathcal{W}_f^*(x,\sigma) := \{y \in \mathcal{W}_f^*(x) \mid d_{\mathcal{W}_f^*}(x,y) < \sigma\}$. Then there exist C^1-uniform constants $h_f > 0$, $\sigma_f \in (0,h_f)$, $C_f > 1$, $\overline{\Lambda}_f > 1$, $\varepsilon_f > 0$, $\theta_f', \theta_f'' \in (0,1)$, and in case f is C^2, also a C^2-uniform constant $\Lambda_f > 0$ satisfying

(1) For $* = c,s,u,cs,cu$, for any $x \in X$, $y \in \mathcal{W}_f^*(x,\sigma_f)$, we have
$$d(x,y) \leq d_{\mathcal{W}_f^*}(x,y) \leq C_f d(x,y).$$
(2) For any $x \in X$, any $y \in B(x,\sigma_f)$, $\mathcal{W}_f^s(x,h_f)$ transversally intersects $\mathcal{W}_f^{cu}(y,h_f)$ at a unique point z, and $d_{\mathcal{W}_f^s}(x,z), d_{\mathcal{W}_f^{cu}}(z,y) < C_f d(x,y)$. If in addition, $y \in \mathcal{W}^{cs}(x,\sigma_f)$, then $d_{\mathcal{W}_f^s}(x,z), d_{\mathcal{W}_f^c}(z,y) < C_f d_{\mathcal{W}_f^{cs}}(x,y)$.
(3) (Center bunching) If $f \in \mathcal{PH}^2(X)$ is center bunched, then for any $x \in X$ and $y \in \mathcal{W}_f^{cs}(x,\frac{\sigma_f}{2})$, the holonomy map $H^s_{f,x,y}\colon \mathcal{W}_f^c(x,\frac{\sigma_f}{2}) \to \mathcal{W}_f^c(y,C_f\sigma_f)$ is well-defined. Moreover, $DH^s_{f,x,y}$ is uniformly continuous and has norm bounded by Λ_f.
(4) (Pinching) f is θ_f'-pinched. Besides, if $f \in \mathcal{PH}^2(X)$, then for any $x \in X$, $y \in \mathcal{W}_f^s(x,\frac{\sigma_f}{2})$, the holonomy map $H^s_{f,x,y}\colon \mathcal{W}_f^{cu}(x,\frac{\sigma_f}{2}) \to \mathcal{W}_f^{cu}(y,1)$ is well-defined and has θ_f'-Hölder norm bounded by Λ_f.
(5) (Hölderness of $\mathfrak{h}$ and H^c) If f satisfies (a), resp. (b), then θ_f'' satisfies (3.1) in place of θ, and the following is true:
   (5.1) any $g \in \mathcal{PH}^1(X)$ with $d_{C^1}(f,g) < \varepsilon_f$ is plaque expansive and for any $x \in X$, any $y \in \mathcal{W}_g^c(x,\frac{\sigma_f}{2})$, the holonomy map $H^c_{g,x,y}$ is defined on $\mathcal{W}_g^*(x,\frac{\sigma_f}{2})$, $* = u,s$, and its θ_f''- (resp. its $(\theta_f'')^2$-) Hölder norm is bounded by $\overline{\Lambda}_f$.
   (5.2) For any $g_1,g_2 \in \mathcal{PH}^1(X)$ such that $d_{C^1}(f,g_i) < \varepsilon_f$, $i=1,2$, the leaf conjugacy $\mathfrak{h}_{g_1,g_2} = \mathfrak{h}_{g_2} \circ \mathfrak{h}_{g_1}^{-1}$ has $(\theta_f'')^2$-Hölder norm bounded by $\overline{\Lambda}_f$. Here $\mathfrak{h}_{g_i}$ is given by Proposition 3.11 for g_i in place of g.

Moreover, we assume that Properties 2, 3, 4 above are also satisfied when we exchange the roles of u and s.

Definition 3.13. Let $f \in \mathcal{PH}^2(X)$ be dynamically coherent and center bunched. We say that f satisfies

$$(ae) \qquad \text{if } (a) \text{ holds and } \theta_f'(\theta_f'')^3 > \frac{c-1}{c}.$$

Similarly, we say that f satisfies

$$(be) \qquad \text{if } (b) \text{ holds and } \theta_f'(\theta_f'')^4 > \frac{c-1}{c}.$$

REMARK 3.14. *If Theorem E(1) holds for f, then by Definition 1.7, Propositions 3.8, 3.11, we can choose θ'_f, θ''_f such that* (ae) *holds. Similarly, if Theorem E(2) holds for f, then we can choose θ'_f, θ''_f such that* (be) *holds.*

Standing hypotheses for the rest of the paper

We denote by X a d-dimensional compact smooth Riemannian manifold with a smooth volume form Vol; r belongs to $\mathbb{N}_{\geq 1} \cup \{\infty\}$; and $f \in \mathrm{Diff}^r(X)$. Whenever f is declaimed to be partially hyperbolic, we denote $c := \dim E^c_f$, $d_s := \dim E^s_f$ and $d_u := \dim E^u_f$. Moreover:

(H1) $f \in \mathcal{PH}^r(X)$ is dynamically coherent and plaque expansive in Sections 4.2-4.4, 5, 8, 9;

(H2) $f \in \mathcal{PH}^r(X)$ is center bunched and $r \geq 2$, in Sections 4.4, 5.

4. Random perturbations

In this section, we will establish some estimates for certain perturbations of the holonomy maps of a dynamically coherent plaque expansive partially hyperbolic diffeomorphism.

4.1. **Basic notions and constructions.**

We start with the following more general situation. The following suspension construction will be used repeatedly.

DEFINITION 4.1 (C^r deformation). Let $a \in \mathbb{R}^I$ for some integer $I > 0$, and let U be an open neighbourhood of a in $\mathbb{R}^I$. A C^r map $\hat{f}\colon U \times X \to X$ satisfying $\hat{f}(a, \cdot) = f$ and $\hat{f}(b, \cdot) \in \mathrm{Diff}^r(X)$, $\forall\, b \in U$ is called a *C^r deformation at (a, f) with I-parameters.* We associate with such $\hat{f}$ the suspension map $T(\hat{f})$ defined by

$$T = T(\hat{f})\colon U \times X \to U \times X, \quad (b, x) \mapsto (b, \hat{f}(b, x)). \tag{4.1}$$

If in addition $\hat{f}(b, \cdot) \in \mathrm{Diff}^r(X, \mathrm{Vol})$ for all $b \in U$, then we say that $\hat{f}$ is *volume preserving.*

DEFINITION 4.2 (Infinitesimal C^r deformation). Given an integer $I > 0$, a C^r map $V\colon \mathbb{R}^I \times X \to TX$ is called an *infinitesimal C^r deformation with I-parameters* if

(1) for each $B \in \mathbb{R}^I$, $V(B, \cdot)$ is a C^r vector field on X;

(2) for each $x \in X$, $B \mapsto V(B, x)$ is a linear map from $\mathbb{R}^I$ to T_xX.

CONSTRUCTION 4.3. Given $I > 0$, $a \in \mathbb{R}^I$, and V, an infinitesimal C^r deformation with I-parameters, then for any sufficiently small $\epsilon > 0$, we associate with V a C^r deformation at (a, f) with I-parameters, denoted by $\hat{f}$, which is defined by

$$\hat{f}(b, x) := \mathscr{F}_{V(b-a,\cdot)}(1, f(x)), \quad \forall\, (b, x) \in U \times X,$$

where $U = B(a, \epsilon) \subset \mathbb{R}^I$ and for any $B \in \mathbb{R}^I$, $\mathscr{F}_{V(B,\cdot)}\colon \mathbb{R} \times X \to X$ denotes the C^r flow generated by the vector field $V(B, \cdot)$. In this case, we say that $\hat{f}$ is *generated by V*. If in addition for each $B \in \mathbb{R}^I$, $V(B, \cdot)$ is divergence-free, then $\hat{f}$ is volume preserving as in Definition 4.1, and we say that V is *volume preserving.*

For any V as in Definition 4.2, we use $\| \cdot \|_X$ to denote the uniform norm of the derivatives of V restricted to $\{0\} \times X$. The following lemma gives bounds on the norms of deformations induced by infinitesimal deformations.

LEMMA 4.4. *Assume that $r \geq 2$. Let $I \in \mathbb{N}_{\geq 1}$ and let $\hat{f}\colon U \times X \to X$ be a C^r deformation at $(0, f)$ generated by some infinitesimal C^r deformation with I-parameters V. Take $T = T(\hat{f})$ as in Definition 4.1. Then there exists a C^2-uniform constant $C_0 = C_0(f) > 0$, such that by possibly taking U smaller, it holds:*

(1) $\|DT\| < C_0(1 + \|\partial_b V\|_X)$ and $\|D^2 T\| < C_0(1 + \|\partial_b \partial_x V\|_X)(1 + \|\partial_b V\|_X)$;
(2) $\pi_X DT((0, x), B) = V(B, f(x))$ for any $(x, B) \in X \times T_0 U$. Here for each $v \in T(U \times X)$, we denote by $\pi_X(v)$ the component of v in TX.

Proof. We defer the proof to Appendix B. □

Some of the estimates will depend on the support of a deformation or of an infinitesimal deformation, which we now define.

DEFINITION 4.5. For an infinitesimal C^r deformation with I-parameters $V\colon \mathbb{R}^I \times X \to TX$, we define

$$\mathrm{supp}_X(V) := \{x \in X \mid \exists\, B \in \mathbb{R}^I \text{ such that } V(B, x) \neq 0\}.$$

Given $a \in \mathbb{R}^I$, an open neighbourhood U of a, and a C^r deformation at (a, f) with I-parameters $\hat{f}\colon U \times X \to X$, we define

$$\mathrm{supp}_X(\hat{f}) := \{x \in X \mid \exists\, b \in U \text{ such that } \hat{f}(b, x) \neq f(x)\}.$$

It is clear from Definitions 4.2 and 4.5 that for any infinitesimal C^r deformation V, if $\hat{f}$ is the C^r deformation of f generated by V, then we have

$$\mathrm{supp}_X(\hat{f}) \subset f^{-1}(\mathrm{supp}_X(V)). \tag{4.2}$$

4.2. **c-disk and c-family.** We first introduce the following notion.

DEFINITION 4.6 (Accessibility class). Let $f\colon X \to X$ be a partially hyperbolic diffeomorphism. For any $x \in X$, any $\ell > 0$ and any integer $k \geq 1$, we let $Acc_f(x, \ell, k)$ be the set of all points $y \in X$ that can be attained from x through a k-legged *accessibility sequence* $x = z_0, z_1, \dots, z_k = y$, where for each $0 \leq i \leq k-1$, $z_{i+1} \in \mathcal{W}^s_f(z_i, \ell) \cup \mathcal{W}^u_f(z_i, \ell)$. We let the *accessibility class* of f at x be $Acc_f(x) := \cup_{\ell > 0, k \geq 1} Acc_f(x, \ell, k)$.

For any $f \in \mathcal{PH}^1(X)$, accessibility classes of f form a partition of X. By Definition 1.4, f is accessible if and only this partition consists of a single class.

In the rest of Section 4, we assume f satisfies (H1).

DEFINITION 4.7 (c-disk). For each $x \in X$ and $\sigma > 0$, we call $\mathcal{C} = \mathcal{W}^c_f(x, \sigma)$ the *center disk* of f (or c-disk of f for short) centered at x with radius σ, and we set $\varrho(\mathcal{C}) := \sigma$. In addition, for any $\theta \in (0, 1]$, we also define $\theta\mathcal{C} := \mathcal{W}^c_f(x, \theta\sigma)$.

DEFINITION 4.8. A collection of disjoint center disks $\mathcal{D} = \{\mathcal{C}_1, \dots, \mathcal{C}_K\}$ is called a *family of center disks for f* (or *c-family for f* for short). In addition, we set

$$\underline{r}(\mathcal{D}) := \inf_{\mathcal{C} \in \mathcal{D}} \{\varrho(\mathcal{C})\}, \quad \overline{r}(\mathcal{D}) := \sup_{\mathcal{C} \in \mathcal{D}} \{\varrho(\mathcal{C})\}, \quad n(\mathcal{D}) := K.$$

Given $\theta \in (0, 1)$ and $k \in \mathbb{N}_{\geq 1}$, we say that $\mathcal{D}$ is a *(θ, k)-spanning* c-family for f if

$$X = \cup_{\mathcal{C} \in \mathcal{D}} \cup_{x \in \theta\mathcal{C}} Acc_f(x, 1, k).$$

Given any subset $\mathcal{C} \subset X$, and $\sigma \geq 0$, we set $(\mathcal{C}, \sigma) := \{x \in X \mid d(x, \mathcal{C}) \leq \sigma\}$. Given a collection $\mathcal{D} = \{\mathcal{C}_1, \dots, \mathcal{C}_K\}$ of subsets of X, we set

$$(\mathcal{D}, \sigma) := \cup_{j=1}^K (\mathcal{C}_j, \sigma), \qquad \forall\, \sigma \geq 0.$$

A collection $\mathcal{D}$ of subsets of X is called *σ-sparse* if for any two distinct $\mathcal{C}, \mathcal{C}' \in \mathcal{D}$, $(\mathcal{C}, \sigma), (\mathcal{C}', \sigma)$ are disjoint. Any c-family for f is σ-sparse for some $\sigma > 0$.

The next lemma is a consequence of the continuity of the invariant foliations with respect to the dynamics. Roughly speaking, it says that any c-family can be slightly perturbed into a c-family for a given nearby map.

LEMMA 4.9. *For any $\theta \in (0,1)$, $\theta' \in (\theta, 1)$, $\theta'' \in (0,1]$, $\rho_M > 0$, $\rho_m \in (0, \rho_M)$, and any $\sigma > 0$, there exists a C^1-open neighbourhood $\mathcal{U}$ of f in $\mathcal{PH}^1(X)$, and C^1-uniform constants $\epsilon > 0$ and $\sigma' > 0$, such that for any c-disk of f, denoted by $\mathcal{C} = \mathcal{W}^c_f(x, \rho)$, with $\rho \in (\rho_m, \rho_M)$; for any $g \in \mathcal{U}$; for any $y \in B(x, \epsilon)$, the c-disk $\mathcal{C}_g = \mathcal{W}^c_g(y, \rho)$ of g satisfies $\mathcal{C}_g \subset (\mathcal{C}, \sigma)$, $\theta''\mathcal{C}_g \subset (\theta''\mathcal{C}, \sigma)$, and*

$$(\theta\mathcal{C}, \sigma') \subset \cup_{y \in \theta'\mathcal{C}_g} Acc_g(y, 1, 2). \tag{4.3}$$

Proof. By letting ϵ be sufficiently small, we clearly have that for any g sufficiently C^1-close to f, $\mathcal{C}_g = \mathcal{W}^c_g(y, \rho)$ satisfies $\mathcal{C}_g \subset (\mathcal{C}, \sigma)$ and $\theta''\mathcal{C}_g \subset (\theta''\mathcal{C}, \sigma)$. By Proposition 3.10, $\mathcal{W}^u_g$, $\mathcal{W}^{cs}_g$ (resp. $\mathcal{W}^s_g$, $\mathcal{W}^{cu}_g$) exist and are uniformly transverse for all g sufficiently C^1-close to f. Then (4.3) follows by letting σ' and $d_{C^1}(f, g)$ be sufficiently small compared to ρ_m, ρ_M and $\theta' - \theta$. □

REMARK 4.10. *Given any $\theta, \theta', \theta'', \rho_m, \rho_M$ as in Lemma 4.9, any $\sigma > 0$, any integer $k \geq 1$, any (θ, k)-spanning c-family for f, denoted by $\mathcal{D}$, such that $[\underline{r}(\mathcal{D}), \overline{r}(\mathcal{D})] \subset (\rho_m, \rho_M)$, the following is true: for any g sufficiently C^1-close to f, there exists a $(\theta', k+2)$-spanning c-family for g, denoted by $\mathcal{D}'$, such that $[\underline{r}(\mathcal{D}'), \overline{r}(\mathcal{D}')] = [\underline{r}(\mathcal{D}), \overline{r}(\mathcal{D})]$, $n(\mathcal{D}') = n(\mathcal{D})$. Indeed, we can apply Lemma 4.9 successively for each $\mathcal{C} \in \mathcal{D}$. Moreover, we can also ensure that for each $\mathcal{C}' \in \mathcal{D}'$, we have $\mathcal{C}' \subset (\mathcal{C}, \sigma)$ and $\theta''\mathcal{C}' \subset (\theta''\mathcal{C}, \sigma)$ for some c-disk $\mathcal{C}$ in $\mathcal{D}$.*

4.3. **Extended map and center subspaces.** Recall that in this subsection, (H1) holds. Let $\bar{\chi}^c, \hat{\chi}^c, \bar{\chi}^s, \bar{\chi}^u$ be as in Definition 1.2 so that (1.3) to (1.6) are satisfied. Let $\xi > 0$ be a constant such that

$$\min(\bar{\chi}^s + \bar{\chi}^c, \bar{\chi}^u - \hat{\chi}^c, \bar{\chi}^s, \bar{\chi}^u) > \xi. \tag{4.4}$$

LEMMA 4.11. *Let $I \in \mathbb{N}_{\geq 1}$, $a \in \mathbb{R}^I$ and let $U \subset \mathbb{R}^I$ be an open neighbourhood of a. Let $\hat{f}\colon U \times X \to X$ be a C^r deformation at (a, f) with I-parameters. If U is chosen sufficiently small, then the map $T = T(\hat{f})$ is a C^r dynamically coherent partially hyperbolic system for some T-invariant splitting*

$$T_bU \oplus T_xX = E^s_T(b,x) \oplus E^c_T(b,x) \oplus E^u_T(b,x), \quad \forall\, (b,x) \in U \times X.$$

Moreover, for any $(b,x) \in U \times X$, we have

$$E^*_T(b,x) = \{0\} \oplus E^*_{\hat{f}(b,\cdot)}(x), \qquad \mathcal{W}^*_T(b,x) = \{b\} \times \mathcal{W}^*_{\hat{f}(b,\cdot)}(x), \qquad \text{for } * = u, s,$$
$$\text{and} \quad E^c_T(b,x) = Graph(\nu_b(x,\cdot)) \oplus E^c_{\hat{f}(b,\cdot)}(x), \tag{4.5}$$

for a unique linear map $\nu_b(x,\cdot)\colon T_bU \to E^{su}_{\hat{f}(b,\cdot)}(x) := E^s_{\hat{f}(b,\cdot)}(x) \oplus E^u_{\hat{f}(b,\cdot)}(x)$.

If in addition (H2) holds, then, after reducing the size of U, u, s-holonomy maps between center leaves of T (within distance 1) are C^1 when restricted to some center-unstable/center-stable leaf, with uniformly continuous, uniformly bounded derivatives.

Proof. For small enough U, the map T is a dynamically coherent partially hyperbolic diffeomorphism (it is C^1-close to $(b,x)\mapsto(b,f(x))$). A detailed treatment for this statement can be found in [36, Section 7].

In the following, let $*=u$ or s, and let U be small. Then for all $b\in U$, $E^*_{\hat f(b,\cdot)}$ is close to E^*_f, and the expansion/contraction rate of $\hat f(b,\cdot)$ along $E^*_{\hat f(b,\cdot)}$ is close to that of f along E^*_f. For any $(b,x)\in U\times X$ and $B+v\in T_bU\oplus T_xX$, we have

$$DT((b,x),B+v)=B+[\partial_b\hat f((b,x),B)+\partial_x\hat f((b,x),v)].$$

Then $DT(b,x)$ maps $\{0\}\oplus E^*_{\hat f(b,\cdot)}(x)$ to $\{0\}\oplus E^*_{\hat f(b,\cdot)}(\hat f(b,x))$, which gives $E^*_T(b,x)=\{0\}\oplus E^*_{\hat f(b,\cdot)}(x)$. It is direct to check that $\{b\}\times\mathcal{W}^*_{\hat f(b,\cdot)}(x)$ integrates E^*_T restricted to $\{b\}\times X$. Thus $\mathcal{W}^*_T(b,x)=\{b\}\times\mathcal{W}^*_{\hat f(b,\cdot)}(x)$. Moreover, it is clear that $\{0\}\oplus E^c_{\hat f(b,\cdot)}(x)\subset E^c_T(b,x)$, and $E^c_T(b,x)\cap(\{B\}\oplus T_xX)\neq\emptyset$ for any $B\in T_bU$. We define $\nu_b(x,B)$ as the unique vector in $E^{su}_{\hat f(b,\cdot)}(x)$ such that $B+\nu_b(x,B)\in E^c_T(b,x)$. It is direct to see that $\nu_b(x,\cdot)$ is a linear map.

Now, if $r\geq 2$ and f is center bunched, by C^1-openness of center bunching, for sufficiently small U, we can verify that T^n is also center bunched for some $n\in\mathbb{N}$. The smoothness of s,u-holonomy maps of T follows from Proposition 3.9. □

Let U, $\hat f$ and T be as in Lemma 4.11. In the following, for any $(b,x)\in U\times X$, we will tacitly use the inclusions $E^*_{\hat f(b,\cdot)}(x)\hookrightarrow\{0\}\oplus E^*_{\hat f(b,\cdot)}(x)\subset T_bU\oplus T_xX$ for $*=s,u,c$, and the isomorphism $\mathbb{R}^I\simeq T_bU\oplus\{0\}\subset T_bU\oplus T_xX$.

For any $(b,x)\in U\times X$ and $v=B+v'\in T_bU\oplus T_xX$, we denote by $\pi_X(v):=v'$ the component in T_xX, and set $\pi_b(v):=B+\nu_b(x,B)$. We also denote by $\pi_*(v)$ the component of v in $E^*_{\hat f(b,\cdot)}$ for $*=u,s,c$. By a slight abuse of notation, we let $\pi_X(b,x):=x$.

We introduce the following definitions, motivated by the need to control return times of a map to the support of a deformation.

Definition 4.12. For any subsets $A,B\subset X$, we define

$$\begin{aligned} R(f,A,B)&:=\inf\{n\geq 0\mid f^n(A)\cap B\neq\emptyset\text{ or }f^{-n}(A)\cap B\neq\emptyset\};\\ R_{\geq 0}(f,A,B)&:=\inf\{n\geq 0\mid f^n(A)\cap B\neq\emptyset\};\\ R_\star(f,A,B)&:=\inf\{n\geq 1\mid f^{\star n}(A)\cap B\neq\emptyset\},\quad \star=+,-.\end{aligned}$$

For any subset $A\subset X$, we use the abbreviation $R_\star(f,A):=R_\star(f,A,A)$, $\star=+,-$.

Similarly, for a C^1 deformation of f, $\hat f\colon U\times X\to X$, we set

$$R_\star(\hat f,A,B):=\inf\{n\geq 1\mid\exists\, b\in U\text{ such that }\hat f(b,\cdot)^{\star n}(A)\cap B\neq\emptyset\},\quad\star=+,-.$$

We define $R(\hat f,A,B)$, $R_{\geq 0}(\hat f,A,B)$, $R_+(\hat f,A)$ and $R_-(\hat f,A)$ in an analogous way. Moreover, it is clear that $R(\hat f,A,B)=\min(R_{\geq 0}(\hat f,A,B),R_-(\hat f,A,B))$.

In the following, for $*=s,u$, and for any $p\in M$, we set

$$\bar\chi^*_k(p):=\begin{cases}\sum_{j=0}^{k-1}\bar\chi^*(f^j(p))>0,&\forall\, k\geq 0,\\ \sum_{j=k}^{-1}\bar\chi^*(f^j(p))>0,&\forall\, k<0.\end{cases}\tag{4.6}$$

We define $\bar\chi^c_k(p)$ and $\hat\chi^c_k(p)$ in a similar way.

The following lemma collects some basic properties of the center bundle E^c_T.

LEMMA 4.13. *We have*

(1) *For any $x \in X \backslash \mathrm{supp}_X(\hat{f})$, any $B \in T_0U$,*
$$DT(B + \nu_0(x, B)) = B + \nu_0(f(x), B).$$
Equivalently, $Df(x, \nu_0(x, \cdot)) = \nu_0(f(x), \cdot)$.

(2) $\sup_{x\in X} \|\nu_0(x, \cdot)\| \leq C_1\|T\|_{C^1}$ *for a C^1-uniform constant $C_1 = C_1(f) > 0$.*

(3) *There is a C^1-uniform constant $C_2 = C_2(f) > 0$ s.t. $\forall\,(x, B) \in X \times T_0U$,*
$$\|\nu_0(x, B)\| \leq C_2\|T\|_{C^1} e^{-\kappa(\hat{f},x)}\|B\|,$$
where for $w \in M$, we let

$$\kappa(\hat{f}, w) := \min(\bar{\chi}^u_{R_{\geq 0}(\hat{f},w,\mathrm{supp}_X(\hat{f}))}(w), \bar{\chi}^s_{-R_-(\hat{f},w,\mathrm{supp}_X(\hat{f}))}(w)) > 0. \tag{4.7}$$

Proof. Proof of (1): For any $x \in X\backslash\mathrm{supp}_X(\hat{f})$, any $B \in T_0U$, we have $DT((0, x), B) = B$. We have $DT(\nu_0(x, B)) \in E^{su}_T(0, f(x))$ and by (4.11), $DT(B + \nu_0(x, B)) \in E^c_T(0, f(x))$. Thus
$$DT(B + \nu_0(x, B)) \in (B + E^{su}_T(0, f(x))) \cap E^c_T(0, f(x)),$$
while the right hand side contains only $B + \nu_0(f(x), B)$.

Proof of (2): For any $(x, B) \in X \times T_0U$, the unstable part of $\nu_0(x, B)$ satisfies
$$\pi_u\nu_0(x, B) = -\sum_{n=1}^{+\infty} Df^{-n}(f^n(x), \pi_u\partial_b\hat{f}((0, f^{n-1}), B)).$$
Then $\|\pi_u\nu_0\| \leq C_1\|T\|_{C^1}$ by $\|\partial_b\hat{f}\| \leq \|T\|_{C^1}$ and $\|Df^{-1}|_{E^u_f}\| < 1$. Arguing similarly for the stable part, we conclude the proof.

Proof of (3): By (1), for any $x \in X$ and $0 \leq n < R_-(\hat{f}, \{x\}, \mathrm{supp}_X(\hat{f}))$, we have
$$\pi_s\nu_0(x, B) = Df^n(\pi_s\nu_0(f^{-n}(x), B)), \quad \forall\, B \in T_0U.$$
By Lemma 4.13(2) and (1.4), for some C^1-uniform constant $C_1 = C_1(f) > 0$, we thus have
$$\|\pi_s\nu_0(x, \cdot)\| \leq C_1\|T\|_{C^1} e^{-\bar{\chi}^s_{-R_-(\hat{f},x,\mathrm{supp}_X(\hat{f}))}(x)}.$$
Similarly, we have $\|\pi_u\nu_0(x, \cdot)\| \leq C_1\|T\|_{C^1} e^{-\bar{\chi}^u_{R_{\geq 0}(\hat{f},x,\mathrm{supp}_X(\hat{f}))}(x)}$. □

4.4. **Holonomy maps.** Recall that in this section, (H1) and (H2) are satisfied. In the following, we fix an integer $I > 0$, and let $\hat{f} : U \times X \to X$ be a C^2 deformation at $(0, f)$ with I-parameters. We set $T = T(\hat{f})$. In the following, we will always take U conveniently small so that by Lemma 4.11, the stable and unstable holonomy maps for T between close center leaves of T are C^1.

We need bounds for the derivatives of holonomy maps with respect to parameters. The following lemma is proved by combining the construction in [35, Proof of Theorem A] and invariant section theorem for jets in [25, Proof of Theorem 3.2]. We defer the technical proof to the appendix.

LEMMA 4.14. *Let $* = u, s$. Take $x \in X$, $y \in \mathcal{W}^{c*}_f(x, \sigma_f/2)$ and set $z := H^*_{f,x,y}(x)$.*

(1) *For any $B \in T_0U$, we have*
$$\pi_b DH^*_{T,(0,x),(0,y)}(B + \nu_0(x, B)) = B + \nu_0(z, B).$$

(2) There exists a C^2-uniform constant $C_3 = C_3(f) > 0$ s.t. for any $B \in T_0U$,

$$\|\pi_c DH^*_{T,(0,x),(0,y)}(B + \nu_0(x, B))\| \leq C_3(\max(e^{-\kappa(\hat{f},x)}, e^{-\kappa(\hat{f},z)})\|DT\| + \|D^2T\| d_{\mathcal{W}^*_f}(x, z))\|B\|.$$

Proof. In Appendix A. □

The following proposition provides fine control of the derivatives of holonomy maps with respect to parameters when we are given certain recurrence condition. We give some illustration in Figure 1.

PROPOSITION 4.15. *There exists a C^2-uniform constant $C_4 = C_4(f) > 0$ such that the following is true. Fix any $R_0, C > 0$, $\sigma \in (0, \sigma_f)$. Assume that $\hat{f}$ is generated by V, an infinitesimal C^2 deformation such that $\sigma\|\partial_b\partial_x V\|_X + \|\partial_b V\|_X < C$. Then there exists a C^1-uniform constant $\xi' > 0$ such that we have the following:*

(1) Let $y \in \mathcal{W}^{cu}_f(x, \sigma)$ and $z := H^u_{f,x,y}(x)$. Assume that $x \notin \mathrm{supp}_X(V)$ and $R_-(\hat{f}, \{x, z\}, \mathrm{supp}_X(V)) > R_0$. Then for any $B \in T_0U$,

$$\|\pi_c DH^u_{T,(0,x),(0,y)}(B + \nu_0(x, B)) - \pi_c V(B, z)\| \leq C_4 C^3 e^{-R_0\xi'}\|B\|.$$

(2) Let $y \in \mathcal{W}^{cs}_f(x, \sigma)$ and $z := H^s_{f,x,y}(x)$. Assume that $R_+(\hat{f}, \{x, z\}, \mathrm{supp}_X(V)) > R_0$. Then for any $B \in T_0U$,

$$\|\pi_c DH^s_{T,(0,x),(0,y)}(B + \nu_0(x, B))\| \leq C_4 C^3 e^{-R_0\xi'}\|B\|.$$

Note that the terms on the RHS of the above inequalities are independent of σ.

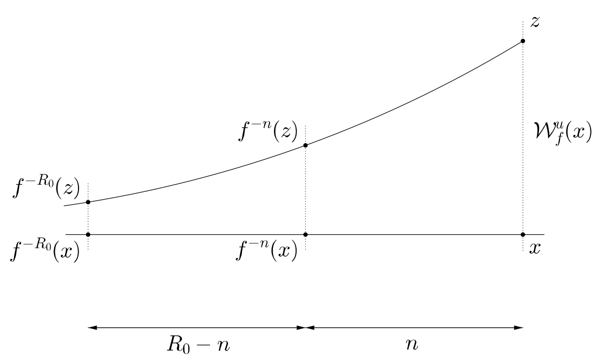

FIGURE 1. The point $f^{-j}(x)$ for any integer $1 \leq j \leq R_0$, and the point $f^{-k}(z)$ for any integer $2 \leq k \leq R_0$ lie outside of $\mathrm{supp}(\hat{f})$. We apply Lemma 4.14 to $f^{-n}(x)$ and $f^{-n}(z)$ where n is a small fraction of R_0.

Proof. We first prove (1). Without loss of generality, we assume that R_0 is sufficiently large.

Let $1 \le n \le R_0 - 1$. Successive application of Lemma 4.13(1) gives

$$DT^{-n}(B + \nu_0(x, B)) = B + \nu_0(f^{-n}(x), B).$$

Then the invariance of the foliations under the dynamics yields

$$\begin{aligned} &\pi_c DH^u_{T,(0,x),(0,y)}(B + \nu_0(x, B)) = \pi_c DH^u_{T,(0,x),(0,z)}(B + \nu_0(x, B)) \\ &= \pi_c DT^n((\pi_b + \pi_c)DH^u_{T,T^{-n}(0,x),T^{-n}(0,z)} DT^{-n}(B + \nu_0(x, B))) \\ &= \pi_c DT^n((\pi_b + \pi_c)DH^u_{T,T^{-n}(0,x),T^{-n}(0,z)}(B + \nu_0(f^{-n}(x), B))), \end{aligned} \tag{4.8}$$

where we have used that $DH^u_{T,T^{-n}(0,x),T^{-n}(0,z)}(B+\nu_0(f^{-n}(x), B)) \in E^c_T(T^{-n}(0, z))$.

Claim 4.16. *There exists a C^2-uniform constant $c_1 = c_1(f) > 0$ such that*

$$\|\pi_c DH^u_{T,T^{-n}(0,x),T^{-n}(0,z)}(B + \nu_0(f^{-n}(x), B))\| \le c_1 C^2 e^{-\min(\bar{\chi}^u_n(f^{-n}(x)), \bar{\chi}^s_{-R_0+n}(f^{-n}(x)))}\|B\|.$$

Proof. By $y \in \mathcal{W}^{cu}_f(x, \sigma)$ and by distortion estimates, for some C^1-uniform constant $c_2 = c_2(f) > 0$, we obtain

$$d_{\mathcal{W}^u_f}(T^{-n}(0, x), T^{-n}(0, z)) < e^{-\bar{\chi}^u_{-n}(x)} d_{\mathcal{W}^u_f}(x, z) < c_2 e^{-\bar{\chi}^u_{-n}(x)}\sigma.$$

By Lemma 4.4, there exist C^2-uniform constants $c_i = c_i(f) > 0$, $i = 3, 4, 5$, so that

$$\|DT\| < c_3 C, \quad \|D^2T\| < c_4(1 + \|\partial_b\partial_x V\|_X)(1 + \|\partial_b V\|_X) < c_5 C^2\sigma^{-1}.$$

Recall that $R_-(\hat{f}, \{x, z\}, \mathrm{supp}_X(V)) > R_0 > n$ by hypothesis, thus by (4.2),

$$\begin{aligned} R_{\ge 0}(\hat{f}, \{f^{-n}(x), f^{-n}(z)\}, \mathrm{supp}_X(\hat{f})) &\ge n - 1, \\ R_-(\hat{f}, \{f^{-n}(x), f^{-n}(z)\}, \mathrm{supp}_X(\hat{f})) &\ge R_0 - n. \end{aligned}$$

Thus

$$\kappa(\hat{f}, f^{-n}(x)) \ge \min(\bar{\chi}^u_{n-1}(f^{-n}(x)), \bar{\chi}^s_{-R_0+n}(f^{-n}(x))). \tag{4.9}$$

Similarly, we have

$$\kappa(\hat{f}, f^{-n}(z)) \ge \min(\bar{\chi}^u_{n-1}(f^{-n}(z)), \bar{\chi}^s_{-R_0+n}(f^{-n}(z))). \tag{4.10}$$

On the other hand, by distortion estimates, we have

$$|\text{RHS of (4.9)} - \text{RHS of (4.10)}| < C'$$

for some C^2-uniform constant $C' > 0$. Thus the claim follows from Lemma 4.14(2). □

There is a C^1-uniform constant $\eta_0 \in (0, 1)$ such that for any integer $n \in (\eta_0 R_0, 2\eta_0 R_0)$, we have

$$\bar{\chi}^u_n(f^{-n}(x)) \le \bar{\chi}^s_{-R_0+n}(f^{-n}(x)).$$

By the above claim, we get

$$\|\pi_c DH^u_{T,T^{-n}(0,x),T^{-n}(0,y)}(B + \nu_0(f^{-n}(x), B))\| \le 2c_1 C^2 e^{-\bar{\chi}^u_{-n}(x)}\|B\|. \tag{4.11}$$

By Lemma 4.14(1), we have

$$\pi_b DH^u_{T,T^{-n}(0,x),T^{-n}(0,z)}(B + \nu_0(f^{-n}(x), B)) = B + \nu_0(f^{-n}(z), B). \tag{4.12}$$

By Lemma 4.13(1), we also have

$$DT^n(B + \nu_0(f^{-n}(z), B)) = DT(B + \nu_0(f^{-1}(z), B)). \tag{4.13}$$

By (4.8)-(4.13), and since for some C^1-uniform constant $c_6 = c_6(f) > 0$,

$$\|DT^n(0,\cdot)|_{E^c_T(w)}\| < c_6 C e^{\hat{\chi}^c_n(w)}, \quad \forall\, w \in M,$$

we deduce that for some C^2-uniform constant $c_7 = c_7(f) > 0$, it holds

$$\begin{aligned}&\|\pi_c DH^u_{T,(0,x),(0,y)}(B + \nu_0(x, B)) - \pi_c DT(B + \nu_0(f^{-1}(z), B))\| \\ &\leq c_7 C^3 e^{\hat{\chi}^c_{-n}(x) - \bar{\chi}^u_{-n}(x)}\|B\| \leq c_7 C^3 e^{-n\xi}\|B\| \leq c_7 C^3 e^{-R_0\eta_0\xi}\|B\|.\end{aligned}$$

We conclude (1) by setting $\xi' := \eta_0\xi$ and by using Lemma 4.4(2), that is,

$$\pi_c DT(B + \nu_0(f^{-1}(z), B)) = \pi_c DT((0, f^{-1}(z)), B) = \pi_c V(B, z).$$

Under condition (2), a similar argument (by choosing some n comparable to R_0) shows that for some C^2-uniform constant $c_8 = c_8(f) > 0$, we have

$$\|\pi_c DH^s_{T,(0,x),(0,y)}(B + \nu_0(x, B))\| \leq c_8 C^3 e^{-(\bar{\chi}^s_n(x) + \hat{\chi}^c_n(x))}\|B\| \leq c_8 C^3 e^{-R_0\xi'}\|B\|.$$

□

5. Submersion from parameter space to phase space

In this section, we will estimate the measure of parameters in a C^r deformation corresponding to certain "unlikely coincidences". First, we need to estimate the derivatives (with respect to parameters) of holonomy maps along certain su-paths.

Throughout this section, we assume that (H1) and (H2) hold.

Definition 5.1. Given $x \in X$, a triplet $\gamma = (x_1, x_2, x_3) \in X^3$ is called a *f-loop* at x if the following holds:

$$x_1 \in \mathcal{W}^u_f(x), \quad x_2 \in \mathcal{W}^s_f(x_1), \quad x_3 \in \mathcal{W}^u_f(x_2), \quad x \in \mathcal{W}^{cs}_f(x_3).$$

The *length* of γ is defined as

$$\ell(\gamma) := d_{\mathcal{W}^u_f}(x, x_1) + d_{\mathcal{W}^s_f}(x_1, x_2) + d_{\mathcal{W}^u_f}(x_2, x_3) + d_{\mathcal{W}^{cs}_f}(x_3, x).$$

By points (1)-(3) in Notation 3.12, for each f-loop γ such that $\ell(\gamma) =: \sigma < \frac{\sigma_f}{2}$, we have a well-defined map $H_{f,\gamma}\colon \mathcal{W}^c_f(x, \Lambda_f^{-4}\sigma) \to \mathcal{W}^c_f(x, (1 + C_f)\sigma)$:

$$H_{f,\gamma} := H^s_{f,x_3,x} H^u_{f,x_2,x_3} H^s_{f,x_1,x_2} H^u_{f,x,x_1}.$$

Let $\hat{f}\colon U \times X \to X$ be a C^1 deformation at $(0, f)$, and let $T = T(\hat{f})$. For any f-loop $\gamma = (x_1, x_2, x_3)$ at x, we define the *lift* of γ for T as $\hat{\gamma} := ((0, x_1), (0, x_2), (0, x_3))$.

Notation 5.2. Recall that $c = \dim E^c_f$, and let

$$K_f := cC_f\Lambda_f^{4c}. \tag{5.1}$$

We fix a C^2-uniform constant $\overline{\sigma}_f \in \left(0, \frac{\sigma_f}{100C_f K_f}\right)$ such that for any $x \in X$, any collection $\{\gamma_j = (x_{j,1}, x_{j,2}, x_{j,3})\}_{j=1,\ldots,c}$ of f-loops at x such that $\ell(\gamma_j) < \overline{\sigma}_f, \forall\, 1 \leq j \leq c$, the map $\prod_{j=1}^c H_{f,\gamma_j}$ is defined on $\mathcal{W}^c_f(x, \overline{\sigma}_f)$. In this case, for any $1 \leq k \leq c+1$ we set $y_k := \prod_{l=1}^{k-1} H_{f,\gamma_l}(x)$, and for $1 \leq j \leq c$ we define

$$y_{j,1} := H^u_{f,x,x_{j,1}}(y_j), \quad y_{j,2} := H^s_{f,x_{j,1},x_{j,2}}(y_{j,1}), \quad y_{j,3} := H^u_{f,x_{j,2},x_{j,3}}(y_{j,2}).$$

The next lemma follows from Notation 3.12, 5.2 by straightforward computations. We thus omit its proof.

LEMMA 5.3. *Let $f, x, \gamma_j, y_j, y_{j,k}$ be as in Notation 5.2. Assume that for some $\sigma \in (0, \overline{\sigma}_f)$, we have $\ell(\gamma_j) < \sigma$, for all $j = 1, \ldots, c$. Then for any $j = 1, \ldots, c$, $d_{\mathcal{W}^c_f}(x, y_{j+1}) \leq C_f\sigma + \Lambda_f^4 d_{\mathcal{W}^c_f}(x, y_j)$. Let $y_{j,0} := y_j$ and $y_{j,4} := y_{j+1}$. We have*

$$d_{\mathcal{W}^c_f}(x, y_j), d_{\mathcal{W}^c_f}(x_{j,k}, y_{j,k}) < K_f\sigma < \tfrac{\sigma_f}{10}, \qquad \forall\, k = 1, 2, 3,$$
$$d_{\mathcal{W}^u_f}(y_{j,k-1}, y_{j,k}), d_{\mathcal{W}^s_f}(y_{j,k}, y_{j,k+1}) < 3C_f K_f\sigma < \tfrac{\sigma_f}{10}, \qquad \forall\, k = 1, 3.$$

In the following, we let $I > 0$ be some integer, let V be an infinitesimal C^2 deformation with I-parameters, and let $\hat{f}\colon U \times X \to X$ be the C^2 deformation at $(0, f)$ generated by V. Set $T = T(\hat{f})$. Let ξ' be defined as in Proposition 4.15.

LEMMA 5.4. *There exists a C^2-uniform constant $C_5 = C_5(f) > 0$ such that the following is true. Let $\sigma \in (0, \overline{\sigma}_f)$, $x \in X$ and let $\gamma = (x_1, x_2, x_3)$ be a f-loop at x with $\ell(\gamma) < \sigma$. Let $x_4 := H_{f,\gamma}(x)$, and let $C, R_0 > 0$ satisfy that:*

(1) $\sigma\|\partial_b\partial_x V\|_X + \|\partial_b V\|_X < C$;
(2) $R(f, \{x, x_2, x_3, x_4\}, \mathrm{supp}_X(V)) > R_0$;
(3) $R_\pm(f, \{x_1\}, \mathrm{supp}_X(V)) > R_0$.

Let $\hat{\gamma}$ be the lift of γ for T. Then, the holonomy map $H_{T,\hat{\gamma}}$ is C^1 in an open neighbourhood of $(0, x)$ in $\mathcal{W}^c_f(x)$, and for any $B \in T_0U$, we have

$$\|\pi_c DH_{T,\hat{\gamma}}(B+\nu_0(x,B)) - D(H^s_{f,x_3,x}H^u_{f,x_2,x_3}H^s_{f,x_1,x_2})(\pi_c V(B, x_1))\| \leq C_5 C^3 e^{-R_0\xi'}\|B\|.$$

Proof. Let $x := x_0 \in X$. By definition, we have

$$H_{T,\hat{\gamma}} = H^s_{T,(0,x_3),(0,x)} H^u_{T,(0,x_2),(0,x_3)} H^s_{T,(0,x_1),(0,x_2)} H^u_{T,(0,x),(0,x_1)}.$$

Since f is center bunched, Lemma 4.11 implies that $H^u_{T,(0,x),(0,x_1)}$, $H^s_{T,(0,x_1),(0,x_2)}$, $H^u_{T,(0,x_2),(0,x_3)}$ and $H^s_{T,(0,x_3),(0,x)}$ are C^1 if U is small enough. Given any $B \in T_0U$, let us calculate $DH_{T,\hat{\gamma}}(B + \nu_0(x, B))$. Set $I_0(B) := B + \nu_0(x, B) \in E^c_T(0, x)$. For $i = 1, \ldots, 4$, we define

$$I_i(B) := DH^{*_i}_{T,(0,x_{i-1}),(0,x_i)}(I_{i-1}(B)) \in E^c_T(0, x_i), \tag{5.2}$$

where $*_i = u$ if $i = 1, 3$ and $*_i = s$ if $i = 2, 4$. In particular, we have $DH_{T,\hat{\gamma}}(B + \nu_0(x, B)) = I_4(B)$. Then by Lemma 4.14(1) and simple induction we deduce that

$$I_i(B) = I^c_i(B) + (B + \nu_0(x_i, B)), \quad \forall\, i = 1, \ldots, 4, \tag{5.3}$$

with $I^c_i(B) := \pi_c(I_i(B))$. By (5.2), we thus obtain

$$I^c_i(B) = DH^{*_i}_{f,x_{i-1},x_i}(I^c_{i-1}(B)) + \pi_c DH^{*_i}_{T,(0,x_{i-1}),(0,x_i)}(B + \nu_0(x_{i-1}, B)). \tag{5.4}$$

By the hypothesis we made on V, $x_0, x_2, x_3, x_4 \notin \mathrm{supp}_X(V)$ and $R_\pm(f, x_i, \mathrm{supp}_X(V)) > R_0$ for $0 \leq i \leq 4$. We can apply Proposition 4.15 to obtain

$$\|\pi_c DH^u_{T,(0,x),(0,x_1)}(B + \nu_0(x, B)) - \pi_c V(x_1, B)\| \leq C_4 C^3 e^{-R_0\xi'}\|B\|,$$
$$\|\pi_c DH^s_{T,(0,x_{i-1}),(0,x_i)}(B + \nu_0(x_{i-1}, B))\| \leq C_4 C^3 e^{-R_0\xi'}\|B\|, \quad i = 2, 4,$$
$$\|\pi_c DH^u_{T,(0,x_2),(0,x_3)}(B + \nu_0(x_2, B))\| \leq C_4 C^3 e^{-R_0\xi'}\|B\|.$$

Combining this with (5.3) and (5.4), we see that there exists a C^2-uniform constant $C_5 = C_5(f) > 0$ such that

$$\|I^c_4(B) - \pi_c D(H^s_{f,x_3,x}H^u_{f,x_2,x_3}H^s_{f,x_1,x_2})(\pi_c V(B, x_1))\| \leq C_5 C^3 e^{-R_0\xi'}\|B\|.$$

□

The following definition is motivated by Lemma 5.3 and Lemma 5.4. The next proposition roughly says that if we have enough control on the magnitude of the deformation and on the return times to the support of the perturbation, then we can obtain a lower bound on the determinant of the differential of a certain map from parameter space to phase space (see Figure 2 for an illustration). This will be important in the parameter exclusion which appears in Section 10.

Definition 5.5. Given any $\sigma \in (0, \overline{\sigma}_f)$, $C, R_0 > 0$, let $\gamma = (x_1, x_2, x_3)$ be a f-loop at a point $x \in X$ with $\ell(\gamma) < \sigma$. We say that V is *adapted to* (γ, σ, C, R_0) if:

(1) $\sigma \|\partial_b \partial_x V\|_X + \|\partial_b V\|_X < C$;
(2) $R(f, \mathcal{W}_f^c(z, K_f\sigma), \mathrm{supp}_X(V)) > R_0$ for $z = x, x_2, x_3$;
(3) $R_\pm(f, \mathcal{W}_f^c(x_1, K_f\sigma), \mathrm{supp}_X(V)) > R_0$.

Proposition 5.6. *For any integer $L > 0$, real numbers $C, \kappa > 0$, there exist C^2-uniform constants $R_0 = R_0(f, L, c, C, \kappa) > 0$ and $\kappa_0 = \kappa_0(f, L, c, C, \kappa) > 0$ such that the following is true.*

Let $x \in X$, $\sigma \in (0, (12C_fK_f)^{-1}\overline{\sigma}_f)$. For each $1 \le i \le L$, $1 \le j \le c$, let $\gamma_{i,j} = (x_{i,j,1}, x_{i,j,2}, x_{i,j,3})$ be a f-loop at x of length at most σ such that V is adapted to $(\gamma_{i,j}, \sigma, C, R_0)$. Denote by $B = (B_\alpha)_{1 \le \alpha \le I}$ an element of $T_0U = \mathbb{R}^I$. Assume that for some integer $1 \le j_0 \le c$, and indices $\{\alpha_{i,j}\}_{1 \le i \le L, 1 \le j \le c} \subset \{1, \dots, I\}$, we have for any $1 \le i, k \le L$ and $1 \le j \le c$ that: if $i \ne k$ or $j \ne j_0$, then for all $z \in \mathcal{W}_f^c(x_{i,j,1}, K_f\sigma)$, we have

$$D_{B_{\alpha_{k,1}}, \dots, B_{\alpha_{k,c}}}(\pi_c V(B, z)) = 0, \tag{5.5}$$

while for any $z \in \mathcal{W}_f^c(x_{i,j_0,1}, K_f\sigma)$, we have

$$\left| \det(B \mapsto D_{B_{\alpha_{i,1}}, \dots, B_{\alpha_{i,c}}}(\pi_c V(B, z))) \right| > 2\kappa. \tag{5.6}$$

Let $\hat{\gamma}_{i,j}$ be the lift of $\gamma_{i,j}$ for T, and set $z_i := \prod_{j=1}^c H_{f,\gamma_{i,j}}(x)$. Then there exists a linear subspace $H \subset T_0U = \mathbb{R}^I$ of dimension Lc such that

$$\det(\Xi|_H) \ge \kappa_0,$$

where we set $\Xi \colon \left\{ \begin{array}{rcl} T_0U & \to & \prod_{i=1}^L E_f^c(z_i), \\ B & \mapsto & \left(\pi_c D(\prod_{j=1}^c H_{T,\hat{\gamma}_{i,j}})(B + \nu_0(x, B)) \right)_{i=1,\dots,L} \end{array} \right.$.

Proof. For each $1 \le i \le L$, and $1 \le j \le c$, we define

$$y_{i,j} := \prod_{l=1}^{j-1} H_{f,\gamma_{i,l}}(x), \qquad y_{i,j,1} := H^u_{f,x,x_{i,j,1}}(y_{i,j}),$$
$$y_{i,j,2} := H^s_{f,x_{i,j,1},x_{i,j,2}}(y_{i,j,1}), \qquad y_{i,j,3} := H^u_{f,x_{i,j,2},x_{i,j,3}}(y_{i,j,2}).$$

By the choice of $\overline{\sigma}_f$ in Notation 5.2, Lemma 5.3 yields

$$y_{i,j} \in \mathcal{W}_f^c(x, K_f\sigma), \quad y_{i,j,k} \in \mathcal{W}_f^c(x_{i,j,k}, K_f\sigma), \quad \forall\, k = 1, 2, 3. \tag{5.7}$$

Denote by $\gamma'_{i,j} = (y_{i,j,1}, y_{i,j,2}, y_{i,j,3})$ the associated f-loop at $y_{i,j}$. Note that we also have $z_i \in \mathcal{W}_f^c(x, K_f\sigma)$, for all $1 \le i \le L$.

By assumption, for any $1 \le i \le L$, $1 \le j \le c$, we have $\ell(\gamma_{i,j}) < \sigma$, thus by Lemma 5.3, $\ell(\gamma'_{i,j}) \le 12C_fK_f\sigma < \overline{\sigma}_f$. Since V is $(\gamma_{i,j}, \sigma, C, R_0)$-adapted, we get

$$R(f, \{y_{i,j}, y_{i,j,2}, y_{i,j,3}\}, \mathrm{supp}_X(V)) > R_0, \quad R_\pm(f, \{y_{i,j,1}\}, \mathrm{supp}_X(V)) > R_0.$$

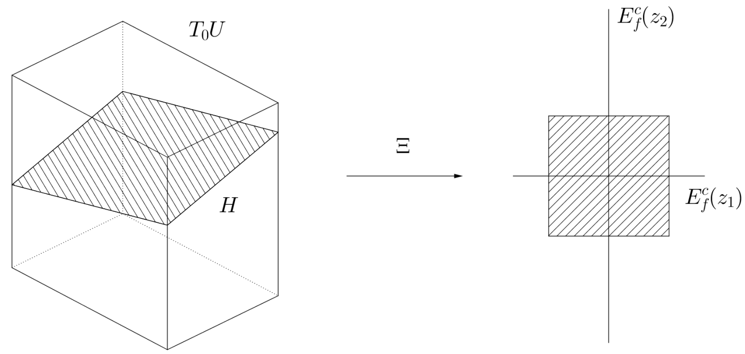

FIGURE 2. The map Ξ is a submersion. Here, the picture is for $L = 2$, H is a $2D$ subspace of T_0U, and $\Xi|_H$ is invertible.

Let $H := \oplus_{i=1}^L \oplus_{j=1}^c \mathbb{R}\partial_{B_{\alpha_{i,j}}}$. Define $\Xi_H \colon H \to \prod_{i=1}^L E^c_f(z_i)$ by

$$\Xi_H(B) = \big(\pi_c D\big(\prod_{j=1}^c H_{T,\hat{\gamma}_{i,j}}\big)(B + \nu_0(x, B))\big)_{i=1,\dots,L}.$$

By Lemma 4.14(1), we have for any $1 \le i, k \le L$:

$$D_{B_{\alpha_{k,1}},\dots,B_{\alpha_{k,c}}}\big(\pi_c D\big(\prod_{j=1}^c H_{T,\hat{\gamma}_{i,j}}\big)(B + \nu_0(x, B))\big) = \sum_{l=1}^c I_{i,k,l}(B),$$

where for each $1 \le l \le c$, we set

$$I_{i,k,l}(B) := D_{B_{\alpha_{k,1}},\dots,B_{\alpha_{k,c}}}\big(\pi_c D\big(\prod_{j=l+1}^c H_{T,\hat{\gamma}_{i,j}}\big)(\pi_c D H_{T,\hat{\gamma}_{i,l}}(B + \nu_0(y_{i,l}, B)))\big).$$

It is clear that for all $1 \le i \le L$, $1 \le l \le c$,

$$\pi_c D H_{T,\hat{\gamma}_{i,l}}(B + \nu_0(y_{i,l}, B)) = \pi_c D H_{T,\hat{\gamma}'_{i,l}}(B + \nu_0(y_{i,l}, B)),$$

where $\hat{\gamma}'_{i,l}$ denotes the lift of $\gamma'_{i,l}$ for T. Since $\gamma'_{i,l}$ and V satisfy the conditions of Lemma 5.4 with (σ, C) replaced by $(12C_fK_f\sigma, 12C_fK_fC)$, there exists a C^2-uniform constant $c_1 = c_1(f) > 0$ (we absorb the term $12C_fK_f$ in c_1) so that

$$\big|I_{i,k,l}(B) - D_{B_{\alpha_{k,1}},\dots,B_{\alpha_{k,c}}}\big(D\big(\prod_{j=l+1}^c H_{f,\gamma_{i,j}} \cdot H^s_{f,x_{i,l,3},x} H^u_{f,x_{i,l,2},x_{i,l,3}} H^s_{f,x_{i,l,1},x_{i,l,2}}\big)$$
$$\cdot\big(\pi_c V(B, y_{i,l,1})\big)\big)\big| \le c_1 C^3 e^{-R_0\xi'}\|B\|.$$

Let $1 \le i, k \le L$ and $1 \le j \le c$. If $i \ne k$ or $j \ne j_0$, then by (5.7), $z = y_{i,j,1}$ satisfies (5.5), hence $|I_{i,k,j}| < c_2 C^3 e^{-R_0\xi'}$. Again by (5.7), $z = y_{i,j_0,1}$ satisfies (5.6), hence

$$|\det(I_{i,i,j_0})| > c_3\kappa - c_4 C^3 e^{-R_0\xi'}.$$

Here $c_2, c_3, c_4 > 0$ are C^2-uniform constants depending only on f, c.

Thus for some C^2-uniform constant $c_5 > 0$ depending only on f, L, c, C, for any sufficiently large R_0 depending only on f, L, c, C, κ, we have $\det(\Xi_H) > c_5\kappa^L$.

Moreover, it is easy to see that R_0 is C^2-uniform with respect to f. Then $\kappa_0 := c_5\kappa^L$ depends only on f, L, c, C, κ and is C^2-uniform in f. This concludes the proof. $\square$

6. Finding suitable spanning c-families

Since we will study a parametrised family of diffeomorphisms, we need a bit more work to find suitable c-families. Let us start by recalling a result presented in [20, Lemma 1.2]. We use here the notations introduced in Subsection 4.2.

LEMMA 6.1 (Accessibility modulo central disks). *Let $f \in \mathcal{PH}^1(X)$ be dynamically coherent. Assume that the fixed points of f^k are isolated, for all $k \geq 1$. Then for every integer $\overline{R} > 0$ there exists $\mathcal{D} = \mathcal{D}(f, \overline{R})$, a c-family for f such that*

$$(1)\ \overline{r}(\mathcal{D}) < \overline{R}^{-1}, \qquad (2)\ R_\pm(f, \mathcal{D}) > \overline{R}, \qquad (3)\ \mathcal{D} \textit{ is } (\frac{1}{80}, 2)\textit{-spanning.}$$

For the convenience of the choices of some constants, we replaced the constant $\frac{1}{2}$ in [20, Lemma 1.2] by $\frac{1}{80}$. This does not introduce any new difficulty into the proof. The following is a consequence of the above lemma.

COROLLARY 6.2. *Assume that $f \in \mathcal{PH}^1(X)$ is dynamically coherent, and the fixed points of f^k are isolated for all $k \geq 1$. Then for every $\overline{R} > 0$, there exist C^1-uniform constants $N = N(f, \overline{R}) > 0$, $\rho = \rho(f, \overline{R}) \in (0, \overline{R}^{-1})$ and $\sigma = \sigma(f, \overline{R}) > 0$ such that the following is true. For all g sufficiently C^1-close to f, there exists $\mathcal{D}_g$, a $(\frac{1}{40}, 4)$-spanning c-family for g such that Lemma 6.1(1) is satisfied for $(\mathcal{D}_g, g)$ in place of $(\mathcal{D}, f)$. Moreover, we have*

$$(1)\ \underline{r}(\mathcal{D}_g) > \rho, \quad (2)\ n(\mathcal{D}_g) < N, \quad (3)\ \mathcal{D}_g \textit{ is } \sigma\textit{-sparse}, \quad (4)\ R_\pm(g, (\mathcal{D}_g, \sigma)) > \overline{R}.$$

Proof. Let $\mathcal{D} = \mathcal{D}(f, 2\overline{R})$ be a $(\frac{1}{80}, 2)$-spanning c-family for f given by Lemma 6.1. Set $N := n(\mathcal{D}) + 1$. Take $\rho \in (0, (2\overline{R})^{-1})$ such that $\underline{r}(\mathcal{D}) > \rho$, and choose $\sigma > 0$ so that (3), (4) above are true for $(\mathcal{D}, f, 4\sigma)$ in place of $(\mathcal{D}_g, g, \sigma)$. Then for any g sufficiently C^1-close to f, for any c-family $\mathcal{D}_g$ for g with $(\mathcal{D}_g, 0) \subset (\mathcal{D}, \sigma)$, items (3),(4) above are satisfied for $(\mathcal{D}_g, g)$.

By Lemma 4.9 applied to f and $(\theta, \theta', \theta'', \rho_m, \rho_M) = (\frac{1}{80}, \frac{1}{40}, \frac{1}{40}, \rho, \overline{R}^{-1})$, and by Remark 4.10, we see that there exists a $\left(\frac{1}{40}, 4\right)$-spanning c-family $\mathcal{D}_g$ for g, satisfying Lemma 6.1(1), Corollary 6.2(1),(2), and $(\mathcal{D}_g, 0) \subset (\mathcal{D}, \sigma)$. Combined with the previous discussions, this concludes the proof. $\square$

While working with a family of diffeomorphisms, we will need to consider several c-families. For that purpose, we use a superposition of a collection of perturbations which are localized in parameter-phase space. The following proposition will allow us to arrange their support in such a way that the interferences between them are very weak; it will serve as a key step in the inductive construction of these localized perturbations in Proposition 10.2.

PROPOSITION 6.3. *Let $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, $J \geq 1$, and let $\{f_a\}_{a \in [0,1]^J}$ be a good (see Definition 3.4) $C^r - J-$family in the space of dynamically coherent, C^r partially hyperbolic diffeomorphisms. Then for any integers $K, R_0 \geq 1$, any real numbers $\vartheta > 0$, $h_0 > 0$, there exists a set Ω_1 compactly contained in $[0,1]^J$ with $\mathrm{Leb}([0,1]^J \backslash \Omega_1) < \vartheta$, an integer $N_0 > 1$, and real numbers $\rho_0 \in (0, h_0)$, $\rho_1 \in (0, \rho_0)$, $\sigma_0, \lambda_0 > 0$ such that the following is true.*

Take any $a \in \Omega_1$, and any integer $0 \leq l \leq K-1$. For any collection of points $\{a_i\}_{i=1}^l \subset B(a, \lambda_0) \cap [0,1]^J$, any $1 \leq i \leq l$, let $\mathcal{D}_i$ be a c-family for f_{a_i} satisfying

$$(1)\ [\underline{r}(\mathcal{D}_i), \overline{r}(\mathcal{D}_i)] \subset (\rho_1, \rho_0); \qquad (2)\ n(\mathcal{D}_i) < N_0.$$

Then there exists a $(\frac{1}{20}, 6)$-spanning c-family for f_a, denoted by $\mathcal{D}_{l+1}$, such that (1), (2) above are satisfied for $i = l+1$, and moreover,

(1) $\mathcal{D}_{l+1}$ is σ_0-sparse, and $(\mathcal{D}_{l+1}, \sigma_0)$ is disjoint from $(\{\mathcal{D}_i\}_{i=1}^l, \sigma_0)$;
(2) for any $a' \in B(a, \lambda_0) \cap [0,1]^J$, we have

- $R(f_{a'}, (\mathcal{D}_{l+1}, \sigma_0), (\{\mathcal{D}_i\}_{i=1}^l, \sigma_0)) > R_0$;
- $R_\star(f_{a'}, (\mathcal{D}_{l+1}, \sigma_0)) > R_0, \qquad \star = +, -$.

Proof. We choose Ω_1 to be any compact set contained in $\mathrm{int}([0,1]^J)$ such that $\mathrm{Leb}([0,1]^J \backslash \Omega_1) < \vartheta$, and for any $a \in \Omega_1$, the fixed points of f_a^k are isolated for any integer $k \geq 1$. The existence of Ω_1 is guaranteed by our hypothesis that $\{f_a\}_{a \in [0,1]^J}$ is a good family.

By the compactness of $[0,1]^J$, there exists $\rho_2 \in (0, h_0)$ such that for any $a \in [0,1]^J$, $x \in X$, the tangent space of $\mathcal{W}^c_{f_a}(x, 4\rho_2)$ is sufficiently close to $E^c_{f_a}(x)$ so that for any $y \in B(x, \rho_2)$, $\mathcal{W}^c_{f_a}(y, 4\rho_2)$ intersects $B(x, \rho_2)$ in a single local center manifold.

Let $\rho_0 \in (0, \frac{\rho_2}{2})$ be small enough so that for any $a \in [0,1]^J$, any $x \in X$, we have

$$f_a^p(\mathcal{W}^c_{f_a}(x, \rho_0)) \subset \mathcal{W}^c_{f_a}(f_a^p(x), \rho_2), \quad \forall -R_0 \leq p \leq R_0. \tag{6.1}$$

By Corollary 6.2 applied to $\overline{R} > \max(\rho_0^{-1}, R_0)$, and by the compactness of Ω_1, there exist $N_0 > 0$, $\rho_1 \in (0, \rho_0)$, $\sigma_1 \in (0, \rho_2)$ such that for all $a \in \Omega_1$ there exists a $(\frac{1}{40}, 4)$-spanning c-family for f_a, denoted by $\widetilde{\mathcal{D}}(a)$, such that $[\underline{r}(\widetilde{\mathcal{D}}(a)), \overline{r}(\widetilde{\mathcal{D}}(a))] \subset (\rho_1, \rho_0)$, $n(\widetilde{\mathcal{D}}(a)) < N_0$, and

$$\widetilde{\mathcal{D}}(a) \text{ is } \sigma_1\text{-sparse}, \quad \text{and} \quad R(f_a, (\widetilde{\mathcal{D}}(a), \sigma_1)) > R_0. \tag{6.2}$$

Take $\sigma_3 > 0$ sufficiently small such that for any $a \in [0,1]^J$, $x \in X$, $y \in B(x, \sigma_3)$, and any $\rho \in (\rho_1, \rho_0)$, we have $\mathcal{W}^c_{f_a}(y, \rho) \subset (\mathcal{W}^c_{f_a}(x, \rho), \sigma_1/2)$ and[8]

$$\mathcal{W}^c_{f_a}(x, \frac{1}{40}\rho) \subset \bigcup_{z \in \mathcal{W}^c_{f_a}(y, \frac{1}{20}\rho)} Acc_{f_a}(z, 1, 2). \tag{6.3}$$

Take $\sigma_2 > 0$ such that for any $a \in [0,1]^J$, any $x \in X$, any collection of $3R_0N_0K$ points $\{x_i\}_{i=1}^{3R_0N_0K} \subset X$, there exists $y \in B(x, \sigma_3)$ such that for all $1 \leq i \leq 3R_0N_0K$, we have $d(\mathcal{W}^c_{f_a}(y, \rho_2), \mathcal{W}^c_{f_a}(x_i, \rho_2)) > 3\sigma_2$.

Take a small constant $\lambda_0 > 0$ such that for any $a \in [0,1]^J$, $a' \in B(a, \lambda_0) \cap [0,1]^J$, $x \in X$, and any $-R_0 \leq p \leq R_0$, we have

$$f_a^p(\mathcal{W}^c_{f_{a'}}(x, \rho_0)) \subset (f_a^p(\mathcal{W}^c_{f_a}(x, \rho_0)), \sigma_2). \tag{6.4}$$

Fix any $a \in \Omega_1$. We denote $\widetilde{\mathcal{D}} := \widetilde{\mathcal{D}}(a) = \{\widetilde{\mathcal{C}}_1, \ldots, \widetilde{\mathcal{C}}_{N_1}\}$ for some $N_1 < N_0$. We take $l, \{a_i\}_{i=1}^l$ and $\{\mathcal{D}_i\}_{i=1}^l$ as in the proposition. We will modify $\widetilde{\mathcal{D}}$ to obtain $\mathcal{D}_{l+1}$ that satisfies the conclusion of the proposition.

[8]As in the case of (4.3), here (6.3) follows from the uniform transversality of $\mathcal{W}^{cs}_{f_a}$ and $\mathcal{W}^u_{f_a}$, respectively $\mathcal{W}^{cu}_{f_a}$ and $\mathcal{W}^s_{f_a}$.

We will define $\mathcal{C}_{l+1,1}, \ldots, \mathcal{C}_{l+1,N_1}$ by induction so that $\mathcal{D}_{l+1} = \{\mathcal{C}_{l+1,1}, \ldots, \mathcal{C}_{l+1,N_1}\}$ is a $(\frac{1}{20}, 6)$-spanning c-family for f_a. Let $0 \leq j \leq N_1 - 1$ be an integer such that for all $1 \leq k \leq j$, $\mathcal{C}_{l+1,k}$ is defined and satisfies $\varrho(\mathcal{C}_{l+1,k}) \in (\rho_1, \rho_0)$,

$$\mathcal{C}_{l+1,k} \subset (\widetilde{\mathcal{C}}_k, \sigma_1/2), \tag{6.5}$$

$$\frac{1}{40}\widetilde{\mathcal{C}}_k \subset \bigcup_{x \in \frac{1}{20}\mathcal{C}_{l+1,k}} Acc_{f_a}(x, 1, 2), \tag{6.6}$$

$$\text{and} \qquad (\mathcal{C}_{l+1,k}, \sigma_2) \cap (\mathcal{C}', \sigma_2) = \emptyset, \quad \forall \mathcal{C}' \in \mathcal{M}_k, \tag{6.7}$$

where

$$\mathcal{M}_k := \bigcup_{\substack{1 \leq i \leq l,\ \mathcal{C} \in \mathcal{D}_i, \\ -R_0 \leq p \leq R_0}} f_a^p(\mathcal{C}) \ \cup \bigcup_{\substack{1 \leq m \leq k-1, \\ -R_0 \leq p \leq R_0}} f_a^p(\mathcal{C}_{l+1,m}).$$

The above is true for $j = 0$. By the choices of ρ_0 and λ_0, by (6.1) and (6.4), for any $1 \leq i \leq l$, $\mathcal{C} \in \mathcal{D}_i$, and $-R_0 \leq p \leq R_0$, there exists $x \in X$ such that $f_a^p(\mathcal{C}) \subset (f_a^p(\mathcal{W}^c_{f_a}(x, \rho_0)), \sigma_2) \subset (\mathcal{W}^c_{f_a}(f_a^p(x), \rho_2), \sigma_2)$. Similarly, for each $1 \leq m \leq j, -R_0 \leq p \leq R_0$, there exists $x \in X$ such that $f_a^p(\mathcal{C}_{l+1,m}) \subset \mathcal{W}^c_{f_a}(x, \rho_2)$. Thus $\mathcal{M}_{j+1}$ is contained in less than $3R_0N_0K$ many σ_2-neighbourhoods of c-disks for f_a of radius ρ_2. Then by (6.1), (6.3), and the choice of σ_2, there exists a center disk $\mathcal{C}_{l+1,j+1} = \mathcal{W}^c_{f_a}(y, \rho')$ for some $\rho' \in (\rho_1, \rho_0)$, satisfying (6.5), (6.6) and (6.7) for $k = j + 1$. We complete the construction of $\mathcal{D}_{l+1}$ by induction.

Since $\widetilde{\mathcal{D}}$ is $(\frac{1}{40}, 4)$-spanning, by (6.6), $\mathcal{D}_{l+1}$ is $(\frac{1}{20}, 6)$-spanning. By taking $\sigma_0 > 0$ sufficiently small, depending only on $\{f_a\}, R_0, \sigma_2$, we can ensure that for any $-R_0 \leq p \leq R_0$, any $\mathcal{C} \in \bigcup_{1 \leq i \leq l+1} \mathcal{D}_i$, we have $f_a^p((\mathcal{C}, 2\sigma_0)) \subset (f_a^p(\mathcal{C}), \sigma_2/4)$.

By further requiring that σ_0 be sufficiently small, depending only on $\{f_a\}, R_0, \sigma_2, \sigma_1$, (6.2), (6.5), (6.7) implies that $\mathcal{D}_{l+1}$ is $2\sigma_0$-sparse, and

- $R(f_a, (\mathcal{D}_{l+1}, 2\sigma_0), (\{\mathcal{D}_i\}_{i=1}^l, 2\sigma_0)) > R_0$,
- $R_\pm(f_a, (\mathcal{D}_{l+1}, 2\sigma_0)) > R_0$.

By continuity, and after possibly taking λ_0 to be even smaller, but depending only on $\{f_a\}$, ρ_0, ρ_1, R_0, N_0, K, σ_0, we can ensure that for all $a' \in B(a, \lambda_0) \cap [0, 1]^J$,

- $R(f_{a'}, (\mathcal{D}_{l+1}, \sigma_0), (\{\mathcal{D}_i\}_{i=1}^l, \sigma_0)) > R_0$,
- $R_\pm(f_{a'}, (\mathcal{D}_{l+1}, \sigma_0)) > R_0$.

□

7. A stable criterion for stable values

7.1. **A criterion for stable values.** In this section we state a topological lemma that is at the core of our construction of open accessibility classes. First we borrow a few definitions from [10].

Definition 7.1. If $f\colon X \to Y$ is a continuous map between metric spaces X and Y, then $y \in Y$ is a *stable value* of f if there is $\epsilon > 0$ such that $y \in \mathrm{Im}(g)$ for every continuous map $g\colon X \to Y$ such that $d_{C^0}(f, g) < \epsilon$.

Definition 7.2. Given a constant $\epsilon > 0$, a continuous map $f\colon X \to Y$ between metric spaces X and Y is called *ϵ-light* if for every $y \in Y$, every connected component of $f^{-1}(y)$ has diameter strictly smaller than ϵ.

Remark 7.3. *This definition is a quantitative version of the notion of light map in* [10] *(a map $f\colon X \to Y$ is called light if all point inverses are totally disconnected).*

Now we state the main topological result in this section.

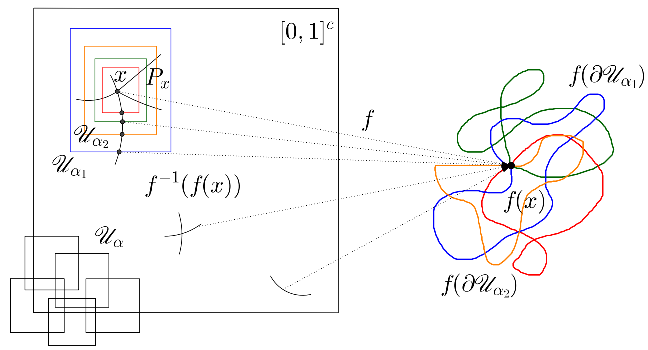

FIGURE 3. Controlling the size of connected components of point inverses.

THEOREM F. *For any integer $n \geq 1$, there exists a constant $\epsilon = \epsilon(n) > 0$ such that any ϵ-light continuous map $f : [0,1]^n \to \mathbb{R}^n$ has a stable value.*

Proof. In Appendix C. □

REMARK 7.4. *Theorem F is a quantitative version of a result due to Bonk-Kleiner: in* [10, Proposition 3.2]*, the authors proved that any light continuous map from a compact metric space of topological dimension at least n to $\mathbb{R}^n$ has stable values. The proof we give is modeled on theirs.*

COROLLARY 7.5. *For any integer $c \geq 1$, let $\{\mathscr{U}_\alpha\}_{\alpha\in\mathcal{A}}$ be an open cover of $[0,1]^c$ s.t.* $\operatorname{diam}(\mathscr{U}_\alpha) < \epsilon(c)$ *for all $\alpha \in \mathcal{A}$, where $\epsilon(c)$ is given by Theorem F. Let $f : [0,1]^c \to \mathbb{R}^c$ be a continuous map s.t. for any $x \in [0,1]^c$, there exists $\mathcal{I} \subset \mathcal{A}$ satisfying*

(1) $\cap_{\alpha\in\mathcal{I}} f(\partial\mathscr{U}_\alpha) = \emptyset$;
(2) $x \in \mathscr{U}_\alpha$ for all $\alpha \in \mathcal{I}$.

Then f has a stable value.

Proof. By Theorem F, it suffices to check that f is $\epsilon(c)$-light. Given any $x \in [0,1]^c$, take $\mathcal{I} \subset \mathcal{A}$ satisfying (1), (2). In particular, there exists $\alpha \in \mathcal{I}$ such that $f(x) \notin f(\partial\mathscr{U}_\alpha)$. We denote by P_x the connected component of $f^{-1}(f(x))$ containing x. We claim that P_x is contained in $\mathscr{U}_\alpha$. Indeed, by the continuity of f, $f^{-1}(f(x))$ has no accumulating point in $\partial\mathscr{U}_\alpha$. If $P_x \cap (\mathscr{U}_\alpha)^c \neq \emptyset$, then we can find two disjoint open sets U, V s.t. $P_x \subset U \cup V$ and $P_x \cap U$, $P_x \cap V$ are both nonempty. This contradicts the connectedness of P_x, hence the claim is true. In particular, the diameter of P_x is not larger than the diameter of $\mathscr{U}_\alpha$ which by hypothesis is strictly smaller than $\epsilon(c)$. Since x is an arbitrary point in $[0,1]^c$, we deduce that f is $\epsilon(c)$-light. □

7.2. **Choosing a cover by disjoint squares.** In this section, we define a cover of $[0,1]^c$ by open cubes, which will later be used when we apply Corollary 7.5 to show the existence of open accessibility classes.

Given an integer $c \geq 1$, a positive constant $\theta \in (\frac{c-1}{c}, 1)$, we set

$$K_0(c,\theta) := \left\lceil \frac{3c+1}{c-(c-1)\theta^{-1}} \right\rceil + 1, \qquad K_1(c,\theta) := cK_0(c,\theta). \tag{7.1}$$

In the following, we will fix c, θ and abbreviate $K_i(c,\theta)$ as K_i, $i = 0, 1$.

By a direct construction, we can fix a cover $\{\mathscr{U}_\alpha\}_{\alpha\in\mathcal{A}}$ of $[0,1]^c$ by open sets in $\mathbb{R}^c$, which satisfies:

1. $\mathcal{A}$ is a finite set and for all $\alpha \in \mathcal{A}$, there exist constants $\{p_{\alpha,i}, q_{\alpha,i}\}_{i=1,\dots,c} \subset [-1,2]$ such that $\mathscr{U}_\alpha = (p_{\alpha,1}, q_{\alpha,1}) \times \cdots \times (p_{\alpha,c}, q_{\alpha,c})$;
2. for any $\alpha \in \mathcal{A}$, $\mathrm{diam}(\mathscr{U}_\alpha) < \epsilon(c)$, where $\epsilon(c)$ is given by Theorem F;
3. for each $x \in [0,1]^c$, there exists a subset $\mathcal{I} \subset \mathcal{A}$ with more than K_1 elements satisfying that $x \in \mathscr{U}_\alpha$ for all $\alpha \in \mathcal{I}$, and $\{\partial\mathscr{U}_\alpha\}_{\alpha\in\mathcal{I}}$ are mutually disjoint;
4. for each $i \in \{1,\dots,c\}$, the points $\{p_{\alpha,i}, q_{\alpha,i}\}_{\alpha\in\mathcal{A}}$ are mutually distinct.

For each integer $i \in \{1,\dots,c\}$, we let $\mathcal{B}_i := \{p_{\alpha,i}, q_{\alpha,i}\}_{\alpha\in\mathcal{A}}$, and for each $\alpha \in \mathcal{A}$, we denote $\partial^i\mathscr{U}_\alpha := [p_{\alpha,1}, q_{\alpha,1}] \times \cdots \times [p_{\alpha,i-1}, q_{\alpha,i-1}] \times \{p_{\alpha,i}, q_{\alpha,i}\} \times [p_{\alpha,i-1}, q_{\alpha,i-1}] \times \cdots \times [p_{\alpha,c}, q_{\alpha,c}]$. Given any $s \in [-1,2]$, we introduce the normalized coordinate

$$\varphi(i,s) := \frac{6i-2+s}{6c} \in \left[\frac{i}{c} - \frac{1}{2c}, \frac{i}{c}\right] \subset (0,1]. \tag{7.2}$$

Note that for any $i < i'$ and any $s, s' \in [-1,2]$, $\varphi(i,s) < \varphi(i',s')$. We also set

$$0 < C_{\min} := 100\Big(\min_{\substack{1\leq i\leq c\\ t\neq t'\in\mathcal{B}_i}} |\varphi(i,t) - \varphi(i,t')|\Big)^{-1} < +\infty. \tag{7.3}$$

Definition 7.6. We define the set

$$\begin{aligned}\Gamma := \big\{(i, \mathcal{B}, \{s_t = (s_{t,1},\dots,s_{t,c})\}_{t\in\mathcal{B}}) \mid i \in \{1,\dots,c\},\ \mathcal{B} \subset \mathcal{B}_i,\ |\mathcal{B}| = K_0,\\ \text{and } s_t \in [-1,2]^{i-1} \times \{t\} \times [-1,2]^{c-i},\ \forall\, t \in \mathcal{B})\big\}.\end{aligned} \tag{7.4}$$

8. Holonomy maps associated to a family of loops

In this section, (H1) holds.

8.1. **Continuous and regular family of loops.**

Definition 8.1. Given $x \in X$, a one-parameter family $\{\gamma(s) = (x_1(s), x_2(s), x_3(s))\}_{s\in[0,1]}$ of f-loops at x is said to be *continuous* if for any $i = 1, 2, 3$, the map $s \mapsto x_i(s)$ is continuous. We define $\ell(\gamma) := \sup_{s\in[0,1]} \ell(\gamma(s))$.

Lemma 8.2 (Continuation of f-loops). *There exist $\mathcal{U}$, a C^1-open neighbourhood of f, as well as $\varsigma_f > 0$ such that the following is true. Let γ be a continuous family of f-loops at $x \in X$ satisfying $\ell(\gamma) < \frac{\varsigma_f}{2}$. Then for any $g \in \mathcal{U}$ and $y \in B(x, \varsigma_f)$, we can define $\gamma_{g,y}$, a continuous family of g-loops at y, such that $\gamma_{f,x} = \gamma$ and each coordinate of $\gamma_{g,y}(s)$ depends continuously on (g, y, s).*

Proof. Let $\gamma = (x_1, x_2, x_3)$. If (g, y) is chosen sufficiently close to (f, x), then for any $s \in [0,1]$, the following leaves intersect at a unique point, and we define

- $\{y_{g,1}(s)\} := \mathcal{W}^u_g(y, h_f) \cap \mathcal{W}^{cs}_g(x_1(s), h_f)$;
- $\{y_{g,2}(s)\} := \mathcal{W}^s_g(y_{g,1}(s), h_f) \cap \mathcal{W}^{cu}_g(x_2(s), h_f)$;
- $\{y_{g,3}(s)\} := \mathcal{W}^u_g(y_{g,2}(s), h_f) \cap \mathcal{W}^{cs}_g(y, h_f)$.

Then we can verify our lemma for $\gamma_{g,y} := (y_{g,1}, y_{g,2}, y_{g,3})$. □

8.2. **A criterion for stable accessibility.** Given $x \in X$, and $\gamma = \{\gamma(s)\}_{s\in[0,1]}$, a continuous family of f-loops at x, satisfying $\ell(\gamma) < \overline{\sigma}_f$ (defined in Notation 5.2), we introduce

$$\psi = \psi(f,x,\gamma)\colon \left\{ \begin{array}{rcl} [-1,2]^c & \to & \mathcal{W}^c_f(x), \\ s=(s_1,\dots,s_c) & \mapsto & (\prod_{j=1}^c H_{f,\gamma(\varphi(j,s_j))})(x). \end{array} \right. \tag{8.1}$$

By Notation 5.2, ψ is well-defined. Besides, it is clear that $\mathrm{Im}(\psi) \subset Acc_f(x)$.

The following property combines a global property, based on the notion of "accessibility modulo central disks" which appears in [20] (see also Section 6), and a local one, based on the notion of ϵ-light maps in Section 7, which together imply the accessibility property.

Definition 8.3 (Property $(\mathcal{P})$). We say that f satisfies property $(\mathcal{P})$ if there exist $0<\theta<\theta'<1$, an integer $k\geq 1$, and $\mathcal{D}$, a (θ,k)-spanning c-family for f, such that for any $\mathcal{C}\in\mathcal{D}$, any $x\in\theta'\mathcal{C}$, there exists a continuous family of f-loops at x, denoted by $\{\gamma_x(s)=(x_1(s),x_2(s),x_3(s))\}_s$, with $\ell(\gamma_x)<\overline{\sigma}_f$, such that the following is true: let $\psi_x := \psi(f,x,\gamma_x)$ be given by (8.1). Then for any $(i,\mathcal{B},\{s_t\}_{t\in\mathcal{B}})\in\Gamma$ (defined in (7.4)), there exist $t,t'\in\mathcal{B}$ such that $\psi_x(s_t)\neq\psi_x(s_{t'})$.

Proposition 8.4. *If f satisfies property $(\mathcal{P})$, then f is C^1-stably accessible.*

Proof. Assume that f satisfies $(\mathcal{P})$ for $0<\theta<\theta'<1$, $k\geq 1$, some (θ,k)-spanning c-family for f, denoted by $\mathcal{D}$, and a set of families of f-loops $\{\gamma_x\}_{x\in\theta'\mathcal{C},\ \mathcal{C}\in\mathcal{D}}$. Take $\mathcal{C}\in\mathcal{D}$, $x\in\theta\mathcal{C}$, and set $\psi_x := \psi_x(f,x,\gamma_x)$. We claim that $Acc_f(x)$ is open.

To see this, take an arbitrary $s\in[0,1]^c$. By property (3) of the open cover $\{\mathscr{U}_\alpha\}_{\alpha\in\mathcal{A}}$ in Subsection 7.2, there exists a subset $\mathcal{I}\subset\mathcal{A}$ with $|\mathcal{I}|\geq K_1$, such that $s\in\mathscr{U}_\alpha$ for all $\alpha\in\mathcal{I}$, and $\{\partial\mathscr{U}_\alpha\}_{\alpha\in\mathcal{I}}$ are mutually disjoint. Let us show that $\cap_{\alpha\in\mathcal{I}}\psi_x(\partial\mathscr{U}_\alpha)=\emptyset$. Assume it is not true. By (7.1) and the pigeonhole principle, we may choose $i\in\{1,\dots,c\}$ and $\mathcal{I}'\subset\mathcal{I}$ with $|\mathcal{I}'|=K_0$, such that $\cap_{\alpha\in\mathcal{I}'}\psi_x(\partial^i\mathscr{U}_\alpha)\neq\emptyset$. Thus there exists $(i,\mathcal{B},\{s_t\}_t)\in\Gamma$ such that

$$\psi_x(s_t)=\psi_x(s_{t'}), \quad \forall\, t,t'\in\mathcal{B},$$

which contradicts $(\mathcal{P})$. Therefore, $\cap_{\alpha\in\mathcal{I}}\psi_x(\partial\mathscr{U}_\alpha)=\emptyset$. Since s can be taken arbitrary in $[0,1]^c$, Corollary 7.5 implies that ψ_x has a stable value y, and thus, $\mathrm{Im}(\psi_x)$ contains an open neighbourhood of $\{y\}$ in $\mathcal{W}^c_f(x)$. But ψ_x takes values in $\mathcal{W}^c_f(x)\cap Acc_f(x)$, hence the latter has non-empty interior. Saturating by local stable and unstable leaves, we deduce that $Acc_f(x)$ has non-empty interior. Then the accessibility class $Acc_f(x)$ is open, and the claim is proved.

Since $\mathcal{D}$ is (θ,k)-spanning, the previous claim implies that for any $x\in X$, $Acc_f(x)$ is open. This shows that f is accessible, since X is connected.

Now it suffices to show that $(\mathcal{P})$ is a C^1-open condition. Let $\varsigma_f>0$ be as in Lemma 8.2 and let $\sigma\in(0,\varsigma_f)$ be a small constant to be determined. Let $\theta''':=\frac{\theta+\theta'}{2}$. By Lemma 4.9 and Remark 4.10, for any g sufficiently C^1-close to f, there exists $\mathcal{D}_g$, a $(\theta''',k+2)$-spanning c-family for g, such that for each $\mathcal{C}_g\in\mathcal{D}_g$, there exists $\mathcal{C}\in\mathcal{D}$ so that $\theta'\mathcal{C}_g\in(\theta'\mathcal{C},\sigma)$. For each $y\in\theta'\mathcal{C}_g$, take $x\in\theta'\mathcal{C}$ with $y\in B(x,\sigma)\subset B(x,\varsigma_f)$. Applying Lemma 8.2 to γ_x, we obtain $\gamma_{g,y}$, a continuous family of g-loops at y which is close to γ_x. By choosing σ sufficiently small, we can ensure that for any g sufficiently close to f in C^1 topology, any $\mathcal{C}_g\in\mathcal{D}_g$, $y\in\theta'\mathcal{C}_g$, any $(i,\mathcal{B},\{s_t\}_{t\in\mathcal{B}})\in\Gamma$, there exists $t,t'\in\mathcal{B}$ such that $\tilde{\psi}_y(s_t)\neq\tilde{\psi}_y(s_{t'})$, where $\tilde{\psi}_y:=\psi(g,y,\gamma_{g,y})$. Thus $(\mathcal{P})$ is a C^1-open condition. □

8.3. **Parametrising an accessibility set using a family of loops.** To optimize the pinching exponents in our theorems, we will mainly consider the class of continuous families of loops as follows.

DEFINITION 8.5 (Regular family). Given $x \in X$ and constants $\sigma \in \left(0, \frac{\overline{\sigma}_f}{2C_f}\right)$, $C > 0$, a continuous family $\{\gamma(s) = (x_1(s), x_2(s), x_3(s))\}_{s\in[0,1]}$ of f-loops at x is said to be (σ, C)-*regular* if it satisfies:

(1) $\ell(\gamma) < \sigma$ and the map $s \mapsto x_1(s)$ is C-Lipschitz (with respect to $d_{\mathcal{W}_f^u}$);
(2) there exists $x' \in \mathcal{W}_f^{cs}(x, \frac{\sigma_f}{2C_f})$ such that $x_2(s) \in \mathcal{W}_f^{cu}(x', \sigma)$ for all $s \in [0,1]$.

In this case, we say that γ is *determined by* x' and $(x_1(s))_{s\in[0,1]}$. Indeed, for any $s \in [0,1]$, $x_2(s)$ is the unique intersection of $\mathcal{W}_f^s(x_1(s), h_f)$ and $\mathcal{W}_f^{cu}(x', h_f)$, and $x_3(s)$ is the unique intersection of $\mathcal{W}_f^u(x_2(s), h_f)$ and $\mathcal{W}_f^{cs}(x, h_f)$.[9]

We now restrict our attention to maps in the region defined as follows.

NOTATION 8.6. Assume that $f_0 \in \mathcal{PH}^2(X)$ is dynamically coherent and center bunched. We consider the following cases:

(1) If $\dim E_{f_0}^c = 1$ and f is plaque expansive, then we let $\mathbb{U}(f_0)$ be a C^1-open neighbourhood of f_0 in which all maps are plaque expansive;
(2) If $\dim E_{f_0}^c \geq 2$ and satisfies (ae) or (be), then we let $\mathbb{U}(f_0)$ be a C^1-open neighbourhood of f_0 such that $d_{C^1}(f_0, f) < \varepsilon_{f_0}$ for any $f \in \mathbb{U}(f_0)$ (see Notation 3.12(5)).

Moreover, we assume that $\mathbb{U}(f_0)$ is small enough so that any $g \in \mathbb{U}(f_0)$ is θ'_{f_0}-pinched and center bunched; and the constants $h_{f_0}, \sigma_{f_0}, C_{f_0}, \overline{\Lambda}_{f_0}$ in Notation 3.12 work for any $g \in \mathbb{U}(f_0)$. By points (1), (2), (4), (5) in Notation 3.12, such $\mathbb{U}(f_0)$ exists. We stress that we do *not* require Λ_f to be uniformly bounded for $f \in \mathbb{U}(f_0)$.

In the rest of this section, we fix a map f_0 as in Notation 8.6, and assume that $f \in \mathbb{U}(f_0)$. We also fix $x \in X$, $C > 0$, $\sigma \in (0, \frac{\overline{\sigma}_f}{2C_f})$, and take $\{\gamma(s) = (x_1(s), x_2(s), x_3(s))\}_{s\in[0,1]}$, a (σ, C)-regular family of f-loops at x, determined by $x' \in \mathcal{W}_f^{cs}(x, \frac{1}{2}\sigma_f)$ and $(x_1(s))_{s\in[0,1]}$. Let $\hat{f}\colon U \times X \to X$ be a C^2-deformation at (a, f), and set $T = T(\hat{f})$. We will always assume that U is conveniently small so that for all $b \in U$, $\hat{f}(b, \cdot) \in \mathbb{U}(f_0)$ and the C^2-uniform constant Λ_f continues to work for $\hat{f}(b, \cdot)$. We now define a continuation of γ as follows.

DEFINITION 8.7. We define a lift of $\{\gamma(s)\}_s$ by

$$\hat{\gamma}(s) = ((a, x_1(s)), (a, x_2(s)), (a, x_3(s))), \quad \forall\, s \in [0,1]. \tag{8.2}$$

Then by continuity, there exists a C^2-uniform constant $\delta_{a,T} = \delta_{a,T}(T) > 0$ such that $B(a, \delta_{a,T}) \subset U$, and for any $(b, y) \in \mathcal{W}_T^c((a, x), \delta_{a,T})$, any $s \in [0,1]$, each of the following intersections exists and is unique:

(1) $\{(b, \hat{x}_1(b, y, s))\} := \mathcal{W}_T^u((b, y), h_f) \cap \mathcal{W}_T^{cs}((a, x_1(s)), h_f)$;
(2) $\{(b, \hat{x}_2(b, y, s))\} := \mathcal{W}_T^s((b, \hat{x}_1(b, y, s)), h_f) \cap \mathcal{W}_T^{cu}((a, x'), h_f)$;
(3) $\{(b, \hat{x}_3(b, y, s))\} := \mathcal{W}_T^u((b, \hat{x}_2(b, y, s)), h_f) \cap \mathcal{W}_T^{cs}((a, x), h_f)$.

[9]Indeed, by $\ell(\gamma) < \sigma$ and Notation 3.12, we have $d(x', x_1(s)) < C_f(\sigma + \frac{1}{2}\sigma_f) < \sigma_f$. Then again by Notation 3.12, $d_{\mathcal{W}^s}(x_2(s), x_1(s)) < \sigma$ and $d_{\mathcal{W}^{cu}}(x_2(s), x') < C_f(\sigma + \frac{1}{2}\sigma_f) < \sigma_f$, we see that $x_2(s)$ is uniquely determined. We argue similarly for $x_3(s)$.

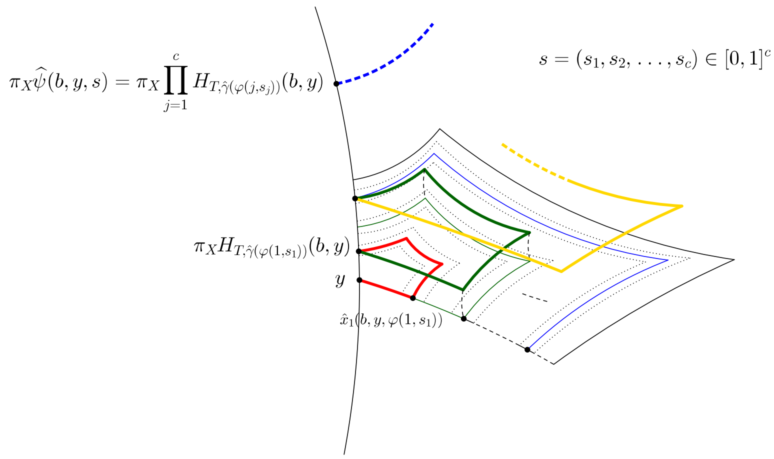

FIGURE 4. Parametrizing a subset of the accessibility class of y for $\hat{f}(b,\cdot)$.

We thus get a continuous family of $\hat{f}(b,\cdot)$-loops at y, denoted by $\{\gamma_{b,y}(s)\}_s$, where

$$\gamma_{b,y}(s) = (\hat{x}_1(b,y,s), \hat{x}_2(b,y,s), \hat{x}_3(b,y,s)), \quad \forall\, s \in [0,1].$$

We further require that $\delta_{a,T} < \varepsilon_{f_0}/10$, and is sufficiently small so that for any $(b,y) \in \mathcal{W}^c_T((a,x),\delta_{a,T})$, we have $\ell(\gamma_{b,y}) < \overline{\sigma}_{\hat{f}(b,\cdot)}$ (see Notation 5.2). Note that in general, $\gamma_{b,y}$ is different from $\gamma_{\hat{f}(b,\cdot),y}$ as defined in Lemma 8.2.

Since $\ell(\gamma) < \sigma < \frac{\overline{\sigma}_f}{2C_f}$, there exists a C^2-uniform constant $\overline{\delta}_{a,T} = \overline{\delta}_{a,T}(T,\sigma) \in (0, \delta_{a,T}/2\Lambda_f)$ such that for any $(b,y) \in \mathcal{W}^c_T((a,x),\overline{\delta}_{a,T})$, we have $\ell(\gamma_{b,y}) < 2\sigma$. In the rest of the proof we will reduce the size of $\overline{\delta}_{a,T}$ finitely many times. By Notation 5.2, the following map is well-defined:

$$\widehat{\psi} = \widehat{\psi}(T)\colon \begin{cases} \mathcal{W}^c_T((a,x),\overline{\delta}_{a,T}) \times [-1,2]^c & \to \quad \mathcal{W}^c_T(a,x), \\ (b,y,s) & \mapsto \quad (\prod_{j=1}^{c} H_{T,\hat{\gamma}(\varphi(j,s_j))})(b,y). \end{cases} \tag{8.3}$$

Moreover, by Lemma 5.3 and Notation 3.12(1) (recall Notation 5.2), we have

$$\pi_X\widehat{\psi}(b,y,\cdot) = \psi(\hat{f}(b,\cdot),y,\gamma_{b,y})(\cdot) \in \mathcal{W}^c_{\hat{f}(b,\cdot)}(y, 2K_f\sigma) \subset B(y, 2C_{f_0}K_f\sigma). \tag{8.4}$$

The following lemma is important. This is the place where several technical conditions introduced earlier come into use.

LEMMA 8.8. *Let $f_0, f, \hat{f}, C, \sigma, x$ be given as above, and assume in addition that $c \geq 2$ (in particular, f_0 satisfies (ae) or (be)) and $\sigma \in \left(0, \frac{1}{100}K_f^{-2}\overline{\Lambda}_{f_0}^{-2}\min(\sigma_{f_0}^2, \delta_{a,T}^2)\right)$. Then there exists a C^2-uniform constant $\widehat{C}_2 = \widehat{C}_2(f) > 0$ such that for any $(b,y) \in \mathcal{W}^c_T((a,x),\overline{\delta}_{a,T})$, any $s, s' \in [-1,2]^c$, we have*

$$d(\widehat{\psi}(b,y,s), \widehat{\psi}(b,y,s')) \leq \widehat{C}_2 C|s-s'|^{\theta_0},$$

where θ_0 is defined for (ae) (resp. for (be)), as

$$\theta_0 = \theta'_{f_0}(\theta''_{f_0})^3 > \frac{c-1}{c} \quad \big(\textit{resp. } \theta_0 = \theta'_{f_0}(\theta''_{f_0})^4 > \frac{c-1}{c}\big). \tag{8.5}$$

Proof. Let $s = (s_k), s' = (s'_k) \in [-1,2]^c$, and for any $j \in \{1,\dots,c\}$, set $t_j := \varphi(j, s_j) \in [0,1]$ and $t'_j := \varphi(j, s'_j) \in [0,1]$. For each $0 \le i \le c$, we let

$$W_i := \prod_{j=1}^{i} H_{T,\hat\gamma(t_j)}(b,y), \quad Z_i := \prod_{j=i+1}^{c} H_{T,\hat\gamma(t'_j)}(W_i).$$

Arguing as in Lemma 5.3, for each $0 \le i \le c$, we see that $\pi_X(W_i) \in \mathcal{W}^c_{\hat f(b,\cdot)}(y, 2K_f\sigma)$, and hence $W_i \in \mathcal{W}^c_T((a,x), \overline{\delta}_{a,T} + 2K_f\sigma) \subset \mathcal{W}^c_T((a,x), \delta_{a,T})$. Similarly, for any $0 \le i \le c-1$, $p \in [0,1]$, we have $H_{T,(a,x),(a,x_1(p))}(W_i) \in \mathcal{W}^c_T((a, x_1(p)), \delta_{a,T})$.

It is direct to see that $Z_0 = \widehat\psi(b,y,s')$, $Z_c = \widehat\psi(b,y,s)$. Thus it is enough to estimate $d(Z_0, Z_c)$. For any $0 \le i \le c-1$, we observe that

$$Z_i = \prod_{j=i+2}^{c} H_{T,\widehat\gamma(t'_j)}(H_{T,\widehat\gamma(t'_{i+1})}(W_i)), \quad Z_{i+1} = \prod_{j=i+2}^{c} H_{T,\widehat\gamma(t'_j)}(H_{T,\widehat\gamma(t_{i+1})}(W_i)).$$

Since f, thus T, are C^2 and center bunched, and the maps $\{H_{T,\widehat\gamma(t'_j)}\}_{j=1,\dots,c}$ are obtained by composing holonomy maps, Notation 3.12(3) yields

$$d(Z_i, Z_{i+1}) \le C_{f_0}\Lambda_f^{4c} d(H_{T,\widehat\gamma(t'_{i+1})}(W_i), H_{T,\widehat\gamma(t_{i+1})}(W_i)).$$

Therefore, it suffices to prove that for some C^2-uniform constant $c_1 = c_1(f_0) > 0$, for any $z \in \mathcal{W}^c_T((a,x), \delta_{a,T})$, any $p, q \in [0,1]$, we have

$$d_{\mathcal{W}^c_{\hat f(b,\cdot)}}(\pi_X(H_{T,\hat\gamma(p)}(z)), \pi_X(H_{T,\hat\gamma(q)}(z))) \le c_1|p-q|^{\theta_0}.$$

Given any $z = (b, \pi_X(z)) \in \mathcal{W}^c_T((a,x), \delta_{a,T})$, any $p, q \in [0,1]$, we set

$$\begin{aligned} z_1 &:= H^u_{T,(a,x),(a,x_1(p))}(z), & z'_1 &:= H^u_{T,(a,x),(a,x_1(q))}(z),\\ z_2 &:= H^s_{T,(a,x_1(p)),(a,x_2(p))}(z_1), & z'_2 &:= H^s_{T,(a,x_1(q)),(a,x_2(q))}(z'_1),\\ z_3 &:= H^u_{T,(a,x_2(p)),(a,x_3(p))}(z_2), & z'_3 &:= H^u_{T,(a,x_2(q)),(a,x_3(q))}(z'_2). \end{aligned}$$

CLAIM 8.9. *We have $d_{\mathcal{W}^u_{\hat f(b,\cdot)}}(\pi_X(z_1), \pi_X(z'_1)) \le \overline{\Lambda}^2_{f_0} C_{f_0} d_{\mathcal{W}^u_f}(x_1(p), x_1(q))^{\theta_1}$. Here $\theta_1 = (\theta''_{f_0})^3$ if (a) is satisfied, otherwise $\theta_1 = (\theta''_{f_0})^4$, when (b) is satisfied.*

Proof. We abbreviate $x_1(p)$ (resp. $x_1(q)$) as x_1 (resp. x'_1). It is clear that $z_1, z'_1 \in \{b\} \times X$. By Notation 3.12(5) and Notation 8.6, we see that there exists a leaf conjugacy between $\mathcal{W}^c_f$ and $\mathcal{W}^c_{\hat f(b,\cdot)}$, denoted by $\mathfrak{h} = \mathfrak{h}_{f,\hat f(b,\cdot)}$, such that $d(\mathfrak{h}(x_1), \mathfrak{h}(x'_1)) < \overline{\Lambda}_{f_0} d_{\mathcal{W}^u_f}(x_1, x'_1)^{(\theta''_{f_0})^2} \le \overline{\Lambda}_{f_0}(2\sigma)^{1/2} < \frac{\sigma_{f_0}}{2}$.

By reducing the size of $\overline{\delta}_{a,T}$ if necessary, we may suppose that both z_1 and $(b, \mathfrak{h}(x_1))$ belong to $W^c_T((a,x_1), \sigma_{f_0}/(4C_{f_0})) \cap \{b\} \times X$. This implies that $\pi_X(z_1) \in \mathcal{W}^c_{\hat f(b,\cdot)}(\mathfrak{h}(x_1), \sigma_{f_0}/2)$. Similarly, $\pi_X(z'_1) \in \mathcal{W}_{\hat f(b,\cdot)}(\mathfrak{h}(x'_1), \sigma_{f_0}/2)$.

By definition, we have $\pi_X(z_1) \in \mathcal{W}^u_{\hat f(b,\cdot)}(\pi_X(z'_1), h_{f_0})$. Then by Notation 3.12(1), (5), we can see that

$$d_{\mathcal{W}^u_{\hat f(b,\cdot)}}(\pi_X(z_1), \pi_X(z'_1)) \le \overline{\Lambda}_{f_0} C_{f_0} d(\mathfrak{h}(x_1), \mathfrak{h}(x'_1))^{\theta_2}, \tag{8.6}$$

where $\theta_2 = \theta''_{f_0}$ if (a) is satisfied; $\theta_2 = (\theta''_{f_0})^2$ if (b) is satisfied. Hence by Notation 3.12(5), the right hand side of (8.6) is at most $\overline{\Lambda}_{f_0}^2 C_{f_0} d_{\mathcal{W}_f^u}(x_1, x_1')^{(\theta''_{f_0})^2\theta_2} = \overline{\Lambda}_{f_0}^2 C_{f_0} d_{\mathcal{W}_f^u}(x_1, x_1')^{\theta_1}$. □

By Notation 3.12(4), Notation 8.6, Claim (8.9) and $\sigma < \left(\frac{\sigma_{f_0}}{10\Lambda_f \overline{\Lambda}_{f_0}^2 C_{f_0}}\right)^2$, we obtain

$$\begin{aligned} d_{\mathcal{W}^{cu}_{\hat{f}(b,\cdot)}}(\pi_X(z_2), \pi_X(z_2')) &\leq \Lambda_f d_{\mathcal{W}^{cu}_{\hat{f}(b,\cdot)}}(\pi_X(z_1), \pi_X(z_1'))^{\theta'_{f_0}} \\ &\leq \Lambda_f \overline{\Lambda}_{f_0}^2 C_{f_0} d_{\mathcal{W}_f^u}(x_1(p), x_1(q))^{\theta'_{f_0}\theta_1} < \sigma_{f_0}. \end{aligned} \tag{8.7}$$

By Notation 3.12(2),(3), and since f is C^2 and center bunched, we get

$$\begin{aligned} d_{\mathcal{W}^{c}_{\hat{f}(b,\cdot)}}(\pi_X(H_{T,\hat{\gamma}(p)}(z)), \pi_X(H_{T,\hat{\gamma}(q)}(z))) &\leq \Lambda_f d_{\mathcal{W}^{c}_{\hat{f}(b,\cdot)}}(\pi_X(z_3), \pi_X(z_3')) \\ &\leq \Lambda_f^2 C_{f_0} d_{\mathcal{W}^{cu}_{\hat{f}(b,\cdot)}}(\pi_X(z_2), \pi_X(z_2')). \end{aligned} \tag{8.8}$$

Since γ is (σ, C)-regular, we have $d_{\mathcal{W}_f^u}(x_1(p), x_1(q)) \leq C|p-q|$. We conclude the proof by (8.7), (8.8) and by noting that $\theta_0 = \theta'_{f_0}\theta_1$. □

9. Constructing charts and vector fields

In order to construct infinitesimal deformations with required properties, we will first introduce coordinates in a neighbourhood of each c-disk. In this section, we assume that (H1) holds, and $r \geq 2$.

In the following, our goal is to define certain vector fields in order to perturb the dynamics and induce a displacement of the holonomies. More precisely, given a small center disk, we define a vector field localized close to the disk. These vector fields will be rich enough for us to apply Proposition 5.6.

Construction 9.1. There exist C^2-uniform constants $\overline{h}_f \in (0, h_f)$, $\overline{C}_f > 1$ such that the following is true. For any c-disk of f, denoted by $\mathcal{C} = \mathcal{W}_f^c(x, h)$, with $x \in X$ and $h \in (0, \overline{h}_f)$, there exists a C^r volume preserving map $\phi = \phi(\mathcal{C})\colon (-h,h)^d \to X$ such that $\phi(0) = x$ and

1. $\frac{1}{5}\mathcal{C} \subset \phi((-h/4, h/4)^c \times \{0\}^{d_u+d_s}) \subset \phi((-2h/3, 2h/3)^c \times \{0\}^{d_u+d_s}) \subset \mathcal{C}$;
2. $\|\phi\|_{C^2} < \overline{C}_f$;
3. $D\phi(0, \mathbb{R}^c \times \{0\}^{d_u+d_s})$, $D\phi(0, \{0\}^c \times \mathbb{R}^{d_u} \times \{0\}^{d_s})$, $D\phi(0, \{0\}^{c+d_u} \times \mathbb{R}^{d_s})$ are respectively equal to $E_f^c(x), E_f^u(x), E_f^s(x)$;
4. for any $y \in \phi((-h,h)^d)$, $\Pi_c D_y(\phi^{-1})\colon E_f^c(y) \to \mathbb{R}^c$ has determinant in $(\overline{C}_f^{-1}, \overline{C}_f)$, where $\Pi_c\colon \mathbb{R}^d \simeq \mathbb{R}^c \times \mathbb{R}^{d_u} \times \mathbb{R}^{d_s} \to \mathbb{R}^c$ is the canonical projection;
5. for any $\zeta > 0$, there exists a C^1-uniform constant $\overline{h}_{f,\zeta} \in (0, \overline{h}_f)$ so that if $h \in (0, \overline{h}_{f,\zeta})$, then for any $y \in \phi((-h,h)^d)$, $\Pi_c D_y(\phi^{-1})\colon E_f^{su}(y) \to \mathbb{R}^c$ has norm smaller than ζ.

The above charts exist. Indeed, for some C^2-uniform constant $\overline{C}'_f > 0$, for any $x \in X$, we can choose a C^∞ volume preserving diffeomorphism $\phi'\colon (-2h, 2h)^d \to X$ satisfying: $\phi'(0) = x$; (2),(3) for $(\phi', \overline{C}'_f)$ in place of $(\phi, \overline{C}_f)$; and that $D\phi'(0)$ is an isometry restricted to $\mathbb{R}^c \times \{0\}^{d_u+d_s}$, etc. Then, for sufficiently small h, $(\phi')^{-1}(\mathcal{C})$ is contained in the graph of a C^r map $\psi\colon (-2h, 2h)^c \to (-2h, 2h)^{d_u+ds}$

with $\psi(0) = 0$ and $\|\psi\|_{C^2} < \overline{C}''_f$ for some C^2-uniform constant $\overline{C}''_f > 0$. For $x = (x_c, x_{us}) \in (-h,h)^c \times (-h,h)^{d_u+d_s}$, we define $\phi(x) = \phi'(x_c, x_{us} + \psi(x_c))$. It is direct to verify (1)-(5) by taking $\overline{h}_f$ sufficiently small, and $\overline{C}_f$ sufficiently large.

For $* = u, s$, set $e_* := (1, 0, \dots, 0) \in \mathbb{R}^{d_*}$. For any $0 < \lambda < h < \overline{h}_f$, we define

$$\mathcal{W}^{cs}(\lambda) := \phi((-h,h)^c \times \{\lambda e_u\} \times (-h,h)^{d_s}),$$
$$\mathcal{W}^{cu}(\lambda) := \phi((-h,h)^c \times (-h,h)^{d_u} \times \{\lambda e_s\}).$$

We construct regular families of loops as follows.

LEMMA 9.2. *There exist C^2-uniform constants $\widetilde{h}_f \in (0, \overline{h}_f)$, $\widetilde{C}_f > 0$ such that the following is true. For any $\rho \in (0, \widetilde{h}_f)$, any $\sigma \in (0, \widetilde{C}_f^{-1}\rho)$, there exist constants $\hat{\varepsilon}_0 = \hat{\varepsilon}_0(f, \rho, \sigma) > 0$, $\hat{\sigma}_0 = \hat{\sigma}_0(f, \rho, \sigma) \in (0, \sigma)$, such that for any c-disk $\mathcal{C}$ of f with radius $h \in (\rho, \widetilde{h}_f)$; any $g \in \mathcal{PH}^1(X)$ such that $d_{C^1}(f, g) < \hat{\varepsilon}_0$; any $x \in (\frac{1}{5}\mathcal{C}, \hat{\sigma}_0)$, there exists $\{\gamma(s) = (x_1(s), x_2(s), x_3(s))\}_{s\in[0,1]}$, a $(\sigma, \widetilde{C}_f)$-regular family of g-loops at x with the following properties: let $\phi = \phi(\mathcal{C})$, $\sigma' := \widetilde{C}_f^{-\frac{1}{2}}\sigma$; then we have*

- *(i) for any $s \in [0,1]$, any $i = 2, 3$, $\mathcal{W}^c_g(x_i(s), K_f\sigma)$ is disjoint from the image $\phi((-h,h)^{c+d_u} \times (-\frac{\sigma'}{2}, \frac{\sigma'}{2})^{d_s})$;*
- *(ii) take $C_{\min}$ as in (7.3). For any $s \in [0,1]$, we have*

$$\mathcal{W}^c_g(x_1(s), K_f\sigma) \subset \phi\big((-h/2, h/2)^c \times (\sigma' s e_u + (-C_{\min}^{-1}\sigma', C_{\min}^{-1}\sigma')^{d_u}) \times (-\frac{\sigma'}{5}, \frac{\sigma'}{5})^{d_s}\big).$$

Proof. Set $\{x'\} := \mathcal{W}^s_g(x, h_f) \cap \mathcal{W}^{cu}(\sigma')$. By Notation 3.12, Construction 9.1(2),(3), and by taking $\widetilde{C}_f$ sufficiently large, $\widetilde{h}_f, \hat{\varepsilon}_0, \hat{\sigma}_0$ sufficiently small, we have:

(1) For each $s \in [0,1]$, each of the following intersections exists and is unique:
   (a) $\{x_1(s)\} := \mathcal{W}^u_g(x, h_f) \cap \mathcal{W}^{cs}(s\sigma')$;
   (b) $\{x_2(s)\} := \mathcal{W}^s_g(x_1(s), h_f) \cap \mathcal{W}^{cu}_g(x', h_f)$;
   (c) $\{x_3(s)\} := \mathcal{W}^u_g(x_2(s), h_f) \cap \mathcal{W}^{cs}_g(x, h_f)$.

(2) For each $s \in [0,1]$, set $\gamma(s) := (x_1(s), x_2(s), x_3(s))$. Then $\gamma := \{\gamma(s)\}_{s\in[0,1]}$ is a $(\sigma, \widetilde{C}_f)$-regular family for g.

Note that for any $s \in [0,1]$, $x_2(s), x_3(s) \in \mathcal{W}^{cu}_g(x', \widetilde{C}_f\sigma')$ for sufficiently large $\widetilde{C}_f$. We get (i) by the continuity of E^{cu}_f, and by letting $\widetilde{h}_f$, $\hat{\varepsilon}_0$, $\hat{\sigma}_0$ be sufficiently small. We get (ii) by the continuity of E^{cs}_f and E^c_f; by $\widetilde{C}_f\sigma < h < \widetilde{h}_f$; by letting $\widetilde{C}_f$ be sufficiently large, and then letting $\widetilde{h}_f, \hat{\varepsilon}_0, \hat{\sigma}_0$ be sufficiently small. ☐

CONSTRUCTION 9.3. For any c-disk $\mathcal{C}$ such that $\varrho(\mathcal{C}) =: h \in (0, \overline{h}_f)$, with $\overline{h}_f$ as in Construction 9.1, for any $\sigma \in (0, h)$, we define a collection of vector fields as follows.

• For each $1 \le j \le c$, let $U_j\colon (-2/3, 2/3)^c \times (-1,1)^{d_u} \times (-1/3, 1/3)^{d_s} \to \mathbb{R}^d$ be a compactly supported C^∞ divergence-free vector field such that U_j restricted to $(-1/2, 1/2)^{c+d_u} \times (-1/5, 1/5)^{d_s}$ is equal to the constant vector, denoted by E_j, that has 1 at j-th coordinate and 0 at the others. Such U_j always exists since $d \ge 3$. Moreover, we can assume that U_j satisfies $\|U_j\|_{C^1} < C_*$ for some constant $C_* = C_*(d) > 0$.

For any $x_c \in \mathbb{R}^c$, $x_u \in \mathbb{R}^{d_u}$, $x_s \in \mathbb{R}^{d_s}$, $a_c, a_u, a_s > 0$, we denote for every $z_c \in \mathbb{R}^c, z_u \in \mathbb{R}^{d_u}, z_s \in \mathbb{R}^{d_s}$:

$$P_{x_c,a_c,x_u,a_u,x_s,a_s}(z_c, z_u, z_s) = (x_c + a_c z_c, x_u + a_u z_u, x_s + a_s z_s).$$

Now, for any $i, j \in \{1, \dots, c\}$, any $t \in \mathcal{B}_i$, we let $U^\sigma_{\mathcal{C},i,t,j} \colon (-h,h)^d \to \mathbb{R}^d$ be the vector field

$$U^\sigma_{\mathcal{C},i,t,j} = U_j(P_{\underline{0^c},h,\varphi(i,t)\sigma e_u,2C^{-1}_{\min}\sigma,\underline{0^{d_s}},\sigma})^{-1}.$$

The support of $U^\sigma_{\mathcal{C},i,t,j}$ is contained in

$$(-2h/3, 2h/3)^c \times (\varphi(i,t)\sigma e_u + 2C^{-1}_{\min}(-\sigma,\sigma)^{d_u}) \times (-\sigma/3, \sigma/3)^{d_s}.$$

Moreover, for any $z_c \in (-h/2, h/2)^c$, $z_u \in \varphi(i,t)\sigma e_u + C^{-1}_{\min}(-\sigma,\sigma)^{d_u}$ and $z_s \in (-\sigma/5, \sigma/5)^{d_s}$, we have

$$U^\sigma_{\mathcal{C},i,t,j}(z_c, z_u, z_s) = E_j.$$

We set

$$V^\sigma_{\mathcal{C},i,t,j} := D\phi(U^\sigma_{\mathcal{C},i,t,j}).$$

By Construction 9.1, and the C^1-bound on U_j above, we see that the vector field $V^\sigma_{\mathcal{C},i,t,j}$ is divergence-free and satisfies:

$$\sigma\|\partial_x V^\sigma_{\mathcal{C},i,t,j}\|_X + \|V^\sigma_{\mathcal{C},i,t,j}\|_X < \widehat{C}_f, \tag{9.1}$$

for some C^2-uniform constant $\widehat{C}_f > 0$, independent of $\mathcal{C}, i, t, j$.

Remark 9.4. *By construction, it is clear that*

$$\mathrm{supp}_X(V^\sigma_{\mathcal{C},i,t,j}) \subset \phi((-2h/3, 2h/3)^c \times (-2\sigma, 2\sigma)^{d_u} \times (-\sigma/3, \sigma/3)^{d_s}).$$

Thus for any $\sigma_0 > 0$, there exists $\sigma > 0$ such that for any $\mathcal{C}, i, j, t$ in Construction 9.3,

$$\mathrm{supp}_X(V^\sigma_{\mathcal{C},i,t,j}) \subset (\mathcal{C}, \sigma_0).$$

The following lemma describes the values taken by $V^\sigma_{\mathcal{C},i,t,j}$ at the corners of the loops that we constructed in Lemma 9.2.

Lemma 9.5. *There exists a C^2-uniform constant $\kappa' = \kappa'(f) > 0$ such that for any $\rho_1 \in (0, \frac{1}{2}\widetilde{h}_f)$, any $\sigma \in (0, \widetilde{C}_f^{-1}\rho_1)$, there exists $\mathcal{U}' = \mathcal{U}'(f, \rho_1, \sigma)$, a C^2 open neighbourhood of f, such that for any $g \in \mathcal{U}'$, for any $\left(\frac{1}{20}, 6\right)$-spanning c-family for f denoted by $\mathcal{D}$, satisfying $[\underline{r}(\mathcal{D}), \overline{r}(\mathcal{D})] \subset (\rho_1, \frac{1}{2}\widetilde{h}_f)$, there exists $\mathcal{D}'$, a $\left(\frac{1}{10}, 8\right)$-spanning c-family for g with $[\underline{r}(\mathcal{D}'), \overline{r}(\mathcal{D}')] \subset (\rho_1, \widetilde{h}_f)$ satisfying the following property.*

For any $\mathcal{C}' \in \mathcal{D}'$, we have $\mathcal{C}' \subset (\mathcal{C}, \sigma)$ for some $\mathcal{C} \in \mathcal{D}$, and for each $x \in \frac{1}{5}\mathcal{C}'$, there exists a $(\sigma, \widetilde{C}_f)$-regular continuous family of g-loops at x, denoted by $\{\gamma(s) = (x_1(s), x_2(s), x_3(s))\}_{s\in[0,1]}$, such that for any $i \in \{1, \dots, c\}$, $s \in [0,1]$, $t \in \mathcal{B}_i$, any $y \in \mathcal{W}^c_g(x, K_f\sigma) \cup \mathcal{W}^c_g(x_2(s), K_f\sigma) \cup \mathcal{W}^c_g(x_3(s), K_f\sigma)$, and any $z \in \mathcal{W}^c_g(x_1(\varphi(i,t)), K_f\sigma)$, by letting $\sigma' := \widetilde{C}_f^{-\frac{1}{2}}\sigma$, we have

$$V^{\sigma'}_{\mathcal{C},i,t,j}(y) = 0,\ \forall\, 1 \le j \le c, \qquad |\det\left((\pi_c V^{\sigma'}_{\mathcal{C},i,t,j}(z))_{j=1,\dots,c}\right)| > \kappa'. \tag{9.2}$$

Proof. Take $\varepsilon_1 := \hat{\varepsilon}_0(f, \rho_1, \sigma)$ and $\sigma_1 := \hat{\sigma}_0(f, \rho_1, \sigma) \in (0, \sigma)$ given by Lemma 9.2.

By Lemma 4.9 and Remark 4.10 applied to f, and $(6, \frac{1}{20}, \frac{1}{10}, \frac{1}{5}, \rho_1, \widetilde{h}_f, \sigma_1)$ in place of $(k, \theta, \theta', \theta'', \rho_m, \rho_M, \sigma)$, for all g sufficiently C^1-close to f, there exists $\mathcal{D}'$, a $(\frac{1}{10}, 8)$-spanning c-family for g, with $[\underline{r}(\mathcal{D}'), \overline{r}(\mathcal{D}')] \subset (\rho_1, \widetilde{h}_f)$, such that for each $\mathcal{C}' \in \mathcal{D}'$, we have $\mathcal{C}' \subset (\mathcal{C}, \sigma_1) \subset (\mathcal{C}, \sigma)$ and $\frac{1}{5}\mathcal{C}' \subset (\frac{1}{5}\mathcal{C}, \sigma_1) \subset (\frac{1}{5}\mathcal{C}, \sigma)$ for some $\mathcal{C} \in \mathcal{D}$.

We can apply Lemma 9.2 to any c-disk $\mathcal{C}$ of f, since it has radius in $(\rho_1, \frac{1}{2}\widetilde{h}_f)$, and any $x \in \frac{1}{5}\mathcal{C}'$, to construct $\{\gamma(s)\}_{s\in[0,1]}$, a $(\sigma, \widetilde{C}_f)$-regular continuous family of g-loops at x such that if i, j, s, t, σ' are as in the lemma, we have

(1) $\mathcal{W}^c_g(x_i(s), K_f\sigma)$, $i = 2, 3$, are disjoint from $\phi((-h, h)^{c+d_u} \times (-\frac{\sigma'}{2}, \frac{\sigma'}{2})^{d_s})$;
(2) $\mathcal{W}^c_g(x, K_f\sigma) \subset \phi((-h/2, h/2)^c \times (-C^{-1}_{\min}\sigma', C^{-1}_{\min}\sigma')^{d_u} \times (-\frac{\sigma'}{5}, \frac{\sigma'}{5})^{d_s})$;
(3) $\mathcal{W}^c_g(x_1(\varphi(i,t)), K_f\sigma) \subset \phi\big((-h/2, h/2)^c \times (\sigma'\varphi(i,t)e_u + + (-C^{-1}_{\min}\sigma', C^{-1}_{\min}\sigma')^{d_u}) \times (-\frac{\sigma'}{5}, \frac{\sigma'}{5})^{d_s}\big)$.

By Construction 9.1(4) and Construction 9.3, we see that (9.2) holds for some C^2-uniform constant $\kappa' > 0$ depending only on $f, \overline{C}_f$. □

10. On the prevalence of the accessibility property

In this section, we fix $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ and an integer $J \geq 1$. In order to avoid repetition, we consider only the volume preserving case in the following. The more general case is handled by repeating exactly the same proof after replacing $\mathrm{Diff}^r(X, \mathrm{Vol})$ by $\mathrm{Diff}^r(X)$, $\mathcal{PH}^r(X, \mathrm{Vol})$ by $\mathcal{PH}^r(X)$, etc.

Let us first give an outline of the construction in this section with an illustration in Figure 5. Given a good $C^r - J-$family $\mathbf{f} := \{f_\omega\}_{\omega\in[0,1]^J}$, we will find a family $\{\hat{\mathbf{f}}^\theta\}_{\theta\in U_1\subset\mathbb{R}^I}$ of $C^r - J-$families which are perturbations of $\mathbf{f}$, in which most $\hat{\mathbf{f}}^\theta$ contain a large proportion of accessible maps. More precisely, we will construct a $C^r - (I+J)-$family $\hat{\mathbf{f}} := \{f_{(\omega,\theta)}\}_{(\omega,\theta)\in[0,1]^J\times U_1}$, where U_1 is a neighbourhood of the origin $0 \in \mathbb{R}^I$, such that $\{f_{(\omega,0)}\}_{\omega\in[0,1]^J} = \mathbf{f}$.

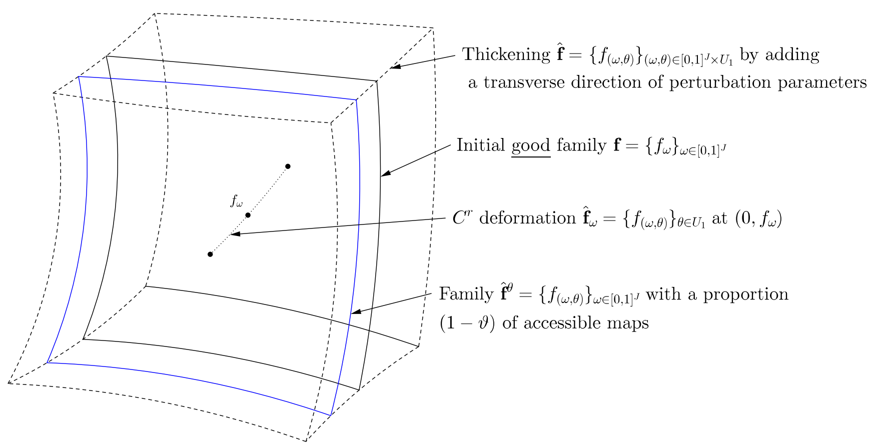

Figure 5. Selection of a perturbed family $\hat{\mathbf{f}}^\theta$ with many accessible maps.

We construct $\hat{\mathbf{f}}$ in the following way. We apply Proposition 6.3 repeatedly to produce a well-distributed finite subset A in $[0,1]^J$ such that for each parameter $a \in A$ we get a $(\frac{1}{20}, 6)$-spanning c-family $\mathcal{D}_a$ for f_a, and produce by Lemma 9.5 a vector field $V^\sigma_{\mathcal{C},i,t,j}$ for each $\mathcal{C} \in \mathcal{D}_a$, $1 \leq i,j \leq c$, $t \in \mathcal{B}_i$ and some small $\sigma > 0$. We construct a C^r map $V\colon [0,1]^J \times \mathbb{R}^I \times X \to TX$ by gluing together the above data in a careful way, and define $\hat{\mathbf{f}} = \{f_{(\omega,\theta)}\}_{(\omega,\theta)\in[0,1]^J\times U_1}$ so that for each $\omega \in [0,1]^J$, $\hat{\mathbf{f}}_\omega := \{f_{(\omega,\theta)}\}_{\theta\in U_1}$ is a C^r deformation at $(0, f_\omega)$ with I-parameters generated by $V(\omega,\cdot,\cdot)$. By choosing V carefully, we may ensure that for a typical parameter $\omega \in [0,1]^J$, the C^r deformation $\hat{\mathbf{f}}_\omega$ exhibits approximately independent perturbations for many different su-paths. Together with a Fubini's argument, this will enable us to verify property $(\mathcal{P})$ for maps at all but extremely small amount of parameters within a typical family $\hat{\mathbf{f}}^\theta := \{f_{(\omega,\theta)}\}_{\omega\in[0,1]^J}$. Notice that the maps in $\hat{\mathbf{f}}$ have uniformly bounded C^r-norms. Thus, by studying carefully the proofs of this section, we can see that throughout this paper we only need to use the fact that h_f, σ_f, C_f and Λ_f are C^2-uniform constants.

10.1. **Constructing perturbations for a family of diffeomorphisms.** In this subsection, we fix a C^r-J-family $\{f_\omega\}_{\omega\in[0,1]^J}$ in the space of dynamically coherent, center bunched C^r partially hyperbolic diffeomorphisms on X.

Let Ω_0 be an open set compactly supported in $(0,1)^J$, let U_1 be an open neighbourhood of the origin in $\mathbb{R}^I$ for some integer $I\geq 1$, and let $\hat{f}\colon \Omega_0\times U_1\times X\to X$ be a C^r map such that $\hat{f}(a,b,\cdot)\in \mathrm{Diff}^r(X)$, for all $(a,b)\in\Omega_0\times U_1$, and $\hat{f}(a,0,\cdot)=f_a$ for all $a\in\Omega_0$. In particular, for any $a\in\Omega_0$, the map $\hat{f}(a,\cdot)\colon U_1\times X\to X$ is a C^r-deformation at $(0,f_a)$. We set $T_a := T(\hat{f}(a,\cdot))$. Moreover, by applying Lemma 4.11 to $\hat{f}(a,\cdot)$ in place of $\hat{f}$, after taking U_1 sufficiently small, for any $(b,x)\in U_1\times X$, we will denote by $\nu^a_b(x,\cdot)\colon \mathbb{R}^I\to E^{su}_{\hat{f}(a,b,\cdot)}(x)$ the unique linear map such that

$$E^c_{T_a}(b,x) = Graph(\nu^a_b(x,\cdot))\oplus E^c_{\hat{f}(a,b,\cdot)}(x). \tag{10.1}$$

Given an element of a C^r-J-family as above, the following notion combines a global property (through spanning c-families) and a local one (existence of deformations which induce an infinitesimal displacement of the holonomies in many directions) which together will be useful to verify Property $(\mathcal{P})$ in Definition 8.3.

Definition 10.1 (Removability). Let $\hat{f}$ be as above. Then for $\rho_m,\rho_M,\sigma,C,\kappa>0$, $a\in\Omega_0$, we say that $\hat{f}$ is $(\rho_m,\rho_M,\sigma,C,\kappa)$-Removable at a if the following is true. There exists $\mathcal{D}$, a $(\frac{1}{10},8)$-spanning c-family for f_a with $[\underline{r}(\mathcal{D}),\overline{r}(\mathcal{D})]\subset(\rho_m,\rho_M)$, such that for each $\mathcal{C}\in\mathcal{D}$, for each $x\in\frac{1}{5}\mathcal{C}$, there exists a (σ,C)-regular continuous family γ of f_a-loops at x with the following properties. Let K_0,Γ be taken as in Subsection 7.2. For any $(i,\mathcal{B},\{s_t\}_{t\in\mathcal{B}})\in\Gamma$ and $(t,j)\in\mathcal{B}\times\{1,\dots,c\}$, we set $\gamma_{t,j}:=\gamma(\varphi(j,s_{t,j}))$, $z_t := (\prod_{j=1}^c H_{f_a,\gamma_{t,j}})(x)$. Let $\hat{\gamma}_{t,j}$ be the lift of $\gamma_{t,}$ for T_a, and

$$\Xi_{a,x}\colon \begin{cases} T_0U_1 & \to \quad \prod_{t\in\mathcal{B}} E^c_{f_a}(z_t)\simeq \mathbb{R}^{K_0c}, \\ \quad B & \mapsto \quad \left[\pi_c\big(D\left(\prod_{j=1}^c H_{T_a,\hat{\gamma}_{t,j}}\right)\cdot(B+\nu^a_0(x,B))\big)\right]_{t\in\mathcal{B}}. \end{cases} \tag{10.2}$$

Then there exists a linear subspace $H\subset\mathbb{R}^I$ of dimension K_0c such that

$$|\det(\Xi_{a,x}|_H)|>\kappa.$$

The main goal of this subsection is the following.

Proposition 10.2. *Assume that $\{f_\omega\}_{\omega\in[0,1]^J}$ is a good C^r-J-family. Then there exist constants $Q,C,\kappa_1>0$ such that for any $\vartheta>0$, any sufficiently small $\widetilde{h}>0$, there exists $\rho_1\in(0,\frac{1}{2}\widetilde{h})$ such that for any sufficiently small $\sigma>0$, there exist*

1. *an open set Ω_0 compactly contained in $(0,1)^J$, with $\mathrm{Leb}([0,1]^J\backslash\Omega_0)<\vartheta$;*
2. *an integer $I>0$, and an open neighbourhood U_1 of the origin in $\mathbb{R}^I$;*
3. *a C^r map $\hat{f}\colon[0,1]^J\times U_1\times X\to X$ such that*

(i) $\hat{f}(a,b,\cdot) \in \mathrm{Diff}^r(X,\mathrm{Vol})$ *and* $\hat{f}(a,0,\cdot) = f_a$, *for all* $(a,b) \in [0,1]^J \times U_1$*;*
(ii) $\|\hat{f}\|_{C^1} < Q$*;*
(iii) $\hat{f}$ *is* $(\rho_1, \widetilde{h}, \sigma, C, \kappa_1)$*-Removable at* a*, for any* $a \in \Omega_0$*.*

Proof. By compactness, we can choose $\widehat{C}_1, C > 0$ so that for all $a \in [0,1]^J$, $\widehat{C}_1 > \widehat{C}_{f_a}$, $C > \widetilde{C}_{f_a}$, where $\widehat{C}_{f_a}$ is given by Construction 9.3; $\widetilde{C}_{f_a}$ is given by Lemma 9.2.

We assume that $\widetilde{h} > 0$ is sufficiently small so that for all $a \in [0,1]^J$, $\widetilde{h} < \widetilde{h}_{f_a} < \overline{h}_{f_a}$, where $\widetilde{h}_{f_a}$ is given by Lemma 9.2 and $\overline{h}_{f_a}$ is given by Construction 9.1. Fix $\rho_1 \in (0, \frac{1}{2}\widetilde{h})$. For any sufficiently small $\sigma > 0$, we choose $\varepsilon_1, \sigma_1 > 0$ so that $\varepsilon_1 < \hat{\varepsilon}_0(f_a, \rho_1, \sigma)$, $\sigma < \hat{\sigma}_0(f_a, \rho_1, \sigma)$ (see Lemma 9.2) for all $a \in [0,1]^J$.

Given any C^r map $V\colon [0,1]^J \times \mathbb{R}^I \times X \to TX$ such that $V(a,\cdot)$ is an infinistesimal C^r deformation with I-parameters for any $a \in [0,1]^J$, we associate with V a C^r map $\hat{f}\colon [0,1]^J \times U_1 \times X \to X$ by

$$\hat{f}\colon (a,b,x) \mapsto \mathscr{F}_{V(a,b,\cdot)}(1, f_a(x)), \qquad \forall\, (a,b,x) \in [0,1]^J \times U_1 \times X, \tag{10.3}$$

where U_1 is a sufficiently small neighbourhood of the origin in $\mathbb{R}^I$, and for any $(a,b) \in [0,1]^J \times U_1$, $\mathscr{F}_{V(a,b,\cdot)}\colon \mathbb{R} \times X \to X$ is the flow generated by $V(a,b,\cdot)$.

To prepare for the proof of Proposition 10.2, we first show the following lemma.

LEMMA 10.3. *Set* $\widehat{K} := \sup_{a\in[0,1]^J} K_{f_a} + 4$. *Then there exist constants* $R_1, \kappa_1 > 0$ *such that the following is true. For any sufficiently small* $\widetilde{h} > 0$*, for any* $\rho_1 \in (0, \frac{1}{2}\widetilde{h})$*, any sufficiently small* $\sigma > 0$*, there exists a constant* $\lambda_1 = \lambda_1(\rho_1, \sigma) > 0$ *such that for any* $a \in [0,1]^J$*, if* $\mathcal{D}$ *is a* $3\widehat{K}\sigma$*-sparse* $(\frac{1}{20}, 6)$*-spanning c-family for* f_a *with* $[\underline{r}(\mathcal{D}), \overline{r}(\mathcal{D})] \subset (\rho_1, \frac{1}{2}\widetilde{h})$*, and if a* C^r *map* $V\colon [0,1]^J \times \mathbb{R}^I \times X \to TX$ *as above satisfies*

(1) $\sigma\|\partial_b\partial_x V\|_X + \|\partial_b V\|_X < \widehat{C}_1$ *and* $R_\pm(f_a, (\mathcal{D}, 3\widehat{K}\sigma), \mathrm{supp}_X(V)) > R_1$*;*
(2) *for* $B = (B_{\mathcal{C},i,t,j})_{\mathcal{C}\in\mathcal{D}, i,j\in\{1,\ldots,c\}, t\in\mathcal{B}_i} \in \mathbb{R}^{2n(\mathcal{D})c^2|\mathcal{A}|}$ *and* $\sigma'_a := \widetilde{C}_{f_a}^{-\frac{1}{2}}\sigma$*,*

$$V(a,\cdot)|_{(\mathcal{D},2\widehat{K}\sigma)} = \sum_{\mathcal{C}\in\mathcal{D}, i,j\in\{1,\ldots,c\}, t\in\mathcal{B}_i} B_{\mathcal{C},i,t,j} V^{\sigma'_a}_{\mathcal{C},i,t,j},$$

then for any $a' \in B(a, \lambda_1) \cap [0,1]^J$*, the* C^r *map* $\hat{f}\colon [0,1]^J \times U_1 \times X \to X$ *given by* (10.3) *is* $(\rho_1, \widetilde{h}, \sigma, C, \kappa_1)$*-Removable at* a'*.*

Proof. Take K_0 as in (7.1). By compactness, we can choose $\kappa > 0$ to be sufficiently small, depending only on $\{f_\omega\}_\omega$, so that for all $a \in [0,1]^J$, $\kappa < \kappa'(f_a)$ (see Lemma 9.5); and choose $R_1 > 0$, $\kappa_1 > 0$, depending only on $\{f_\omega\}_\omega$, so that for any $a \in [0,1]^J$, $R_1 > R_0(f_a, K_0, c, \widehat{C}_1, \frac{\kappa}{2})$, $\kappa_1 < \kappa_0(f_a, K_0, c, \widehat{C}_1, \frac{\kappa}{2})$ as in Proposition 5.6.

By hypothesis (1) and Lemma 9.5, we can choose $\lambda_1 > 0$ to be sufficiently small, depending only on $\{f_\omega\}_\omega$, ρ_1, σ, R_1, such that for any $a, \mathcal{D}$ given in the lemma, for any $a' \in B(a,\lambda_1) \cap [0,1]^J$, we have $f_{a'} \in \mathcal{U}'(f_a, \rho_1, \sigma)$ (see Lemma 9.5) and $R_\pm(f_{a'}, (\mathcal{D}, 2\widehat{K}\sigma), \mathrm{supp}_X(V)) > R_1$. Then we apply Lemma 9.5 to $\mathcal{D}$, and $(f_{a'}, f_a)$ in place of (g,f), to obtain $\mathcal{D}'$, a $(\frac{1}{10}, 8)$-spanning c-family for $f_{a'}$ with $[\underline{r}(\mathcal{D}'), \overline{r}(\mathcal{D}')] \subset (\rho_1, \widetilde{h})$ such that the conclusion of Lemma 9.5 holds.

For any $\mathcal{C}' \in \mathcal{D}'$, any $x \in \frac{1}{5}\mathcal{C}'$, let γ be a $(\sigma, \widetilde{C}_{f_a})$-regular (hence (σ, C)-regular) continuous family of $f_{a'}$-loops at x satisfying the conclusion of Lemma 9.5. We claim that for any $s \in [0,1]$, the vector field $V(a,\cdot)$ in the lemma is adapted to $(\gamma(s), \sigma, \widehat{C}_1, R_1)$. Indeed, by $\widehat{K} \geq K_{f_{a'}} + 4$ and $\mathcal{C}' \subset (\mathcal{D}, \sigma)$, we have $\mathcal{W}^c_{f_{a'}}(z, K_{f_{a'}}\sigma) \subset$

$(\mathcal{D}, 2\widehat{K}\sigma)$ for any $z \in \{x, x_1(s), x_2(s), x_3(s)\}$. Then by the choice of λ_1 and (9.2), we verify (2), (3) in Definition 5.5 for $(\gamma(s), \sigma, \widehat{C}_1, R_1)$ in place of (γ, σ, C, R_0). We verify (1) in Definition 5.5 by (9.1) and the hypothesis on V. This verifies the claim.

For any $(i, \mathcal{B}, \{s_t\}_{t\in\mathcal{B}}) \in \Gamma$, any $(t, j) \in \mathcal{B} \times \{1, \dots, c\}$, we define $\gamma_{t,j}$ as in Definition 10.1. Note that (5.5), (5.6) are satisfied by (9.2), thus for all $\sigma > 0$ sufficiently small (depending only on $\{f_\omega\}_\omega$), we can apply Proposition 5.6 to $(f_a, K_0, \widehat{C}_1, \frac{\kappa}{2}, V(a, \cdot), \{\gamma_{t,j}\}_{\substack{t\in\mathcal{B}\\ 1\le j\le c}})$ in place of $(f, L, C, \kappa, V, \{\gamma_{i,j}\}_{\substack{1\le i\le L\\ 1\le j\le c}})$, which concludes. $\square$

We now continue with the proof of Proposition 10.2.

Let $R_1, \kappa_1 > 0$ be given by Lemma 10.3. Take any $\vartheta > 0$. We set $K := (20J)^J$. Then by applying Proposition 6.3 to $r, J, \{f_\omega\}_\omega, K, \vartheta$ and to $(R_1, \frac{1}{2}\widetilde{h})$ in place of (R_0, h_0), we obtain a set Ω_1 compactly contained in $(0,1)^J$ and constants $N_0, \rho_0 \in (0, \frac{1}{2}\widetilde{h}), \rho_1 \in (0, \rho_0), \sigma_0, \lambda_0 > 0$ satisfying the conclusion of Proposition 6.3. For any sufficiently small $\sigma > 0$ in Proposition 10.2, we let $\lambda_1 = \lambda_1(\rho_1, \sigma)$ be taken as in Lemma 10.3.

Let $T > 0$ be some large integer such that $\lambda := \frac{100J}{T} < \min(\lambda_0, \lambda_1)$, and set $W_0 := (-\frac{1}{2T}, \frac{1}{2T})^J$. We choose points $\{a_1, \dots, a_{M_0}\} \subset \Omega_1$, $M_0 \le T^J$, such that $W_i := a_i + 2W_0$ is compactly contained in $[0,1]^J$, the collection $\{W_i\}_{1\le i\le M_0}$ forms an open cover of Ω_1, and the cover multiplicity of $\{a_i + 10JW_0\}_{1\le i\le M_0}$ is bounded by $K = (20J)^J$.

Let $\Theta\colon \mathbb{R}^J \to [0,1]$ be a smooth function such that $\Theta|_{2W_0} \equiv 1$ and $\operatorname{supp}(\Theta) \subset 3W_0$. Let $\Theta_i := \Theta(\cdot - a_i)$ for all $1 \le i \le M_0$, so that $\operatorname{supp}(\Theta_i) \subset a_i + 3W_0$.

For each $1 \le i \le M_0$, we will inductively define a $(\frac{1}{20}, 6)$-spanning c-family for f_{a_i}, denoted by $\mathcal{D}_i$, in the following way. Assume that for some $k \in \{1, \dots, M_0\}$, and for all $1 \le i \le k-1$, we have defined $\mathcal{D}_i$ satisfying:

(1) $\mathcal{D}_i$ is a σ_0-sparse $(\frac{1}{20}, 6)$-spanning c-family for f_{a_i};
(2) $[\underline{r}(\mathcal{D}_i), \overline{r}(\mathcal{D}_i)] \subset (\rho_1, \rho_0)$;
(3) $n(\mathcal{D}_i) < N_0$.

This assumption is always true for $k = 1$.

Let $\{i_1, \dots, i_l\}$ be the set of all indices $p \in \{1, \dots, k-1\}$ such that $W_p \subset B(W_k, \frac{5c}{T})$. By the choice of $\{a_i\}$, we have $l < K$. Then we can apply Proposition 6.3 to obtain a spanning c-family for f_{a_k}, denoted by $\mathcal{D}_k$, such that (1), (2), (3) above are true for $i = k$. Moreover, for any $1 \le j \le l$, $(\mathcal{D}_{i_j}, \sigma_0)$ is disjoint from $(\mathcal{D}_k, \sigma_0)$, and for all $a \in W_k \subset B(a_k, \lambda_0)$, we have $R(f_a, (\mathcal{D}_k, \sigma_0), (\{\mathcal{D}_{i_j}\}_{j=1}^l, \sigma_0)) > R_1$ and $R_\pm(f_a, (\mathcal{D}_k, \sigma_0)) > R_1$.

Having constructed $\{\mathcal{D}_i\}_{1\le i\le M_0}$, for each $1 \le k \le M_0$, we set $I_k := n(\mathcal{D}_k)c\sum_{i=1}^c |\mathcal{B}_i| = 2n(\mathcal{D}_k)c^2|\mathcal{A}|$, set $\sigma'_{a_k} := \widetilde{C}_{f_{a_k}}^{-\frac{1}{2}}\sigma$, and let $V^{(k)}\colon \mathbb{R}^{I_k} \times X \to TX$ be the infinitesimal C^r deformation defined as follows:

$$V^{(k)}(B, \cdot) := \sum_{\mathcal{C}\in\mathcal{D}_k, i,j\in\{1,\dots,c\}, t\in\mathcal{B}_i} B_{\mathcal{C},i,t,j} V_{\mathcal{C},i,t,j}^{\sigma'_{a_k}}, \ \forall\, B = (B_{\mathcal{C},i,t,j}) \in \mathbb{R}^{I_k}. \tag{10.4}$$

By Remark 9.4, for all sufficiently small $\sigma > 0$, we have $\operatorname{supp}_X(V^{(k)}) \subset (\mathcal{D}_k, \sigma_0)$. Let $I := \sum_{k=1}^{M_0} I_k$. For any $B = (B_k)_{k=1}^{M_0} \in \mathbb{R}^I$, where $B_k \in \mathbb{R}^{I_k}$ for each $1 \le k \le$

M_0, we define a C^r map $V\colon [0,1]^J \times \mathbb{R}^I \times X \to TX$ as follows:

$$V(a,B,\cdot) := \sum_{k=1}^{M_0} \Theta_k(a) V^{(k)}(B_k,\cdot).$$

By definition, the map V is linear in B. For each $a \in [0,1]^J$, let $\{i_1,\dots,i_l\}$ be the set of indices p such that $\Theta_p(a) \neq 0$. Note that $l \leq K$. Moreover, by construction, we see that the sets $(\mathcal{D}_{i_j},\sigma_0)$ are mutually disjoint for $j \in \{1,\dots,l\}$, and

$$\operatorname{supp}_X(V(a,\cdot)) \subset \bigsqcup_{j=1}^{l} (\mathcal{D}_{i_j},\sigma_0). \tag{10.5}$$

By the choice of $\widehat{C}_1$ above, (9.1), (10.5), and since $\mathcal{D}_k$ is σ_0-sparse for all $1 \leq k \leq M_0$,

$$\sigma\|\partial_b\partial_x V\|_X + \|\partial_b V\|_X < \widehat{C}_1. \tag{10.6}$$

Again, by the above construction, we see that

$$R_\pm(f_a,(\{\mathcal{D}_{i_j}\}_{j=1}^l,\sigma_0)) > R_1. \tag{10.7}$$

Take any $k \in \{1,\dots,M_0\}$. By construction, $\mathcal{D}_k$ is a σ_0-sparse $\left(\frac{1}{20},6\right)$-spanning c-family for f_{a_k}, and for any $a \in W_k$, for each $B = (B_l)_{1\leq l\leq M_0} \in \mathbb{R}^I$, we see that

$$V(a,B,\cdot)|_{(\mathcal{D}_k,\sigma_0)} = V^{(k)}(B_k,\cdot)|_{(\mathcal{D}_k,\sigma_0)}. \tag{10.8}$$

We define $\hat{f}$ by (10.3) for V given as above. It is clear that $\hat{f}$ is C^r, and for each $a \in [0,1]^J$, $\hat{f}(a,\cdot)\colon U_1 \times X \to X$ is the C^r deformation at $(0,f_a)$ generated by $V(a,\cdot)$. By (10.3), (10.6), and Lemma 4.4(1), we obtain $\|\hat{f}\|_{C^1} < Q$ for some $Q > 0$ depending only on $\{f_a\}_a$ and $\widehat{C}_1$, after possibly reducing the size of U_1.

By (10.4)-(10.8), for each $1 \leq k \leq M_0$, for any $a' \in W_k \subset B(a_k,\lambda_1)$, the assumptions of Lemma 10.3 are satisfied for all sufficiently small $\sigma > 0$, hence $\hat{f}$ is $(\rho_1,\widetilde{h},\sigma,C,\kappa_1)$-Removable at a'. The set $\Omega_0 := \bigcup_{i=1}^{M_0} W_i$ is compactly contained in $[0,1]^J$. By our choices of $\{a_i\}$, we have $\Omega_1 \subset \Omega_0 \subset [0,1]^J$, thus it is clear that $\mathrm{Leb}([0,1]^J\backslash\Omega_0) < \vartheta$. Then $\hat{f}$ is $(\rho_1,\widetilde{h},\sigma,C,\kappa_1)$-Removable at a for all $a \in \Omega_0$. This concludes the proof. □

10.2. **Getting accessibility by perturbation.** In this subsection, we fix a map $f_0 \in \mathcal{PH}^r(X,\mathrm{Vol})$ which is dynamically coherent and center bunched.

Let $f \in \mathbb{U}(f_0) \cap \mathrm{Diff}^r(X,\mathrm{Vol})$ ($\mathbb{U}(f_0)$ is defined in Notation 8.6). Let $\mathcal{C}$ be a c-disk of f with radius h in $(0,\overline{h}_f)$, and set $\phi = \phi(\mathcal{C})$ (see Construction 9.1). Let $\hat{f}\colon U \times X \to X$ be a C^r deformation at (a,f), and set $T = T(\hat{f})$. Then for $C > 0$, $\sigma \in (0,\frac{\overline{\sigma}_f}{2})$, $x \in \frac{1}{5}\mathcal{C}$, and γ, a (σ,C)-regular continuous family of f-loops at x, let $\hat{\gamma}$ be given by (8.2), and let $\widehat{\psi}$ be given by (8.3). We let $\sigma > 0$ be sufficiently small, and by (8.4), we have $\pi_X\widehat{\psi}(b,y,s) \in \phi((-h,h)^d)$ for all $(b,y,s) \in \mathcal{W}_T^c((a,x),\overline{\delta}_{a,T}) \times [-1,2]^c$. Let Π_c be as in Construction 9.1(4); we define

$$\Phi\colon \left\{ \begin{array}{rcl} \mathcal{W}_T^c((a,x),\overline{\delta}_{a,T}) \times [-1,2]^c & \to & \mathbb{R}^c, \\ (b,y,s) & \mapsto & \Pi_c\phi^{-1}\pi_X\widehat{\psi}(b,y,s), \end{array} \right. \tag{10.9}$$

where $\Pi_c\colon \mathbb{R}^d \simeq \mathbb{R}^c \times \mathbb{R}^{d_u} \times \mathbb{R}^{d_s} \to \mathbb{R}^c$ denotes the canonical projection.

LEMMA 10.4. *Let $f, \hat{f}, \mathcal{C}, \gamma, \hat{\psi}$ be as above. Then there is a C^2-uniform constant $\widehat{C}_0 = \widehat{C}_0(f) > 0$ such that, after possibly reducing the size of U, the following is true:*

(1) for any $s \in [-1,2]^c$, the map $(b,y) \mapsto \Phi(b,y,s)$ is C^1, and $D\Phi(b,y,s)$ is uniformly continuous, uniformly bounded by $\widehat{C}_0$;
(2) if $c \geq 2$, there is a C^2-uniform constant $\theta_0 = \theta_0(f_0) \in \left(\frac{c-1}{c}, 1\right)$ such that for any $(b,y) \in \mathcal{W}^c_T((0,x), \overline{\delta}_{a,T})$, the map $s \mapsto \Phi(b,y,s)$ has θ_0-Hölder norm less than $\widehat{C}_0$.

We remark that Lemma 10.4(2) is needed to "discretize" Property ($\mathcal{P}$) when $c \geq 2$.

Proof. Point (1) follows from the fact that f is C^2, center bunched, and Lemma 4.11. Point (2) follows from Lemma 8.8. □

The main technical result of this section is the following. It provides estimates on the volume of "bad" parameters under some removability condition.

PROPOSITION 10.5. *Let $\{f_\omega\}_{\omega \in [0,1]^J}$ be a good C^r-J-family in $\mathbb{U}(f_0) \cap \mathrm{Diff}^r(X, \mathrm{Vol})$. For any $Q, C, \kappa_1 > 0$, all sufficiently small $h > 0$, for any $\rho_1 \in (0,h)$, and for all sufficiently small $\sigma > 0$, the following is true. Assume that there exist an open set Ω_0 compactly supported in $(0,1)^J$; and integer $I > 0$; an open neighbourhood U_1 of the origin in $\mathbb{R}^I$; and a C^r map $\hat{f}\colon [0,1]^J \times U_1 \times X \to X$, such that*

(i) $\hat{f}(a,b,\cdot) \in \mathrm{Diff}^r(X, \mathrm{Vol})$ and $\hat{f}(a,0,\cdot) = f_a$, for all $(a,b) \in [0,1]^J \times U_1$;
(ii) $\|\hat{f}\|_{C^1} < Q$;
(iii) $\hat{f}$ is $(\rho, h, \sigma, C, \kappa_1)$-Removable at a, for all $a \in \overline{\Omega_0}$.

Then for any sufficiently small $\epsilon > 0$, any $\delta > 0$, there exists a subset $\mathcal{E} = \mathcal{E}(\epsilon, \delta) \subset \Omega_0 \times U_1$ of the parameter space such that $\mathrm{Leb}(\mathcal{E}) < \delta$, and for any $(a,b) \in (\Omega_0 \times (U_1 \cap B(0,\epsilon))) \backslash \mathcal{E}$, $\hat{f}(a,b,\cdot)$ is C^1-stably accessible.

Proof. We only detail the case where $c = 2$. We will sketch the adaptation needed for $c = 1$ at the end of the proof.

Let us assume for now that $c = 2$. Consequently, either (ae) or (be) holds. In the following, we take $\theta = \theta_0(f_0)$ as in Lemma 10.4, and set $K_0 := K_0(c,\theta)$ as in (7.1). By Lemma 10.4, $\theta > \frac{c-1}{c}$ and thus $K_0 \geq 2$.

Let $\hat{f}, Q, C, \kappa_1, h, \rho_1, \sigma$ be as in the proposition. Let $T\colon [0,1]^J \times U_1 \times X \to [0,1]^J \times U_1 \times X$ be the C^r map $T\colon (a,b,x) \mapsto (a,b,\hat{f}(a,b,x))$.

For any $a \in \overline{\Omega_0}$, $\hat{f}$ can be regarded as a C^r deformation at $((a,0), f_a)$ with $J+I$ parameters. Let $\nu_{a,0}(x,\cdot)\colon \mathbb{R}^{J+I} \to E^{su}_{f_a}(x)$ be the (unique) linear map given by Lemma 4.11. Set $T_a := T(\hat{f}(a,\cdot))$ and let $\nu^a_0(x,\cdot)\colon \mathbb{R}^I \to E^{su}_{f_a}(x)$ be the unique linear map satisfying (10.1) for $b = 0$. It is direct to see that $\nu^a_0(x,B) = \nu_{a,0}(x, \{0\}^J \times B)$ for all $x \in X, B \in \mathbb{R}^I$. In the following we tacitly use the inclusion $\mathbb{R}^I \subset \{0\}^J \times \mathbb{R}^I$ and for any $B \in \mathbb{R}^I$, we abbreviate $\nu_{a,0}(x, \{0\}^J \times B)$ as $\nu_{a,0}(x,B)$.

Fix $a \in \overline{\Omega_0}$. By the hypothesis of $(\rho_1, h, \sigma, C, \kappa_1)$-Removability, we can choose $\mathcal{D} = \mathcal{D}(a)$, a $(\frac{1}{10}, 8)$-spanning c-family for f_a, and for any $\mathcal{C} \in \mathcal{D}$, any $x \in \frac{1}{5}\mathcal{C}$, we let $\gamma = \gamma(a,x)$ be a (σ, C)-regular continuous family of f_a-loops at x. By compactness, we can take a small constant $\overline{\delta} \in (0, \min(\overline{\delta}_{(a',0),T}, \overline{\delta}_{0,T_{a'}}))$ where $\overline{\delta}_{(a',0),T}, \overline{\delta}_{0,T_{a'}}$ are given by Definition 8.7. Let $\hat{\gamma}$ be the lift of γ for T. Take $\sigma > 0$ sufficiently small, so that the map associated to γ and T as in (10.9), denoted by $\Phi\colon \mathcal{W}^c_T((a,0,x),\overline{\delta})\times[-1,2]^c\to\mathbb{R}^c$ is well-defined. For each $(i,\mathcal{B},\{s_t\}_{t\in\mathcal{B}})\in\Gamma$, set

$$\Psi=\Psi_{a,\mathcal{C},x,i,\mathcal{B},\{s_t\}}\colon \begin{cases} \mathcal{W}^c_T((a,0,x),\overline{\delta}) & \to \quad \mathbb{R}^{K_0 c},\\ \qquad (a',b',y) & \mapsto \quad (\Phi(a',b',y,s_t))_{t\in\mathcal{B}}. \end{cases} \tag{10.10}$$

By Lemma 10.4(1), we can differentiate Ψ at $(a,0,x)$, and obtain for each $B\in\mathbb{R}^I$:

$$\begin{aligned} D\Psi((a,0,x),B+\nu_{a,0}(x,B)) &= \big(D\Phi((a,0,x,s_t),B+\nu_{a,0}(x,B))\big)_{t\in\mathcal{B}}\\ &= \big(\Pi_c D\phi^{-1}\pi_X D\big(\prod_{j=1}^c H_{T,\hat{\gamma}_{t,j}}\big)(B+\nu_{a,0}(x,B))\big)_{t\in\mathcal{B}}, \end{aligned}$$

where $\gamma_{t,j}=\gamma(\varphi(j,s_{t,j}))$, and $\hat{\gamma}_{t,j}$ is the lift of $\gamma_{t,j}$ for T. Let $\widetilde{\gamma}_{t,j}$ be the lift of $\gamma_{t,j}$ for T_a. Then by definition and by Lemma 4.14, we obtain

$$\pi_X D\big(\prod_{j=1}^c H_{T,\hat{\gamma}_{t,j}}\big)(B+\nu_{a,0}(x,B))=\pi_c D\big(\prod_{j=1}^c H_{T_a,\widetilde{\gamma}_{t,j}}\big)(B+\nu^a_0(x,B))+\nu^a_0(z,B),$$

where we have set $z:=\prod_{j=1}^c H_{f_a,\gamma_{t,j}}(x)$.

Let $\zeta>0$ be a small constant to be determined. Let $h>0$ be sufficiently small such that for any $a'\in[0,1]^J$, we have $h<\overline{h}_{f_{a'},\zeta}$ (as in Construction 9.1(5)). Then by Lemma 4.13(2), Construction 9.1 and hypothesis (ii), there exists a constant $D_1>0$ depending only on $\{f_\omega\}_{\omega\in[0,1]^J}$ such that for any $B\in\mathbb{R}^I$,

$$\|\Pi_c D\phi^{-1}\nu^a_0(z,B)\|\le D_1\zeta Q\|B\|. \tag{10.11}$$

By $(\rho_1,h,\sigma,C,\kappa_1)$-Removability at a, there exists a linear subspace $H\subset\mathbb{R}^I$ of dimension K_0c such that

$$\Big|\det\big(H\ni B\mapsto \pi_c D\big(\prod_{j=1}^c H_{T_a,\widetilde{\gamma}_{t,j}}\big)(B+\nu^a_0(x,B))\big)_{t\in\mathcal{B}}\Big|>\kappa_1. \tag{10.12}$$

Then by (10.11), we can choose $\zeta>0$ sufficient small, depending only on $(Q,\kappa_1,\{f_\omega\}_{\omega\in[0,1]^J})$, such that for some constant $D_2>0$ depending only on $\{f_\omega\}_{\omega\in[0,1]^J}$, it holds

$$\big|\det(H\ni B\mapsto D\Psi((a,0,x),B+\nu_{a,0}(x,B))\in\mathbb{R}^{K_0c})\big|>D_2^{-1}\kappa_1.$$

Now, by Lemma 10.4 and the pre-compactness of Ω_0, $D\Psi$ is uniformly continuous, with norms uniformly bounded for all choices of $a,\mathcal{C},x,i,\mathcal{B},\{s_t\}$. Then, by possibly reducing the size of $\overline{\delta}$ independently of the choices of $a,\mathcal{C},x,i,\mathcal{B},\{s_t\}$, we can assume that for any $(a',b',y)\in\mathcal{W}^c_T((a,0,x),\overline{\delta})$, there exists a subspace $H'\subset T_{a',b'}(\Omega_0\times U_1)$ of dimension K_0c such that

$$|\det(D\Psi(a,'b',y)|_{H'})|>\frac{1}{2}D_2^{-1}\kappa_1. \tag{10.13}$$

By compactness, for any $\mathcal{C}\in\mathcal{D}$, we can choose a finite set $\mathcal{A}(a,\mathcal{C})\subset\frac{1}{5}\mathcal{C}$ s.t.

$$\mathcal{V}(a,\mathcal{C}):=\bigcup_{x\in\mathcal{A}(a,\mathcal{C})}\mathcal{W}^c_T((a,0,x),\overline{\delta}) \tag{10.14}$$

is an open neighbourhood of $\{a\}\times\{0\}\times\frac{1}{5}\mathcal{C}$ in $\mathcal{W}^c_T$.

By compactness, Lemma 4.9 and Remark 4.10 (for $f=f_a,\mathcal{D},(k,\theta,\theta',\theta'',\rho_m,\rho_M)=(8,\frac{1}{10},\frac{1}{9},\frac{1}{8},\rho_1,h)$ and $\sigma<\frac{1}{2}d(\mathcal{V}(a,\mathcal{C})^c,\{a\}\times\{0\}\times\frac{1}{5}\mathcal{C})$), there exists a constant $\delta_0>0$ (independent of the choices of $a,\mathcal{C},x,i,\mathcal{B},\{s_t\}$) such that for any $(a',b')\in B(a,\delta_0)\times(U_1\cap B(0,\delta_0))$, there exists $\mathcal{D}'$, a $(\frac{1}{9},10)$-spanning c-family for $\hat{f}(a',b',\cdot)$, such that for any $\mathcal{C}' \in \mathcal{D}'$, $\{a'\} \times \{b'\} \times \frac{1}{8}\mathcal{C}' \subset \mathcal{V}(a, \mathcal{C})$ for some $\mathcal{C} \in \mathcal{D}$. Without loss of generality, we assume that $U_1 \subset B(0, \delta_0)$ and set $\mathcal{U}(a) := B(a, \delta_0) \times U_1$.

Now, by $\Omega_0 \subset [0, 1]^J$, we can find a finite set $\mathcal{K} \subset \Omega_0$ such that

$$\Omega_0 \times U_1 \subset \cup_{a \in \mathcal{K}} \mathcal{U}(a). \tag{10.15}$$

By (7.1) we have $\frac{\theta(cK_0-2c-1)}{(c-1)K_0} > 1$. Let $\beta \in \left(0, \min\left(\frac{\theta(cK_0-2c-1)}{(c-1)K_0} - 1, 1\right)\right)$, so that $-(c-1)K_0\frac{1+\beta}{\theta} + cK_0 - 2c > 1$. Then, choose $\eta > 0$ small enough such that

$$\upsilon := -K_0\frac{1+\beta}{\theta}(c-1) + K_0 c - 2c - (K_0+1)\eta - 1 > 0. \tag{10.16}$$

For any sufficiently small $\delta > 0$, for each $i \in \{1, \dots, c\}$, for any $t \in \mathcal{B}_i$, let $\mathcal{N}_{t,i}$ be a $\delta^{\frac{1+\beta}{\theta}}$-net in $[-1, 2]^{i-1} \times \{t\} \times [-1, 2]^{c-i}$ such that $|\mathcal{N}_{t,i}| < \delta^{-\frac{1+\beta}{\theta}(c-1)-\eta}$.

We denote by Σ the diagonal of $(\mathbb{R}^c)^{K_0} \simeq \mathbb{R}^{K_0 c}$, that is,

$$\Sigma := \{(y, \dots, y) \in \mathbb{R}^{K_0 c} \mid y \in \mathbb{R}^c\}, \tag{10.17}$$

and for any $\delta > 0$, we let Σ_δ be the δ-neighbourhood of Σ defined by

$$\Sigma_\delta := \{(y_i)_{1 \le i \le K_0} \in (\mathbb{R}^c)^{K_0} \mid \exists\, y \in \mathbb{R}^c, |y_i - y| < \delta,\ \forall\, 1 \le i \le K_0\}.$$

For any $a \in \mathcal{K}$, let $\mathcal{D} = \mathcal{D}(a)$ be the $(\frac{1}{10}, 8)$-spanning c-family for f_a given above. For any $\mathcal{C} \in \mathcal{D}$, $x \in \mathcal{A}(a, \mathcal{C})$ and $(i, \mathcal{B}, \{s_t\}_{t \in \mathcal{B}}) \in \Gamma$, set $\Psi := \Psi_{a,\mathcal{C},x,i,\mathcal{B},\{s_t\}}$. By (10.13), the map $D\Psi$ is a submersion from $\mathcal{W}^c_T((a, 0, x), \overline{\delta})$ to its image, and is uniformly transverse to Σ, i.e., whenever $w = (a', b', y) \in \mathcal{W}^c_T((a, 0, x), \overline{\delta}) \cap \Psi^{-1}(\Sigma)$,

$$T_{\Psi(w)}\Sigma + D\Psi(T_w\mathcal{W}^c_T((a, 0, x), \overline{\delta})) \simeq \mathbb{R}^{K_0 c}.$$

Therefore, $\Psi^{-1}_{a,\mathcal{C},x,i,\mathcal{B},\{s_t\}}(\Sigma)$ is a submanifold of $\mathcal{W}^c_T((a, 0, x), \overline{\delta})$ of dimension[10] $J + I + 2c - K_0 c$. Besides, by uniform transversality, there exists $\delta_1 > 0$ independent of the choice of $a, \mathcal{C}, x, i, \mathcal{B}, \{s_t\}$ such that for any $0 < \delta < \delta_1$, we have

$$\mathscr{N}(\Psi^{-1}_{a,\mathcal{C},x,i,\mathcal{B},\{s_t\}}(\Sigma_\delta), \delta) < \delta^{K_0 c - 2c - I - J - \eta}, \tag{10.18}$$

where for any set $\mathcal{S}$, $\mathscr{N}(\mathcal{S}, \delta)$ is the minimal number of δ-balls required to cover $\mathcal{S}$. For any $0 < \delta < \delta_1$, let $\mathcal{E} = \mathcal{E}(\delta)$ be the subset of "bad" parameters in $\Omega_0 \times U_1$:

$$\mathcal{E} := \bigcup_{\substack{a \in \mathcal{K},\ \mathcal{C} \in \mathcal{D}, \\ x \in \mathcal{A}(a, \mathcal{C}), \\ (i, \mathcal{B}, \{s_t\}_{t \in \mathcal{B}}) \in \Gamma \ s.t.\ \forall\, t \in \mathcal{B},\ s_t \in \mathcal{N}_{t,i}}} \pi_{[0,1]^J \times U_1}(\Psi^{-1}_{a,\mathcal{C},x,i,\mathcal{B},\{s_t\}}(\Sigma_\delta)). \tag{10.19}$$

Since in the above collection, only the last item, $\mathcal{N}_{t,i}$, depends on δ, there exists a constant $D_3 > 0$ such that for any $0 < \delta < \delta_1$,

$$\mathscr{N}(\mathcal{E}, \delta) < D_3 \delta^{-K_0\frac{1+\beta}{\theta}(c-1) - K_0\eta} \delta^{K_0 c - 2c - I - J - \eta} = (D_3 \delta^\upsilon)\delta^{-I-J+1}. \tag{10.20}$$

By (10.16), there exists $0 < \delta_2 < \delta_1$ such that $D_3\delta_2^\upsilon < 1$. We deduce that $\mathrm{Leb}(\mathcal{E}) \le (2\delta)^{J+I}\mathscr{N}(\mathcal{E}, \delta) < \delta$ for all $0 < \delta < \delta_2$.

We claim that for all sufficiently small $\epsilon > 0$, for all sufficiently small $\delta > 0$, and any $(a, b) \in (\Omega_0 \times (U_1 \cap B(0, \epsilon))) \backslash \mathcal{E}$, $\hat{f}(a, b, \cdot)$ is C^1-stably accessible. This would finish the proof for the case $c \ge 2$.

[10] See [23, Theorem 3.3]; by transversality, $\Psi^{-1}(\Sigma) \subset \mathcal{W}^c_T((a, 0, x), \overline{\delta})$ has same codimension as Σ in $\mathbb{R}^{K_0 c}$. Here, transversality is w.r.t. the parameter b', and by (10.13), it is uniform in variables a, x, which gives a uniform bound in (10.18) on the number of balls needed to cover $\Psi^{-1}(\Sigma_\delta)$.

Indeed, by (10.15), for any $(a,b) \in (\Omega_0 \times (U_1 \cap B(0,\epsilon)))\backslash\mathcal{E}$, there exists $a_0 \in \mathcal{K}$ such that $(a,b) \in \mathcal{U}(a_0)$. Let $\mathcal{D}_0 = \mathcal{D}(a_0)$. Then by the definition of $\mathcal{U}(a_0)$, there exists a $(\frac{1}{9},10)$-spanning c-family for $\hat{f}(a,b,\cdot)$, denoted by $\mathcal{D}$, such that for each $\mathcal{C} \in \mathcal{D}$, there exists $\mathcal{C}_0 \in \mathcal{D}_0$ such that $\{a\}\times\{b\}\times\frac{1}{8}\mathcal{C} \subset \mathcal{V}(a_0,\mathcal{C}_0)$. Then for each $x \in \frac{1}{8}\mathcal{C}$, by (10.14), there exists $x_0 \in \mathcal{A}(a_0,\mathcal{C}_0)$, such that $(a,b,x) \in \mathcal{W}^c_T((a_0,0,x_0),\overline{\delta})$.

CLAIM 10.6. *For any sufficiently small $\delta > 0$, the following is true. For any $(i,\mathcal{B},\{s_t\}_{t\in\mathcal{B}}) \in \Gamma$, take $\Psi := \Psi_{a_0,\mathcal{C}_0,x_0,i,\mathcal{B},\{s_t\}}$ as in* (10.10)*. Then, $\Psi(a,b,x) \notin \Sigma$.*

Proof. Indeed, for any $(i,\mathcal{B},\{s_t\}_{t\in\mathcal{B}}) \in \Gamma$, there exists $\{w_t\}_{t\in\mathcal{B}}$ such that for all $t \in \mathcal{B}$, $w_t \in \mathcal{N}_{t,i}$ and $|s_t - w_t| < \delta^{\frac{1+\beta}{\theta}}$. Since $(a,b) \notin \mathcal{E}$, and by (10.10) and (10.19), there exist $t,t' \in \mathcal{B}$ such that $|\Phi(a,b,x,w_t) - \Phi(a,b,x,w_{t'})| > \delta$, where Φ is defined as in (10.9) for $((a_0,0),x_0)$ in place of (a,x). By Lemma 10.4, we get $|\Phi(a,b,x,w_t) - \Phi(a,b,x,s_t)|$, $|\Phi(a,b,x,w_{t'}) - \Phi(a,b,x,s_{t'})| < D_4\delta^{1+\beta}$ for some constant $D_4 > 0$ independent of δ. The claim follows, since for sufficiently small $\delta > 0$, we then have $|\Phi(a,b,x,s_t) - \Phi(a,b,x,s_{t'})| > \delta - 2D_4\delta^{1+\beta} > 0$. □

Let $\gamma_0 = \gamma(a_0,x_0)$ be the (σ,C)-regular continuous family of f_{a_0}-loops at x_0 associated to (a_0,x_0). Since $\overline{\delta} < \overline{\delta}_{(a,0),T} < \delta_{(a,0),T}$ (see Definition 8.7) and $(a,b,x) \in \mathcal{W}^c_T((a_0,0,x_0),\overline{\delta})$, let $\gamma_{a,b,x}$ be the continuous family of $\hat{f}(a,b,\cdot)$-loops at x associated to γ_0 according to Definition 8.7. By (8.3), (8.4), (10.9) and (10.10), we see that $\hat{f}(a,b,\cdot)$ satisfies $(\mathcal{P})$ (for $(\theta,\theta',k) = (\frac{1}{9},\frac{1}{8},10)$, $\mathcal{D}$, $\{\gamma_{a,b,x}\}_{x\in\frac{1}{8}\mathcal{C},\ \mathcal{C}\in\mathcal{D}}$ and $\psi = \psi(\hat{f}(a,b,\cdot),x,\gamma_{a,b,x})$ as in (8.1). Thus, our claim follows from Proposition 8.4.

Now we consider the case where $c = 1$. We can just choose $K_0 = K_1 = 5$ in (7.1). It suffices to notice that for each $i \in \{1,\dots,c\}$, for each $t \in \mathcal{B}_i$, the set $[-1,2]^{i-1} \times \{t\} \times [-1,2]^{c-i}$ is reduced to the singleton $\{t\}$. Thus we can choose in the above proof that for any $\delta > 0$, $\mathcal{N}_{t,i} = \{t\}$. It is straightforward to verify that the above proof for $c \geq 2$ (below (10.16)) carries over to the case $c = 1$. For instance, (10.20) becomes

$$\mathscr{N}(\mathcal{E},\delta) < D_3\delta^{K_0c-2c-I-J-\eta} = (D_3\delta^{\upsilon'})\delta^{-I-J+1}$$

where

$$\upsilon' = K_0c - 2c - \eta - 1 > 0.$$

This finishes the proof. □

11. THE PROOF OF THEOREM E

Combining Propositions 10.2 and 10.5, we are ready to prove Theorem E.

Proof of Theorem E. We only detail the volume preserving case, for the proof of the other one follows the same line after replacing $\mathrm{Diff}^r(X,\mathrm{Vol})$ by $\mathrm{Diff}^r(X)$.

By Notation 3.12, for any map f in Theorem E, f is dynamically coherent, center bunched, and satisfies (ae) or (be). Set $\mathcal{U} := \mathbb{U}(f) \cap \mathrm{Diff}^r(X,\mathrm{Vol})$ (see Notation 8.6), and let $\{f_a\}_{a\in[0,1]^J}$ be a good $C^r - J-$family of diffeomorphisms in $\mathcal{U}$. By Proposition 10.2 and Proposition 10.5, for any $\vartheta > 0$, there exist an open set $\Omega_0 \subset [0,1]^J$ with $\mathrm{Leb}([0,1]^J\backslash\Omega_0) < \vartheta$, an open neighbourhood U_1 of the origin in $\mathbb{R}^I$, and a C^r map $\hat{f}\colon [0,1]^J \times U_1 \times X \to X$ with $f_a = \hat{f}(a,0,\cdot)$ for all $a \in [0,1]^J$, such that for all sufficiently small $\epsilon > 0$, and any $\delta > 0$, there exists $\mathcal{E} \subset \Omega_0 \times U_1$ such that $\mathrm{Leb}(\mathcal{E}) < \delta$ and $\hat{f}(a,b,\cdot)$ is C^1-stably accessible for all $(a,b) \in (\Omega_0 \times (U_1 \cap B(0,\epsilon)))\backslash\mathcal{E}$. Now, given any sufficiently small $\epsilon > 0$, let

$\delta \in (0, \epsilon^I \vartheta)$ and $\mathcal{E} = \mathcal{E}(\epsilon, \delta)$ be as above, and for any $b \in U_1$, set $\mathcal{E}^b := \mathcal{E} \cap (\Omega_0 \times \{b\})$. Then, by Fubini's theorem, there exists $b \in U_1 \cap B(0, \epsilon)$ such that

$$\mathrm{Leb}\big((([0,1]^J \backslash \Omega_0) \times \{b\}) \cup \mathcal{E}^b\big) \lesssim \mathrm{Leb}([0,1]^J \backslash \Omega_0) + \epsilon^{-I} \mathrm{Leb}(\mathcal{E}) \lesssim \vartheta. \tag{11.1}$$

For any integer $n \geq 1$, we consider the following collection of $C^r - J-$families in $\mathcal{U}$:

$$\mathcal{F}_n := \{\{f_a\}_{a \in [0,1]^J} \in C^r([0,1]^J, \mathcal{U}) \mid \text{the set of } a \in [0,1]^J \text{ such that}$$
$$f_a \text{ is not } C^1\text{-stably accessible has measure less than } \tfrac{1}{n}\}.$$

It follows from $J \geq J_0$, Proposition 3.5 and (11.1) above that for any $n \geq 1$, the set $\mathcal{F}_n$ is C^1-open and C^r-dense in the space $C^r([0,1]^J, \mathcal{U})$. In particular, $\mathcal{G} := \bigcap_{n \geq 1} \mathcal{F}_n$ is residual. By definition, for any $\{f_a\}_{a \in [0,1]^J} \in \mathcal{G}$, the set of $a \in [0,1]^J$ such that f_a is not C^1-stably accessible has measure zero. This concludes the proof. □

Appendix A.

Proof of Lemma 4.14. We detail the case where $* = u$. The other one is handled similarly.

Proof of (1): for any $b \in U$, $x \in X$, we have $\mathcal{W}_T^u(b, x) \subset \{b\} \times X$. Hence the image of $H^u_{T,(0,x),(0,y)}$ is contained in $\{b\} \times X$. Then for any $B \in T_0 U$, we have $DH^u_{T,(0,x),(0,y)}(B + \nu_0(x, B)) \in B + T_z X$, while $\pi_b(B + T_z X) = B + \nu_0(z, B)$.

To show (2), we need the following lemma.

LEMMA A.1 (*A priori* estimates). *There exists a C^1-uniform constant $\tilde{\sigma}_f > 0$, and a C^2-uniform constant $C_\star = C_\star(f) > 0$ such that the following is true. Take any $x \in X$, $w \in \mathcal{W}_f^{cu}(x, \tilde{\sigma}_f/2)$. Then $z := H^u_{f,x,w}(x)$ is well-defined, and for any $B \in T_0 U$, we have*

$$\begin{aligned} &\|\pi_c DH^u_{T,(0,x),(0,w)}(B + \nu_0(x, B))\| \\ &\leq C_\star(\|\nu_0(x, \cdot)\| + \|\nu_0(z, \cdot)\| + \|D^2 T\| d_{\mathcal{W}_f^u}(x, z))\|B\|. \end{aligned} \tag{A.1}$$

We have an analogous statement for any x, w in a local center stable leaf.

Proof. We follow the construction in [35, Proof of Theorem 4.1] and refer to [35] for many details.

Let $\widetilde{E}^u$ (resp. $\widetilde{E}^{cs}$) be a smooth bundle of TX that closely approximates E_f^u (resp. E_f^{cs}) and let $\delta = \delta(f) > 0$ be a small C^1-uniform constant, to be chosen in due course. Following the proof of [35, Theorem 4.1, page 530], we embed $\widetilde{E}^u$, $\widetilde{E}^{cs}$ via C^∞ maps $\imath_1 \colon \widetilde{E}^u \to X \times \mathbb{R}^{m_1}$ and $\imath_2 \colon \widetilde{E}^{cs} \to X \times \mathbb{R}^{m_2}$, where for $i = 1, 2$, $m_i \in \mathbb{N}$, and $\mathbb{R}^{m_i}$ is equipped with a metric $\|\cdot\|_i$ such that the Lipschitz constant of $\imath_i$ is uniformly bounded by some C^1-uniform constant $c_0 = c_0(f) > 0$. As in [35, Proof of Theorem 4.1], we can choose $\widetilde{E}^u, \widetilde{E}^{cs}, \imath_1, \imath_2, \|\cdot\|_1, \|\cdot\|_2, U, \delta$ so that there is a C^1 bundle contraction $F_\#$ satisfying

$$\begin{array}{ccc} U \times X \times Y & \xrightarrow{F_\#} & U \times X \times Y \\ \downarrow & & \downarrow \\ U \times X & \xrightarrow{T} & U \times X \end{array}$$

Here Y is defined as

$$Y := \{g \in C^0(\mathbb{R}^{m_1}(2c_0\delta), \mathbb{R}^{m_2}) \mid g(0) = 0,\ \mathrm{Lip}(g) \leq 1\}$$

equipped with the norm

$$\|g\| := \sup_x \frac{\|g(x)\|_2}{\|x\|_1}, \quad \forall\, g \in Y.$$

Recall that by [35, (11)], we have

$$\|(F_\#)_p g - (F_\#)_p h\| \leq C e^{-\bar{\chi}^u(p)+\hat{\chi}^c(p)} \|g-h\|, \quad \forall\, p \in X.$$

Moreover, we have:

(1) the unique invariant section of $F_\#$ is a family of Lipschitz functions $\{\gamma_p \colon \mathbb{R}^{m_1}(2c_0\delta) \to \mathbb{R}^{m_2}\}_{p\in U\times X}$ parametrizing local unstable manifolds: for any $p=(b,x) \in U\times X$, we have

$$\gamma_p(\imath_1(\widetilde{E}^u(x,2c_0\delta))) \subset \imath_2(\widetilde{E}^{cs}(x)),$$

and $\mathcal{W}^u_{\hat{f}(b,\cdot)}(x,\delta/c') \subset \exp_x(Graph(\imath_2^{-1}\gamma_p\imath_1|_{\widetilde{E}^u(x,\delta)})) \subset \mathcal{W}^u_{\hat{f}(b,\cdot)}(x,c'\delta)$,

for some C^1-uniform constant $c'=c'(f)>0$;

(2) $F_\#$ preserves the sub-bundle Y_0 where

$$Y_0(p) := \{g \in Y \mid g(\imath_1(\widetilde{E}^u(x,2c_0\delta))) \subset \imath_2(\widetilde{E}^{cs}(x))\}, \quad \forall\, p=(b,x)\in U\times X,$$

and satisfies that for some C^2-uniform constant $c_*=c_*(f)>0$, for any sufficiently close points p,q in the same center-unstable leaf of T, for any $g\in Y$, for any $z\in\mathbb{R}^{m_1}(2c_0\delta)$,[11]

$$\|(F_\#)_p(g)(z)-(F_\#)_q(g)(z)\|_2 \leq c_*\|D^2T\|d(p,q)\|z\|_1. \tag{A.2}$$

LEMMA A.2. *For any sufficiently close $p,q\in U\times X$ in a local center-unstable manifold of T,*

$$\|\gamma_p(z)-\gamma_q(z)\|_2 \leq c_*\|D^2T\|d(p,q)\|z\|_1, \quad \forall\, z\in B(0,2c_0\delta)\subset\mathbb{R}^{m_1}. \tag{A.3}$$

Proof. The proof follows [25, Proof of Theorem 3.2]. For each $p\in U\times X$, we let $\widehat{\mathcal{J}}_p$ be the set of C^0 sections $g\colon V_{p,g}\to Y$ – where $V_{p,g}$ is an open neighbourhood of p – such that

$$g(p)=\gamma_p, \qquad \limsup_{\mathcal{W}^{cu}_T(p)\ni q\to p} \frac{\|g(q)-g(p)\|}{d(q,p)} < \infty.$$

Define a semi-norm on $\widehat{\mathcal{J}}_p$ by

$$d(g,h) := \limsup_{\mathcal{W}^{cu}_T(p)\ni q\to p} \frac{\|g(q)-h(q)\|}{d(q,p)}, \qquad \forall\, g,h\in\widehat{\mathcal{J}}_p.$$

We declare that $g,h\in\widehat{\mathcal{J}}_p$ are equivalent if $d(g,h)=0$. Then the semi-norm d descends to a distance on the equivalence classes of $\widehat{\mathcal{J}}_p$, denoted by $\mathcal{J}_p$. Notice that $\mathcal{J}_p$ is the space of *Lipschitz jets*, defined in [25, Chapter 3], at p.

We define a Banach bundle $\mathcal{J}$ over $U\times X$ (equipped with the discrete topology) by setting

$$\mathcal{J} := \bigcup_{p\in U\times X} \mathcal{J}_p.$$

The map $F_\#$ gives rise to a bundle map J of $\mathcal{J}$. More precisely, for any $g\colon V_{T^{-1}(p),g}\to Y$ in $\widehat{\mathcal{J}}_{T^{-1}(p)}$, we define $Jg\in\widehat{\mathcal{J}}_p$ by

$$Jg(q) := (F_\#)_{T^{-1}(q)}(g(T^{-1}(q))), \quad \forall\, q\in U\times X\cap T(V_{T^{-1}(p),g}).$$

[11] See [35, Page 533].

It is clear that J descends to a map from $\mathcal{J}_{T^{-1}(p)}$ to $\mathcal{J}_p$. Consequently, J is a bundle map of $\mathcal{J}$ over T. For any $g, h \in \widehat{\mathcal{J}}_{T^{-1}(p)}$, we have

$$\begin{aligned}
d(Jg, Jh) &= \limsup_{\mathcal{W}^{cu}_T(p) \ni q \to p} \frac{\|Jg(q) - Jh(q)\|}{d(p,q)} \\
&= \limsup_{\mathcal{W}^{cu}_T(p) \ni q \to p} \frac{\|(F_\#)_{T^{-1}(q)}(g(T^{-1}(q))) - (F_\#)_{T^{-1}(q)}(h(T^{-1}(q)))\|}{d(p,q)} \\
&= \limsup_{\mathcal{W}^{cu}_T(T^{-1}(p)) \ni u \to T^{-1}(p)} \frac{\|(F_\#)_u(g(u)) - (F_\#)_u(h(u))\|}{\|g(u) - h(u)\|} \\
&\quad \cdot \limsup_{\mathcal{W}^{cu}_T(p) \ni q \to p} \frac{d(T^{-1}(p), T^{-1}(q))}{d(p,q)} \cdot \limsup_{\mathcal{W}^{cu}_T(p) \ni u \to T^{-1}(p)} \frac{\|g(u) - h(u)\|}{d(T^{-1}(p), u)} \\
&\le \hat{c} e^{-\bar{\chi}^u(T^{-1}(p)) + \hat{\chi}^c(T^{-1}(p)) - \bar{\chi}^c(T^{-1}(p))} d(g,h),
\end{aligned}$$

for some constant $\hat{c} > 0$, as the points p, q above are in the same center-unstable manifold. Moreover, by center bunching, $-\bar{\chi}^u + \hat{\chi}^c - \bar{\chi}^c < 0$, thus $\mathcal{J}$ has a unique J-invariant section h_0. On the other hand, as γ is $F_\#$-invariant, it is clear that γ must be a representative of h_0.

We say that $\bar{g} \in \mathcal{J}_p$ is the constant section if $\bar{g}$ admits a representative $g \in \widehat{\mathcal{J}}_p$ such that

$$g(q) = \gamma_p, \quad \forall\, q \in V_{p,g}.$$

We denote by $h_* \colon U \times X \to \mathcal{J}$ the section where $h(p)$ is the constant section in $\mathcal{J}_p$. By (A.2), the space of sections $h \colon U \times X \to \mathcal{J}$ that is within distance $c_*\|D^2 T\|$ to h_* is invariant under J if c_* is sufficiently large. Consequently, $d(h_0, h_*) \le c_*\|D^2T\|$. □

Let $x \in X$ and $p = (0, x)$. In a small neighbourhood of x, we can define a C^∞ coordinate chart $\tau_p \colon (-1,1)^d \to X$ with $\tau_p(0) = x$ and the following properties:

(1) $D\tau_p(0, \cdot)$ maps $\mathbb{R}^{d_u} \times \{0\}$ (resp. $\{0\} \times \mathbb{R}^{c+d_s}$) to $E^u_f(x)$ (resp. $E^{cs}_f(x)$);
(2) $\widetilde{E}^u$ is close to the tangent space of $\tau_p((-1,1)^{d_u} \times \{z_{cs}\})$, $\forall\, z_{cs} \in (-1,1)^{c+d_s}$;
(3) $\widetilde{E}^{cs}$ is close to the tangent space of $\tau_p(\{z_u\} \times (-1,1)^{c+d_s})$, $\forall\, z_u \in (-1,1)^{d_u}$.

The required closeness in (2),(3) will be made evident from the proof.

Such chart τ_p is obtained by first choosing a C^∞ chart $\tilde{\tau}_p$ satisfying $\tilde{\tau}_p(0) = x$ and (1), and then considering the restriction of $\tilde{\tau}_p$ to a sufficiently small neighbourhood of 0. We can also choose τ_p to depend continuously on p, with $\|\tau_p\|_{C^1}$, $\|\tau_p^{-1}\|_{C^1}$ bounded by a C^1-uniform constant $c'' = c''(f) > 0$.

In the following, we fix $p = (0, x)$ and denote $\tau = \tau_p$. We will not distinguish a point $z \in \tau((-1,1)^d)$ and its coordinate under τ^{-1} e.g. we denote $p = (0, 0)$. Besides, we identify a tangent vector $v \in T_z X$ with its preimage $D\tau^{-1}(v)$, whenever it is defined. Without loss of generality, we assume that δ is small compared to the size of $\tau((-1,1)^d)$.

Let $p' = (0, x') \in \mathcal{W}^u_T(p)$ be a point sufficiently close to p such that there exists $w_0 \in \imath_1(\widetilde{E}^u(p, \delta/2)) \subset \mathbb{R}^{m_1}$ satisfying

$$x' = \exp_x(\imath_1^{-1}(w_0) + \imath_2^{-1}\gamma_p(w_0)). \tag{A.4}$$

Fix any $B \in T_0U$. Recall that $B + \nu_0(x, B) \in E^c_T(p)$. Then let $t > 0$ be any sufficiently small constant, and define $q = q(t) \in \mathcal{W}^c_T(p, \delta)$ by

$$q := (tB, y) = p + t(B, \nu_0(x, B)) + o(t), \qquad y := t\nu_0(x, B) + o(t), \tag{A.5}$$

where $o(t)$ denotes a vector of modulus sublinear in t. Here (A.4) is interpreted as follows: $D\tau^{-1}(t(B, \nu_0(x, B)))$ is a vector in $T_0(-1, 1)^d$, and adds up to $\tau^{-1}(p)$ using the isomorphism $T_0(-1, 1)^d \simeq \mathbb{R}^d$. Several other expressions in this proof shall be understood in the same way. Since $\widetilde{E}^u, \widetilde{E}^{cs}$ are C^1 embedded into $X \times \mathbb{R}^{m_1}$, $X \times \mathbb{R}^{m_2}$ respectively, then for sufficiently small t, there exists $w_2 \in \imath_1(\widetilde{E}^u(y, \delta))$ such that

$$\|w_2 - w_0\|_2 < c_1(\|y\| + t\|B\|)\|w_0\|_1$$

for some C^2-uniform constant $c_1 = c_1(f) > 0$.

We define a point in $\mathcal{W}^u_{\hat{f}(tb,\cdot)}(y, c'\delta)$ by

$$y'' := \exp_y(\imath_1^{-1}(w_2) + \imath_2^{-1}\gamma_q(w_2)).$$

Recall that $TX = \widetilde{E}^u \oplus \widetilde{E}^{cs}$ is C^∞ embedded into $X \times \mathbb{R}^{m_1+m_2}$, and $\exp\colon TX \to X$ is a C^∞ map. Then, by (A.3), (A.5), and since $\mathrm{Lip}(\imath_1), \mathrm{Lip}(\imath_2) \le c_0$, we deduce that

$$\begin{aligned}\|x' - y''\| &< c_2(\|y\| + \|w_0 - w_2\|_1 + \|\gamma_p(w_0) - \gamma_q(w_0)\|_2)\\ &< c_3(t\|\nu_0(x, B)\| + t\|D^2T\|(\|B\| + \|\nu_0(x, B)\|)\|w_0\|_1),\end{aligned} \tag{A.6}$$

for C^2-uniform constants $c_2 = c_2(f), c_3 = c_3(f) > 0$.

Let $q' := H^u_{T,p,p'}(q)$. By definition, $\{q'\} = \mathcal{W}^{cs}_T(p') \cap \mathcal{W}^u_T(q)$. Since $\mathcal{W}^u_T(q) = \{tB\} \times \mathcal{W}^u_{\hat{f}(tB,\cdot)}(y)$, we have $q' = (tB, y')$ for some $y' \in \mathcal{W}^u_{\hat{f}(tB,\cdot)}(y, 2c'\delta)$. On $U \times \tau((-1, 1)^d)$, $\mathcal{W}^{cs}_T(p')$ is closely approximated by $E^{cs}_T(p') = Graph(\nu_0(x', \cdot)) \oplus E^{cs}_f(x')$. Thus $y' = x' + t\nu_0(x', B) + v^{cs}(t) + o(t)$ for some $v^{cs}(t) \in E^{cs}_f(x')$. Hence

$$(y' - y'') - v^{cs}(t) = t\nu_0(x', B) + x' - y'' + o(t). \tag{A.7}$$

Since $y', y'' \in \mathcal{W}^u_{\hat{f}(tB,\cdot)}(y)$, we also know that $y' - y''$ is close to $E^u_{\hat{f}(tB,\cdot)}(y'')$. By conditions (2),(3) in the choice of τ_p, the angle between $E^u_{\hat{f}(tb,\cdot)}(y'')$ and $E^{cs}_f(x')$ is uniformly bounded from below. By (A.7), for some C^2-uniform constant $c_4 = c_4(f) > 0$, it holds

$$\|v^{cs}(t)\|, \|y' - y''\| \le c_4(t\|\nu_0(x', B)\| + \|x' - y''\|). \tag{A.8}$$

Combining estimates (A.6) and (A.8), and since $\|\pi_X p' - \pi_X q'\| \le \|x' - y''\| + \|y' - y''\|$, we deduce that for some C^2-uniform constant $c_5 = c_5(f) > 0$,

$$\|\pi_X p' - \pi_X q'\| \le c_5(\|\nu_0(x, \cdot)\| + \|\nu_0(x', \cdot)\| + \|D^2T\| d_{\mathcal{W}^u_{\hat{f}(b,\cdot)}}(x, x'))t\|B\|.$$

We then conclude our proof by noting that

$$\|\pi_c DH^u_{T,p,p'}(B + \nu_0(x, B))\| = \lim_{t\to 0} \frac{\|\pi_X p' - \pi_X q'\|}{t}.$$

□

We now continue with the proof of (2). Without loss of generality, we may assume that $\sigma_f \in (0, \tilde{\sigma}_f)$. By Lemma A.1 and Lemma 4.13(3), there exist C^2, resp.

C^1-uniform constants $c_2 = c_2(f) > 0$, resp. $c_1 = c_1(f) > 0$, such that (recall (4.7))

$$\|\pi_c DH^u_{T,(0,x),(0,y)}(B + \nu_0(x,B))\| \leq c_2(\|\nu_0(x,\cdot)\| + \|\nu_0(z,\cdot)\| + \|D^2T\| d_{\mathcal{W}^u_f}(x,z))\|B\|$$
$$\leq c_2c_1(\max(e^{-\kappa(\hat{f},x)}, e^{-\kappa(\hat{f},z)})\|T\|_{C^1} + \|D^2T\| d_{\mathcal{W}^u_f}(x,z))\|B\|.$$

□

Appendix B.

Proof of Lemma 4.4. Let $V\colon \mathbb{R}^I \times X \to TX$ be a C^r vector field as in Definition 4.2, and let $U \subset \mathbb{R}^I$ be a small neighbourhood of the origin. We let $\mathscr{F}\colon \mathbb{R}\times U \times X \to X$ be the associate flow; it is defined by the following equation:

$$\partial_t \mathscr{F}(t,b,x) = V(b, \mathscr{F}(t,b,x)), \tag{B.1}$$

with initial condition $\mathscr{F}(0,b,x) = f(x)$. For any $(s,b,x) \in \mathbb{R}\times U \times X$, we have

- $\partial_b\mathscr{F}(0,b,x) = 0$,
- $\partial_b^2\mathscr{F}(0,b,x) = \partial_b\partial_x\mathscr{F}(0,b,x) = 0$,
- $\partial_x\mathscr{F}(0,b,x) = \partial_x f(x)$,
- $\partial_x^2\mathscr{F}(0,b,x) = \partial_x^2 f(x)$,
- $\hat{f}(b,x) = \mathscr{F}(1,b,x)$,
- $f(x) = \mathscr{F}(s,0,x)$.

By differentiating (B.1), we obtain the following equations:

$$\partial_t\partial_x\mathscr{F} = \partial_x V \partial_x \mathscr{F}, \quad \partial_t\partial_b\mathscr{F} = \partial_b V + \partial_x V\partial_b\mathscr{F},$$
$$\partial_t\partial_x^2\mathscr{F} = \partial_x^2 V(\partial_x\mathscr{F}, \partial_x\mathscr{F}) + \partial_x V \partial_x^2\mathscr{F},$$
$$\partial_t\partial_b\partial_x\mathscr{F} = \partial_b\partial_x V(\partial_b, \partial_x\mathscr{F}) + \partial_x^2 V(\partial_x\mathscr{F}, \partial_b\mathscr{F}) + \partial_x V\partial_b\partial_x\mathscr{F},$$
$$\partial_t\partial_b^2\mathscr{F} = \partial_b^2 V + 2\partial_b\partial_x V(\partial_b, \partial_b\mathscr{F}) + \partial_x^2 V(\partial_b\mathscr{F}, \partial_b\mathscr{F}) + \partial_x V\partial_b^2\mathscr{F}.$$

In particular, for all $t \in (0,1)$, $x \in X$ and $B \in T_0U$, we have

$$\partial_t\partial_b\mathscr{F}((t,0,x),B) = \partial_b V((0,\mathscr{F}(t,0,x)),B) + \partial_x V((0,\mathscr{F}(t,0,x)), \partial_b\mathscr{F}((t,0,x),B)).$$

In the above equality, the first term on the RHS equals $V(B, \mathscr{F}(t,0,x)) = V(B, f(x))$; and the second term on the RHS equals 0. Thus

$$\pi_X DT((0,x),B) = \partial_b\hat{f}((0,x),B) = \partial_b\mathscr{F}((1,0,x),B) = V(B, f(x)).$$

This concludes the proof of (2).

By a slight abuse of notations, we use $\|\cdot\|$ to denote the uniform norm for:

(a) derivatives of f, $\partial_b V$ and $\partial_b\partial_x V$ as functions on X; (b) derivatives of $\hat{f}$ and V as functions on $U\times X$; (c) derivatives of $\mathscr{F}$ as functions on $[0,1]\times U\times X$.

To prove (1), we need to bound the norms of $\|D\hat{f}\|$ abd $\|D^2 f\|$. Since $B \mapsto V(B,\cdot)$ is linear, it is clear that by reducing the size of U, we can assume that $\|\partial_x V\| < \frac{1}{10}$. Then by Grönwall's inequality and possibly reducing the size of U, there exists an absolute constant $c_0 > 0$ such that

$$\|\partial_x\mathscr{F}\| < c_0\max(1, \|\partial_x f\|), \quad \|\partial_b\mathscr{F}\| < c_0\|\partial_b V\|. \tag{B.2}$$

Since $\partial_b^2 V \equiv 0$, by Grönwall's inequality and (B.2), there exists an absolute constant $c_1 > 0$ such that

$$\|\partial_x^2\mathscr{F}\| \leq \|\partial_x^2 f\| + c_0\|\partial_x^2 V\|\|\partial_x\mathscr{F}\|^2 \leq \|\partial_x^2 f\| + c_1\|\partial_x^2 V\|\max(1, \|\partial_x f\|^2),$$
$$\|\partial_b\partial_x\mathscr{F}\| \leq c_0(\|\partial_b\partial_x V\|\|\partial_x\mathscr{F}\| + \|\partial_x^2 V\|\|\partial_x\mathscr{F}\|\|\partial_b\mathscr{F}\|)$$
$$\leq c_1(\|\partial_b\partial_x V\| + \|\partial_x^2 V\|\|\partial_b V\|)\max(1, \|\partial_x f\|),$$
$$\|\partial_b^2\mathscr{F}\| \leq 2c_0(\|\partial_b\partial_x V\|\|\partial_b\mathscr{F}\| + \|\partial_x^2 V\|\|\partial_b\mathscr{F}\|^2) \leq c_1(\|\partial_b\partial_x V\|\|\partial_b V\| + \|\partial_x^2 V\|\|\partial_b V\|^2).$$

Note that by possibly reducing the size of U, we can ensure that

$$\|\partial_x^2 V\| < \min(\max(1, \|\partial_x f\|^2)^{-1}, \|\partial_b V\|^{-2}, \|\partial_b V\|^{-1}).$$

Thus there exists an absolute constant $c_2 > 0$ such that

$$\|\hat{f}\|_{C^2} \le \|\mathscr{F}(1,\cdot)\|_{C^2} < c_2(1 + \|\partial_b \partial_x V\|)(1 + \|f\|_{C^2} + \|\partial_b V\|).$$

We conclude the proof of (1) by noticing that $\|D^i T\| \lesssim \|D^i \hat{f}\|$ for $i = 1, 2$. $\square$

Appendix C.

Proof of Theorem F. We repeat the proof of [10, Proposition 3.2] for ϵ-light maps, instead of light maps, for some $\epsilon = \epsilon(n)$.

Recall that the order of a cover $\mathcal{O} = \{O_k\}_{k\in K}$ is the supremum of all numbers $\#K'$ such that $\cap_{k\in K'} O_k \neq \emptyset$. Let then $\mathcal{V}_0$ be a cover of $X := [0,1]^n$ such that $\mathcal{V}_0$ does not admit an open refinement of order less than or equal to n. Take $\delta > 0$ a Lebesgue number of the cover $\mathcal{V}_0$ and define $\epsilon := \delta/2$.

Assume that $f : X \to Y$ is ϵ-light. Let T be some triangulation of Y and denote by $\mathcal{U} = \{U_i\}_{i\in I}$ its open star cover. For every $i \in I$, $f^{-1}(U_i)$ can be written as a disjoint union of connected open sets; they form an open cover of X, denoted by $\mathcal{V} = \{V_j\}_{j\in J}$. For each $j \in J$, we let $\alpha(j) \in I$ be such that $V_j \subset f^{-1}(U_{\alpha(j)})$. By assumption, f is ϵ-light, hence we can choose T fine enough such that each V_j has diameter smaller than $2\epsilon = \delta$. Therefore $\mathcal{V}$ is an open refinement of $\mathcal{V}_0$; in particular, any open refinement of $\mathcal{V}$ has order at least $n+1$.

We define $\mathrm{Ner}(\mathcal{U})$ as the collection of subsets $I' \subset I$ such that $\cap_{i\in I'} U_i \neq \emptyset$. We define $\mathrm{Ner}(\mathcal{V})$ in a similar way. Both $\mathrm{Ner}(\mathcal{U})$ and $\mathrm{Ner}(\mathcal{V})$ are simplicial n-complexes, and we identify them with their geometric realization.

Given a partition of unity $\{\rho_i\}$ subordinate to $\mathcal{U}$, we get a homeomorphism $\rho : Y \to \mathrm{Ner}(\mathcal{U})$, while α induces a local homeomorphism $\phi : \mathrm{Ner}(\mathcal{V}) \to \mathrm{Ner}(\mathcal{U})$. Then, the functions $\nu_j := \chi_{V_j} \cdot (\rho_{\alpha(j)} \circ \phi)$, $j \in J$, define a partition of unity subordinate to $\mathcal{V}$. We let $\nu : X \to \mathrm{Ner}(\mathcal{V})$ be the associate map, where $\nu(x)$ has barycentric coordinates $(\nu_j(x))_j$. By construction, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle \nu}\big\downarrow & & \big\downarrow{\scriptstyle \rho} \\ \mathrm{Ner}(\mathcal{V}) & \xrightarrow[\phi]{\quad\quad} & \mathrm{Ner}(\mathcal{U}) \end{array}$$

Let us see that some n-simplex σ of $\mathrm{Ner}(\mathcal{V})$ has an interior point ξ which is a stable value of $\nu : X \to \mathrm{Ner}(\mathcal{V})$. Assume it is not the case; then we may form a set $\mathcal{S}$ by choosing one interior point from each n-simplex of $\mathrm{Ner}(\mathcal{V})$ and perturb ν on a small neighbourhood of $\nu^{-1}(\mathcal{S})$ to get $\nu' : X \to \mathrm{Ner}(\mathcal{V})\backslash\mathcal{S}$. Denote by $p : \mathrm{Ner}(\mathcal{V})\backslash\mathcal{S} \to [\mathrm{Ner}(\mathcal{V})]_{n-1}$ the radial projection to the $(n-1)$-skeleton of $\mathrm{Ner}(\mathcal{V})$, s.t. the barycentric coordinates of $\nu'' := p \circ \nu' : X \to [\mathrm{Ner}(\mathcal{V})]_{n-1}$ are subordinate to $\mathcal{V}$. Pulling back the open star cover of $\mathrm{Ner}(\mathcal{V})$ by ν'', we get a refinement of $\mathcal{V}$ of order at most n, a contradiction. Thus, $\rho^{-1}(\phi(\xi))$ is a stable value of f, which concludes. $\square$

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