

# MARKED LENGTH SPECTRAL DETERMINATION OF ANALYTIC CHAOTIC BILLIARDS WITH AXIAL SYMMETRIES

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ABSTRACT. We consider billiards obtained by removing from the plane finitely many strictly convex analytic obstacles satisfying the non-eclipse condition. The restriction of the dynamics to the set of non-escaping orbits is conjugated to a subshift, which provides a natural labeling of periodic orbits. We show that under suitable symmetry and genericity assumptions, the Marked Length Spectrum determines the geometry of the billiard table.

## CONTENTS

Introduction	2
1. Definitions and statement of our main results	5
1.1. Spectral determination	7
1.2. Idea of the proof	11
1.3. Organization of the paper	14
1.4. Previous results	15
2. The Birkhoff Normal Form in a neighborhood of period two orbits	15
2.1. Canonical choice of the conjugacy map $R$	18
2.2. The time reversal involution in Birkhoff coordinates	22
3. Extension of the Birkhoff coordinates along the separatrices	24
4. Marked Lyapunov Spectrum and Birkhoff invariants	26
4.1. Preliminary estimates on the parameters	26
4.2. Lyapunov exponents and asymptotic expansions of the parameters	30
4.3. Determination of the scaled coefficients $\{\bar{g}_\ell, \bar{\gamma}_\ell, \bar{a}_\ell\}_{\ell \geq 0}$	43
4.4. Change of Lyapunov exponents in the horseshoe	45
5. Further estimates on the Marked Length Spectrum	46
5.1. Basic facts about twist maps and generating functions	46
5.2. Estimates on the Marked Length Spectrum	47
5.3. $\mathcal{MLS}$ -determination of the Birkhoff data	49
6. Reconstructing the geometry from the Marked Length Spectrum	52
6.1. A construction for symmetric billiard tables	53
6.2. Recovering the geometry of a symmetric billiard table	55
7. Conclusions and further questions	60
Appendix A. The Marked Lyapunov Spectrum	60
References	63

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## INTRODUCTION

Billiard dynamics studies the behavior of a point particle which moves freely on some domain and undergoes elastic reflections upon collision with the domain boundary. In this article we consider the class of domains obtained by removing from the plane a number  $m \geq 3$  of strongly convex<sup>1</sup> compact sets with analytic boundary. Systems of this type were first considered in [GR], where the authors studied the classical scattering of a point particle from three circular disks on the plane. Strong convexity of the obstacles allows to construct stable and unstable cones for the billiard map, hence implies that the corresponding billiard dynamics enjoys strong hyperbolicity properties, see e.g. [CM, Section 4.4], [Sin, W1, W2]. Hyperbolicity, together with a standard non-eclipse condition (see Section 1), allows to encode the dynamics of non-escaping trajectories as a subshift of finite type on  $m$  symbols (see e.g. [Mor] or [PS2, Section 2.2]). This observation provides, in particular, a natural marking of each periodic orbit with the associated encoding. The *Marked Length Spectrum* is then defined as the set of all lengths of periodic orbits together with their marking (see Definition 1.1).

Two domains that have the same Marked Length Spectrum are said to be *marked-length-isospectral*. For instance, two isometric domains are necessarily isospectral. On the other hand, it is a fascinating problem to characterize marked-length-isospectral billiards modulo isometries. We refer to this problem as the *dynamical inverse spectral problem*.

The main result of this article is that, under suitable symmetry and genericity assumptions, the Marked Length Spectrum of a domain is sufficient to reconstruct the domain (up to isometries).

In order to describe our results in some context, let us first present some related classical problems and corresponding results.

**The Laplace inverse spectral problem.** The dynamical inverse spectral problem introduced above is deeply related to the question that M. Kac (see [K]) famously phrased as: “*Can one hear the shape of a drum?*”, i.e. is the shape of a planar domain determined by its Laplace Spectrum (with either Dirichlet or Neumann boundary conditions)? The relation between the dynamical and the Laplace problem is apparent, for instance, in the thesis of Colin de Verdière [CdV1, CdV2], where it is shown that the Laplace Spectrum determines the Length Spectrum of a generic manifold, and in the trace formula proved by Andersson–Melrose [AM]: generalizing previous results by Chazarain [Ch], Duistermaat–Guillemin [DG], they showed that, for strictly convex<sup>2</sup>  $C^\infty$  domains, the singular support of the wave trace is contained in the Length Spectrum (in fact, generically, the two sets coincide). In particular, in this setting, the Laplace Spectrum generically determines the Length Spectrum. Similarly, there is a connection between Laplace Spectrum and Length Spectrum in hyperbolic situations: indeed, the Selberg trace formula [Se] shows that the Laplace Spectrum determines the Length Spectrum on hyperbolic manifolds, or for generic Riemannian metrics. In particular, in the case of closed hyperbolic surfaces, the stronger statement that the Laplace Spectrum determines the Length Spectrum and vice-versa holds (see [H1, H2]).

<sup>1</sup> A set is strongly convex if its boundary has strictly positive curvature.

<sup>2</sup> The trace formula (and its consequences) also holds without the convexity assumption for planar domains, and for convex higher dimensional domains (see e.g. [PS2]).

**Spectral determination and spectral rigidity for convex domains with symmetries.** It has been famously proven by Zelditch in a series of papers (see [Z1, Z2, Z3]) that the Laplace Spectrum completely determines (modulo isometries) the domain in a generic class of analytic  $\mathbb{Z}_2$ -symmetric (i.e., symmetric with respect to some axis of reflection) planar convex domains. Hezari–Zelditch [HZ2] have obtained a higher dimensional analogue of this result: bounded analytic domains in  $\mathbb{R}^n$  with reflection symmetries across all coordinate axes, and with one axis height fixed (satisfying some generic non-degeneracy conditions) are spectrally determined among other such domains. Notably, the same authors have shown in a very recent preprint (see [HZ3]) that ellipses of small eccentricity are identified by their Laplace spectrum among smooth domains with the symmetries of the ellipse.

Spectral determination results are, currently, far beyond reach for smooth (non-elliptic) domains. In the last decade, however, interesting results have appeared for a deformation version of spectral determination known as *spectral rigidity*<sup>3</sup>. In [HZ1], Hezari–Zelditch have shown the following result: given a domain bounded by an ellipse (of arbitrary eccentricity), any one-parameter isospectral  $C^\infty$  deformation which additionally preserves the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry group of the ellipse is necessarily flat (i.e., all derivatives have to vanish at the initial parameter).

The problem of dynamical spectral determination was studied by Colin de Verdière; in [CdV3], he has shown that, in the class of convex analytic billiards with the symmetries of the ellipse, the Marked Length Spectrum determines the domain geometry. In the smooth category, in [DKW], the authors proved that any sufficiently smooth  $\mathbb{Z}_2$ -symmetric strictly convex domain sufficiently close to a circle is dynamically spectrally rigid, i.e., all deformations among domains in the same class which preserve the length of all periodic orbits of the associated billiard flow must necessarily be isometries; see also the numerical exploration [AD] by Ayub-De Simoi for ellipses of eccentricity smaller than 0.30. Moreover, recently, for smoothly conjugate billiard maps of Birkhoff billiards, Kaloshin-Koudjina [KK] studied rigidity in the form of Marvizi-Melrose invariants.

**The Inverse Problem for flat billiards.** Spectral rigidity questions have also been explored in the context of flat billiards. In [DELS], the authors study the relationship between the shape of a Euclidean polygon and the symbolic dynamics of its billiard flow. They introduce a Bounce Spectrum for polygons, and show that two simply connected Euclidean polygons with the same Bounce Spectrum are either similar or right-angled and affinely equivalent.

**Determination of the geometry of a negatively curved surface by the Marked Length Spectrum.** Another natural setting where similar questions were investigated is for geodesic flows on negatively curved surfaces. As in the present work, the dynamics of such flows is hyperbolic.<sup>4</sup> A famous result due independently to Otal [O] and Croke [Cr] show that if  $g_0$  and  $g_1$  are negatively curved metrics on a closed surface with the same Marked Length Spectrum, then  $g_1$  is isometric to  $g_0$ . In higher dimension, Guillarmou–Lefeuvre in [GL] recently proved that the Marked

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<sup>3</sup> Recall that a domain is said to be *spectrally rigid* if any Laplace isospectral *continuous deformation* is necessarily isometric.

<sup>4</sup> Yet, in that case, the geodesic flow is a genuine *Anosov* flow, while for open dispersing billiards, the interesting dynamics occurs only on a Cantor set.

Length Spectrum of a Riemannian manifold  $(M, g)$  with Anosov geodesic flow and non-positive curvature locally determines the metric.

**The Laplace Inverse Resonance Problem.** A dual (or exterior) formulation of the inverse spectral problem is the inverse resonance problem, in which one attempts to reconstruct an unbounded domain (e.g. the complement of a finite number of convex scatterers) by the resonances (i.e., the poles) of the resolvent  $(\Delta - z^2)^{-1}$  (see e.g. [PS2, Zw2, Z4]). In particular, in [Z4], Zelditch showed that a  $\mathbb{Z}_2$ -symmetric configuration of two convex analytic obstacles in the plane  $\mathbb{R}^2$  is determined by its Dirichlet or Neumann resonance poles. This result is the direct analogue, for exterior domains, of the proof that a  $\mathbb{Z}_2$ -symmetric bounded simply connected analytic plane domain is determined by the Laplace eigenvalue spectrum. The proof is based on the fact that wave invariants of an exterior domain are resonance invariants and on the method of [Z2, Z3] for calculating the wave invariants explicitly in terms of the boundary defining function. In [ISZ], the authors give another proof of the inverse result with two symmetries using Birkhoff Normal Forms of the billiard map and quantum monodromy operator rather than the Laplacian, applying some results on semi-classical trace formulae and on quantum Birkhoff normal forms for semi-classical Fourier integral operators to inverse problems. Generalizing results of [G], they showed that the classical Birkhoff Normal Form can be recovered from semi-classical spectral invariants, and in fact, that the full quantum Birkhoff Normal Form of the quantum Hamiltonian near a closed orbit, and infinitesimally with respect to the energy can be recovered.

**Dynamical inverse spectral problems for billiards with hyperbolic dynamics.** We finally come to the setting that will be explored in this article. In [BDKL], the authors (together with P. Bálint) study billiards obtained by removing  $m \geq 3$  strictly convex finitely smooth obstacles from  $\mathbb{R}^2$ ; it is stated (without a complete proof) that the Marked Length Spectrum determines the Lyapunov exponent of each periodic orbit; in Appendix A we give a proof of a slightly weaker result that suffices for our current purposes. The proof is similar to the one sketched for [BDKL, Corollary E].

It is a crucial observation that, unlike billiards inside convex domains, billiards inside the domains considered in this article are *open systems*, i.e. there exist initial conditions (in fact, a full-measure set) for which the billiard ball escapes to infinity after finitely many bounces. As a consequence, it will be the case that no periodic trajectory visits certain regions of the configuration space. This implies that periodic spectral data will never suffice to recover the geometry of the unexplored region. Hence, one can either attempt to recover the full geometry under additional rigidity assumptions (e.g. analyticity of the scatterers), or consider the question of determination restricted to the explored region.

In this paper, we pursue the first strategy and assume that all scatterers have real analytic curves as boundary. Our goal is achieved once we recover the full jet of the curvature at some point on each scatterer. Due to analyticity, this entirely determines the geometry of the scatterers. The result stated below as Main Theorem, asserts that this is indeed possible, provided that two scatterers have some symmetries (similar to the “bi-symmetric” setting of [Z1]).

In our proof, we reconstruct from spectral data the classical (hyperbolic) Birkhoff Normal Form of the two-periodic orbit associated to the symmetric scatterers. This can be done, provided that a genericity condition is satisfied, by analyzing some asymptotics in the Marked Length Spectrum relative to periodic orbits that approximate homoclinic orbits of the two-periodic orbit.

Once the normal form has been obtained, exploiting the symmetries of our system, and some extra information that can be obtained by the Marked Length Spectrum, it is possible to reconstruct the geometry of the billiard. A more detailed explanation of the proof will be given in Section 1.2, after introducing some necessary preliminaries.

Our results are an analogue of those presented in [CdV3] for the class of chaotic billiards under consideration, or an analogue in terms of the Marked Length Spectrum of [Z1] (see also [ISZ]).

Note that due to the convexity of the obstacles, the presence of more than two scatterers in our case is crucial to guarantee the existence of a large set of periodic orbits. On the other hand, billiard trajectories in the exterior of only two strictly convex domains in the plane were considered by Stoyanov in [Sto].

## 1. DEFINITIONS AND STATEMENT OF OUR MAIN RESULTS

In the present paper, we consider billiard tables  $\mathcal{D} \subset \mathbb{R}^2$  given by  $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^m \mathcal{O}_i$ , for some integer  $m \geq 3$ , where each  $\mathcal{O}_i$  is a convex domain with analytic boundary  $\partial\mathcal{O}_i$ . We refer to each of the  $\mathcal{O}_i$ 's as *obstacle* or *scatterer*. We let  $\ell_i := |\partial\mathcal{O}_i|$  be the corresponding lengths, set  $\mathbb{T}_i := \mathbb{R}/\ell_i\mathbb{Z}$ , and parametrize each  $\partial\mathcal{O}_i$  in arc-length, for some analytic map  $\Upsilon_i \in C^\omega(\mathbb{T}_i, \mathbb{R}^2)$ ,  $s \mapsto \Upsilon_i(s)$ . We assume that the following condition holds:

**NON-ECLIPSE CONDITION:** The convex hull of any two scatterers is disjoint from the other  $m - 2$  scatterers.

The set of all billiard tables obtained by removing from the plane  $m$  strictly convex analytic obstacles satisfying the non-eclipse condition will be denoted by  $\mathbf{B}(m)$ .

Fix  $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^m \mathcal{O}_i \in \mathbf{B}(m)$ . We denote the collision space by

$$\mathcal{M} = \bigcup_i \mathcal{M}_i, \quad \mathcal{M}_i = \{(q, v), q \in \partial\mathcal{O}_i, v \in \mathbb{R}^2, \|v\| = 1, \langle v, n \rangle \geq 0\},$$

where  $n$  is the unit normal vector to  $\partial\mathcal{O}_i$  pointing inside  $\mathcal{D}$ . For each  $x = (q, v) \in \mathcal{M}$ ,  $q$  is associated with the arclength parameter  $s \in [0, \ell_i]$  for some  $i \in \{1, \dots, m\}$ , i.e.,  $q = \Upsilon_i(s)$ . We let  $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$  be the oriented angle between  $n$  and  $v$  and set  $r := \sin(\varphi)$ . In other words, each  $\mathcal{M}_i$  can be seen as a cylinder  $\mathbb{T}_i \times [-1, 1]$  endowed with coordinates  $(s, r)$ . In the following, given a point  $x = (q, v) \in \mathcal{M}$  associated with the pair  $(s, r)$ , we also denote by  $\Upsilon(s) := q$  the point of the table defined as the projection of  $x$  onto the  $q$ -coordinate. Moreover, for each pair  $(s, r), (s', r') \in \mathcal{M}$ , we denote by

$$(1.1) \quad h(s, s') := \|\Upsilon(s') - \Upsilon(s)\|$$

the Euclidean length of the segment connecting the associated points of the table.

Set  $\Omega := \{(q, v) \in \mathcal{D} \times \mathbb{S}^1\}$ . Denote by  $\Phi^t: \Omega \rightarrow \Omega$  the flow of the billiard and let

$$\mathcal{F} = \mathcal{F}(\mathcal{D}): \mathcal{M} \rightarrow \mathcal{M}, \quad x \mapsto \Phi^{\tau(x)+0}(x)$$

be the associated billiard map, where  $\tau: \mathcal{M} \rightarrow \mathbb{R}_+ \cup \{+\infty\}$  is the first return time, and  $\Phi^{\tau(x)+0}(x)$  is the image of the point  $x$  right after the collision at time  $\tau(x)$ . For any point  $x = (s, r) \in \mathcal{M}$  such that  $(s', r') := \mathcal{F}(s, r)$  is well-defined, we denote by  $\mathcal{L} := h(s, s')$  the distance between the two points of collision, we let  $\mathcal{K} := \mathcal{K}(s)$ ,  $\mathcal{K}' := \mathcal{K}(s')$  be the respective curvatures, and set  $\nu := \sqrt{1 - r^2}$ ,  $\nu' := \sqrt{1 - (r')^2}$ . It follows from [CM, (2.26)] that

$$(1.2) \quad D\mathcal{F}_{(s,r)} = - \begin{pmatrix} \frac{1}{\nu'}(\mathcal{L}\mathcal{K} + \nu) & \frac{\mathcal{L}}{\nu\nu'} \\ \mathcal{L}\mathcal{K}\mathcal{K}' + \mathcal{K}\nu' + \mathcal{K}'\nu & \frac{1}{\nu}(\mathcal{L}\mathcal{K}' + \nu') \end{pmatrix} \in \text{SL}(2, \mathbb{R}),$$

and the map  $\mathcal{F}$  is symplectic for the form  $ds \wedge dr$ .

Due to the convexity of the obstacles, for each  $i, j \in \{1, \dots, m\}$ , with  $i \neq j$ , there exist  $0 \leq a_i^j \leq b_i^j \leq \ell_i$ , and for each parameter  $s \in [a_i^j, b_i^j]$ , there exists a non-empty closed interval  $I_i^j(s) \subset [-1, 1]$  such that  $\tau(x) < +\infty$ , if  $x = (s, r) \in \widetilde{\mathcal{M}}_i := \bigcup_{k \neq i} \widetilde{\mathcal{M}}_i^k$ , and  $\tau(x) = +\infty$ , if  $x \in \mathcal{M}_i \setminus \widetilde{\mathcal{M}}_i$ , where

$$\widetilde{\mathcal{M}}_i^j := \{(s, r) \in \mathcal{M}_i : s \in [a_i^j, b_i^j], r \in I_i^j(s)\} = \mathcal{M}_i \cap \mathcal{F}^{-1}(\mathcal{M}_j).$$

In other words,  $[a_i^j, b_i^j]$  is the interval of parameters  $s$  of points sitting at  $\partial\mathcal{O}_i$  which can be joined to some point on  $\partial\mathcal{O}_j$  for some suitable angle, namely  $r \in I_i^j(s)$ .

In particular, the set of trajectories that do not escape to infinity is given by

$$\bigcap_{k \in \mathbb{Z}} \mathcal{F}^{-k}(\widetilde{\mathcal{M}}), \quad \widetilde{\mathcal{M}} := \bigcup_{j \neq i} \widetilde{\mathcal{M}}_j,$$

and is homeomorphic to a Cantor set. Due to the non-eclipse condition, the restriction of the dynamics to this set is conjugated to a subshift of finite type associated with the transition matrix

$$\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}.$$

In other words, any word  $(\varsigma_j)_j \in \{1, \dots, m\}^{\mathbb{Z}}$  such that  $\varsigma_{j+1} \neq \varsigma_j$  for all  $j \in \mathbb{Z}$  can be realized by an orbit, and by hyperbolicity of the dynamics, this orbit is unique. Such a word is called *admissible*. Besides, this marking is unique provided that we fix a starting point in the orbit and an orientation.

In particular, any periodic orbit of period  $p$  (observe that necessarily  $p \geq 2$ ) can be labeled by a finite admissible word  $\sigma = (\sigma_1 \sigma_2 \dots \sigma_p) \in \{1, \dots, m\}^p$ , such that the infinite word  $\sigma^\infty := \dots \sigma \sigma \sigma \dots$  is admissible (or equivalently, such that  $\sigma_j \neq \sigma_{j+1 \bmod p}$ , for all  $j \in \{1, \dots, p\}$ ). We denote by  $\text{Adm}$  the set of finite admissible words  $\sigma \in \bigcup_{p \geq 2} \{1, \dots, m\}^p$ .

Given any word  $\sigma = (\sigma_1 \sigma_2 \dots \sigma_p) \in \text{Adm}$ , we denote by  $\bar{\sigma}$  be the transposed word

$$\bar{\sigma} := (\sigma_p \sigma_{p-1} \dots \sigma_1).$$

The word  $\bar{\sigma}$  encodes the same periodic trajectory as  $\sigma$ , but with opposite orientation.

As explained above, for any  $j \in \{1, \dots, p\}$ , the  $j^{\text{th}}$  symbol  $\sigma_j$  of  $\sigma$  corresponds to a point  $x(j)$  in the trajectory, where  $x(j) = (s(j), r(j)) \in \mathcal{M}_{\sigma_j}$  is represented by position and angle coordinates. For all  $k \in \mathbb{Z}$ , we also extend the previous notation by setting  $\sigma_k := \sigma_{k \bmod p}$ , and similarly for  $x(k)$ ,  $s(k)$  and  $r(k)$ .

**Definition 1.1.** *The Marked Length Spectrum  $\mathcal{MLS}(\mathcal{D})$  of  $\mathcal{D}$  is defined as the function*

$$(1.3) \quad \mathcal{L}: \text{Adm} \rightarrow \mathbb{R}_+, \quad \sigma \mapsto \mathcal{L}(\sigma),$$

where  $\mathcal{L}(\sigma)$  is the length of the periodic orbit identified by  $\sigma$ , obtained as the sum of the lengths of all the line segments that compose it.

An object is said to be a  $\mathcal{MLS}$ -invariant if it can be obtained by the sole knowledge of the Marked Length Spectrum.

For any periodic orbit  $(x_1, \dots, x_p)$  encoded by a word  $\sigma$  of length  $p \geq 2$ , we have  $D_{x_j} \mathcal{F}^p \in \text{SL}(2, \mathbb{R})$ , for  $j \in \{1, \dots, p\}$ . Due to the strong convexity of the obstacles, each of the matrices  $D_{x_j} \mathcal{F}^p$  is hyperbolic, and we denote by  $\lambda(\sigma) < 1 < \lambda(\sigma)^{-1}$  their (common) eigenvalues. We define the *Lyapunov exponent* of this orbit as

$$(1.4) \quad \text{LE}(\sigma) := -\frac{1}{p} \log \lambda(\sigma) > 0.$$

**Definition 1.2.** *The Marked Lyapunov Spectrum of the billiard table  $\mathcal{D}$  is defined as the function*

$$(1.5) \quad \text{LE}: \text{Adm} \rightarrow \mathbb{R}_+, \quad \sigma \mapsto \text{LE}(\sigma).$$

We conclude this section by recalling an important symmetry of the billiard dynamics, which will be crucial in the following. Let us denote by  $\mathcal{I}$  the involution  $\mathcal{I}: (s, r) \mapsto (s, -r)$ . It conjugates the billiard map  $\mathcal{F}$  with its inverse  $\mathcal{F}^{-1}$ , according to the time reversal property of the billiard dynamics:

$$\mathcal{I} \circ \mathcal{F} \circ \mathcal{I} = \mathcal{F}^{-1}.$$

A periodic orbit of period  $p \geq 2$  is called *palindromic* if it can be labeled by an admissible word  $\sigma \in \{1, \dots, m\}^p$  such that  $\sigma = (\sigma_1 \dots \sigma_{q-1} \sigma_q \sigma_{q-1} \dots \sigma_1 \sigma_0)$  for certain symbols  $(\sigma_0, \sigma_1, \dots, \sigma_q) \in \{1, \dots, m\}^{q+1}$ . Observe that  $p = 2q$ , hence we gather that palindromic orbit always have even period. As we shall see later, there is a connection between the palindromic symmetry and the time reversal property recalled above. In particular, by the palindromic symmetry and by expansiveness of the dynamics, the associated trajectory hits the billiard table perpendicularly at the points associated to the symbols  $\sigma_0$  and  $\sigma_q$ .

For more details about chaotic billiards and inverse spectral problems, we refer the reader to the books of Chernov–Markarian [CM] and Petkov–Stoyanov [PS2].

**1.1. Spectral determination.** Recall that  $\mathbf{B}(m)$  is the set of all billiard tables  $\mathcal{D}$  formed by  $m \geq 3$  convex analytic obstacles satisfying the non-eclipse condition, that  $\mathcal{F}(\mathcal{D})$  denotes the associated billiard map, and that  $\mathcal{K}$  is the curvature function. We introduce a class of tables with two additional symmetries, which, without loss of generality, are assumed to be associated with the obstacles  $\mathcal{O}_1, \mathcal{O}_2$ .

**Definition 1.3.** *We let  $\mathbf{B}_{\text{sym}}(m) \subset \mathbf{B}(m)$  be the subset of all billiard tables  $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^m \mathcal{O}_i$  which are symmetric in the following sense:*

- *the jets of  $\mathcal{K}$  are the same at the endpoints of the 2-periodic orbit (12);*
- *the jets of  $\mathcal{K}|_{\mathbb{T}_1}, \mathcal{K}|_{\mathbb{T}_2}$  are even, assuming that  $0_1 \in \mathbb{T}_1, 0_2 \in \mathbb{T}_2$  are the arc-length parameters of the endpoints of the orbit (12).*

In particular, by analyticity, the pair of obstacles  $\mathcal{O}_1, \mathcal{O}_2$  has some  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetry:  $\mathcal{O}_1, \mathcal{O}_2$  are images of each other by the reflection along the line segment bisector of the trace of the orbit (12), and each of them is symmetric with respect to the line through the endpoints of (12).

The reason for requiring the two symmetries will be clarified in Remark 1.8 below. In the following, we let  $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ , and let  $C^\omega(\mathbb{T}, \mathbb{R})$  be the space of  $2\pi$ -periodic real analytic functions. Given  $r > 0$ , we denote by  $C_r^\omega(\mathbb{T}, \mathbb{R}) \subset C^\omega(\mathbb{T}, \mathbb{R})$  the subspace of (bounded) analytic functions defined on the strip<sup>5</sup>  $\{z \in \mathbb{C}/(2\pi\mathbb{Z}) : |\Im z| < r\}$  and extending continuously to the boundary, and for any  $f \in C_r^\omega(\mathbb{T}, \mathbb{R})$ , we let  $|f|_r := \sup_{\{|\Im z| < r\}} |f(z)|$ . We denote by  $C^\omega(\mathbb{T}, \mathbb{R}^2)$  the space of analytic functions  $f: \theta \mapsto (f_1(\theta), f_2(\theta))$ , with  $f_1, f_2 \in C^\omega(\mathbb{T}, \mathbb{R})$ . Similarly, for any  $r > 0$ , we denote by  $C_r^\omega(\mathbb{T}, \mathbb{R}^2) \subset C^\omega(\mathbb{T}, \mathbb{R}^2)$  the subspace of functions  $f = (f_1, f_2) \in (C_r^\omega(\mathbb{T}, \mathbb{R}))^2$  endowed with the norm  $\|\cdot\|_r$ , where  $\|f\|_r := \max(|f_1|_r, |f_2|_r)$ .

**Definition 1.4** (Topology on  $\mathbf{B}_{\text{sym}}(m, r)$ ). *Let  $\mathbf{Conv} \subset C^\omega(\mathbb{T}, \mathbb{R}^2)$  be the set of all functions  $f \in C^\omega(\mathbb{T}, \mathbb{R}^2)$  such that  $f(\mathbb{T})$  is a simple closed curve such that the interior region bounded by  $f(\mathbb{T})$  is strongly convex, i.e., the curvature of  $f(\mathbb{T})$  never vanishes. We denote by  $\mathcal{O}(f)$  the interior region bounded by  $f(\mathbb{T})$ . For any  $r > 0$ , we let  $\mathbf{Conv}_r := \mathbf{Conv} \cap C_r^\omega(\mathbb{T}, \mathbb{R}^2)$ . For any integer  $m \geq 3$ , we thus get a map*

$$\Phi = \Phi(m): \mathbf{Conv}^m \ni (f^{(i)})_{i=1, \dots, m} \mapsto \mathcal{D} := \mathbb{R}^2 \setminus \bigcup_{i=1}^m \mathcal{O}(f^{(i)}).$$

Given  $r > 0$ , let  $\mathbf{W}_{\text{sym}}(m, r) := \Phi^{-1}(\mathbf{B}_{\text{sym}}(m)) \cap \mathbf{Conv}_r^m$ , and endow it with the topology induced by the product topology on  $(C_r^\omega(\mathbb{T}, \mathbb{R}^2))^m$ . Then we let  $\mathbf{B}_{\text{sym}}(m, r) := \Phi(\mathbf{W}_{\text{sym}}(m, r))$  and equip it with the topology coinduced by the map  $\Phi$ . In particular, for any  $\mathcal{D} \in \mathbf{B}_{\text{sym}}(m)$ , we have  $\mathcal{D} \in \mathbf{B}_{\text{sym}}(m, r)$  for some  $r > 0$ .

We are now ready to state the main result of this paper.

**Main Theorem.** *For any  $m \geq 3$  and for any  $r > 0$ , there exists an open and dense set of billiard tables  $\mathbf{B}_{\text{sym}}^*(m, r) \subset \mathbf{B}_{\text{sym}}(m, r)$  so that if  $\mathcal{D}, \mathcal{D}' \in \mathbf{B}_{\text{sym}}^*(m, r)$  verify  $\mathcal{MLS}(\mathcal{D}) = \mathcal{MLS}(\mathcal{D}')$ , then  $\mathcal{D}$  is isometric to  $\mathcal{D}'$ .*

**Remark 1.5.** *In fact, the open and dense condition we need is an explicit non-degeneracy condition. Namely: we require that after a change of coordinates, the first coefficient in the expansion of the dynamics near a certain two-periodic orbit is non-vanishing (see Remark 1.9 and condition  $(\star)$  in Lemma 6.6).*

*In particular, given a table  $\mathcal{D}_0 = \Phi((f^{(i)})_{i=1, \dots, m}) \in \mathbf{B}_{\text{sym}}(m, r)$  for some  $r > 0$ , with  $(f^{(i)})_{i=1, \dots, m} \in \mathbf{Conv}_r^m$ , and a “non-degenerate” family  $(\mathcal{D}_x)_x$  of perturbations of  $\mathcal{D}_0$  within  $\mathbf{B}_{\text{sym}}(m, r')$  for some  $r' \in (0, r]$ , we can give an explicit description of the set of tables  $\mathcal{D}_x$  which are not in  $\mathbf{B}_{\text{sym}}^*(m, r')$ . More precisely, let  $\mathbb{D} \subset \mathbb{C}$  be the unit disk, let  $\Delta := \mathbb{D}^{\mathbb{N}}$ , and let  $\varepsilon: \mathbb{N} \rightarrow \mathbb{R}_+ \setminus \{0\}$  be an arbitrary function that decays exponentially fast, with  $\sup_{n \in \mathbb{N}} \varepsilon(n) < \varepsilon_0$  for some small  $\varepsilon_0 > 0$ . For some  $r' \in (0, r]$ , we thus get a family  $(\mathcal{D}_x^\varepsilon)_{x \in \Delta}$  of billiards in  $\mathbf{B}_{\text{sym}}(m, r')$ , where for*

<sup>5</sup> Given a complex number  $z$ , we denote its real (resp. imaginary) part with  $\Re z$  (resp.  $\Im z$ ).

$x = (x_m)_{m \in \mathbb{N}} \in \Delta$ , the billiard  $\mathcal{D}_x^\varepsilon$  is given by

$$\mathcal{D}_x^\varepsilon := \Phi(f^{(1)} + g_x^\varepsilon, f^{(2)} + g_x^\varepsilon, f^{(3)}, \dots, f^{(m)}),$$

$$g_x^\varepsilon := \left[ \theta \mapsto \sum_{m \geq 1} \varepsilon(m) \Re \left( x_m e^{2\pi i m \theta} \right) \right] \in C_{r'}^\omega(\mathbb{T}, \mathbb{R}^2).$$

Let us consider the map  $\mathbf{a}_1^\varepsilon: \Delta \ni x \mapsto a_1(\mathcal{D}_x^\varepsilon) \in \mathbb{R}$  in Lemma 6.6. The map  $\mathbf{a}_1^\varepsilon$  is a submersion (see the proof of Lemma 6.6 for more details), hence the set  $\Delta_{\text{bad}}^\varepsilon := \{x \in \Delta : \mathcal{D}_x^\varepsilon \notin \mathbf{B}_{\text{sym}}^*(m, r')\} = (\mathbf{a}_1^\varepsilon)^{-1}(\{0\})$  of parameters for which the associated table is not in  $\mathbf{B}_{\text{sym}}^*(m, r')$  is a codimension one submanifold of  $\Delta$ .

**Remark 1.6.** We have stated our Main Theorem in the case of  $m \geq 3$  scatterers, but indeed it suffices to prove the statement for  $m = 3$ . In fact, fix  $m > 3$ ,  $r > 0$ , and let

$$\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^m \mathcal{O}_i \in \mathbf{B}_{\text{sym}}(m, r);$$

for  $2 < i \leq m$ , define

$$\mathcal{D}_i := \mathbb{R}^2 \setminus (\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_i).$$

It is immediate to show that  $\mathcal{D}_i \in \mathbf{B}_{\text{sym}}(3, r)$  (since the non-eclipse condition holds automatically). Assume that the Main Theorem has been proved for  $m = 3$ , so that  $\mathbf{B}_{\text{sym}}^*(3, r)$  is defined, and let

$$\mathbf{B}_{\text{sym}}^*(m, r) := \{\mathcal{D} \in \mathbf{B}_{\text{sym}}(m, r) \text{ s.t. } \forall 2 < i \leq m, \mathcal{D}_i \in \mathbf{B}_{\text{sym}}^*(3, r)\}.$$

It is easy to check that  $\mathbf{B}_{\text{sym}}^*(m, r)$  is open and dense. Since  $\mathcal{MLS}(\mathcal{D}_i)$  is the restriction of  $\mathcal{MLS}(\mathcal{D})$  to the periodic orbits that only collide with  $\mathcal{O}_1$ ,  $\mathcal{O}_2$  and  $\mathcal{O}_i$ , we can apply our Main Theorem for  $m = 3$  to  $\mathcal{D}_i$  and recover the geometry of  $\mathcal{O}_1$ ,  $\mathcal{O}_2$  and  $\mathcal{O}_i$  for any  $i$ . Since  $i$  was arbitrary, we proved the Main Theorem for  $m$ .

It is a standard observation that any continuous deformation of smooth domains which preserves the (unmarked) Length Spectrum  $\mathcal{LS}(\mathcal{D})$  automatically preserves the Marked Length Spectrum (see e.g. [Sib2, Proposition 3.2.2]); therefore our result also implies a spectral rigidity result. Let us first introduce a definition.

**Definition 1.7.** Given an integer  $m \geq 3$  and  $r > 0$ , a family  $(\mathcal{D}_t)_{t \in (-1, 1)}$  of billiards is called an *iso-length-spectral deformation* in  $\mathbf{B}_{\text{sym}}^*(m, r)$  if

- each  $\mathcal{D}_t$  is in  $\mathbf{B}_{\text{sym}}^*(m, r)$ , and the map  $(-1, 1) \ni t \mapsto \mathcal{D}_t$  is continuous;
- $\mathcal{LS}(\mathcal{D}_t) = \mathcal{LS}(\mathcal{D}_0)$ , for all  $t \in (-1, 1)$ .

Our Main Theorem thus yields the following result.

**Theorem.** For  $m \geq 3$  and  $r > 0$ , any iso-length-spectral deformation in  $\mathbf{B}_{\text{sym}}^*(m, r)$  is isometric.

The proof of the Main Theorem (for  $m = 3$ ) is given in Corollary 6.7 and Corollary 6.8 in Section 6, based on the constructions provided in detail in the next sections. From now on, we will only consider the case of three scatterers. We will abbreviate  $\mathbf{B} := \mathbf{B}(3)$  and  $\mathbf{B}_{\text{sym}} := \mathbf{B}_{\text{sym}}(3)$  and similarly for  $\mathbf{B}_{\text{sym}}(r) := \mathbf{B}_{\text{sym}}(3, r)$  and  $\mathbf{B}_{\text{sym}}^*(r) := \mathbf{B}_{\text{sym}}^*(3, r)$ .

Let  $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^3 \mathcal{O}_i \in \mathbf{B}$  be a billiard table, and let  $\mathcal{F} := \mathcal{F}(\mathcal{D})$ . A key object in our study is the so-called *Birkhoff Normal Form* for saddle fixed points of symplectic local surface diffeomorphisms, whose definition we now recall. We introduce it for period two orbits since this is the case we will consider in the following, but the same can be done for any periodic orbit (given a periodic orbit of period  $p \geq 2$ , each point in the orbit is a saddle fixed point of  $\mathcal{F}^p$ ). Let  $j \neq k \in \{1, 2, 3\}$ , and let  $(s(j, k), 0)$  be the  $(s, r)$ -coordinates of the point of  $\mathcal{O}_j$  in the orbit  $(jk)$ . Recall that by [Mos, Ste], there exists an analytic symplectomorphism  $R: \mathcal{U} \rightarrow \mathcal{V}$  from a neighborhood  $\mathcal{U} \subset \mathcal{M}$  of  $(s(j, k), 0)$  to a neighborhood  $\mathcal{V} \subset \mathbb{R}^2$  of  $(0, 0)$  and a unique analytic map  $\Delta = \Delta(\mathcal{D}, j, k) \in C^\omega(\mathbb{R}, \mathbb{R}^*)$ , with  $\Delta(z) = \lambda + \sum_{\ell \geq 1} a_\ell z^\ell$ , s.t.

$$R \circ \mathcal{F}^2|_{\mathcal{U}} = N \circ R|_{\mathcal{U}},$$

where  $N$  is the *Birkhoff Normal Form* of  $\mathcal{F}^2|_{\mathcal{U}}$ :

$$N = N(\mathcal{D}, j, k): (\xi, \eta) \mapsto (\Delta(\xi\eta)\xi, \Delta(\xi\eta)^{-1}\eta).$$

In the following, we refer to  $(a_\ell)_\ell$  as the *Birkhoff invariants* or *coefficients* of  $N$ .

**Remark 1.8.** *The two symmetries described above are needed because of two different issues. Let us consider a billiard table  $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^3 \mathcal{O}_i \in \mathbf{B}_{\text{sym}}$ .*

- *The axial symmetry between  $\mathcal{O}_1$  and  $\mathcal{O}_2$  we ask for is similar to the one that appears for instance in the work of Zelditch. It is explained by the fact that in order to speak about Birkhoff Normal Forms, we need a fixed point. As the billiard map has no such fixed point, we need at least to consider its square. In the process, some information is lost, unless  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are the mirror image of each other; otherwise, we are a priori only able to recover some averaged information between  $\mathcal{O}_1$  and  $\mathcal{O}_2$ .*
- *The second symmetry we require is due to a well known observation made in [CdV3]. Indeed, the reason why we ask each of the obstacles  $\mathcal{O}_1, \mathcal{O}_2$  to be themselves symmetric follows from the fact that the Birkhoff Normal Form has some intrinsic symmetries (it has two axes of symmetry), and only conveys partial information on the billiard dynamics, which has a priori only one natural symmetry, given by the time reversal property. Roughly speaking, we lose half of the information on the billiard map, unless the map itself has some additional symmetry: this symmetry of the map can be ensured provided that  $\mathcal{O}_1, \mathcal{O}_2$  have a  $\mathbb{Z}_2$ -symmetry.*

**Remark 1.9.** *In the following, we show that the Lyapunov exponents of the orbits  $(h_n)_n$  can be expanded as a series indexed by  $\mathbb{Z}^2$  (see (1.6)), each of whose coefficients is a  $\mathcal{MLS}$ -invariant. The expression of these coefficients combines three different sets of geometric data, including the Birkhoff invariants we want to reconstruct. We show that under some open and dense condition, it is possible to extract enough information from the first three lines of the coefficients of the series in order to recover separately the three sets of data. More precisely, the condition we need is the non-vanishing of the first Birkhoff invariant (see condition  $(\star)$  in Lemma 6.6), which can be seen dynamically as some twist condition. It guarantees that certain linear systems in the data we want to recover are invertible. Note that Lemma 4.10 gives an effective way of checking whether a given billiard table  $\mathcal{D} \in \mathbf{B}_{\text{sym}}$  satisfies*

*this twist condition; in other words, the property “ $\mathcal{D} \in \mathbf{B}_{\text{sym}}^*$ ” itself is a  $\mathcal{MLS}$ -invariant. Besides, this condition comes from the particular subset of coefficients we consider (which is easiest to work with), and it is likely that considering other subsets of coefficients would produce another non-degeneracy condition involving different Birkhoff invariants. In particular, it seems reasonable to believe that as long as the Birkhoff Normal Form is not degenerate (i.e., is not linear), our construction can be adapted to produce invertible systems in the coefficients we want to reconstruct.*

**1.2. Idea of the proof.** Let us give an idea of the proof of the above results. We fix a billiard table  $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^3 \mathcal{O}_i \in \mathbf{B}_{\text{sym}}$ , and let  $\mathcal{F} := \mathcal{F}(\mathcal{D})$ . Note that it is natural to focus on 2-periodic orbits, since we may hope to determine  $\mathcal{F}^2$  instead of  $\mathcal{F}^p$  for some higher exponent  $p \geq 3$ , and also because of the additional symmetries of such orbits. As we shall see, the Birkhoff coefficients  $(a_\ell)_{\ell \geq 1}$  above can be obtained by the asymptotics of Lyapunov exponent for certain periodic orbits which spend a lot of time near the periodic orbit (12). We then define a sequence of periodic orbits  $(h_n)_n$  with a certain palindromic symmetry that accumulate<sup>6</sup> an orbit  $h_\infty$  which is homoclinic to (12).

A key step in the construction is the extension of the coordinates given by the conjugacy  $R$  between  $\mathcal{F}^2$  and its Birkhoff Normal Form, which is initially defined only in a neighborhood of the saddle fixed point. Indeed, in order to make the connection with the Lyapunov exponent of the orbits  $(h_n)_n$ , it is crucial to extend the conjugacy to describe them globally. The construction of the extension follows a classical procedure, by using the dynamics to propagate  $R$  along the separatrices, i.e., the stable and unstable manifolds of the origin. This is actually sufficient to describe all points in the orbits  $(h_n)_n$ , since for  $n$  large enough, these orbits stay in a small neighborhood of the separatrices. In this way, we produce convenient coordinates to describe the dynamics in a neighborhood of the separatrices, which can be seen as a hyperbolic analogue of the coordinates provided by the Birkhoff Normal Form near the boundary of the billiard table, in the elliptic case, and which were used for instance in [CdV3]. The problem is that we are extending our coordinates along two different directions, and at some point, since the trajectory is periodic, these two extensions will overlap in the collision space. In particular, we will need to perform a “gluing” of the two charts obtained in this way in a neighborhood of the homoclinic point on the third scatterer, and we will explain how to take care of this issue in the sequel.

By the palindromic property, we can write an equation for the images under the conjugacy map  $R$  of the points in the periodic orbits  $(h_n)_n$  (see Lemma 3.2). This allows us to find an implicit expression of the parameters of those points in terms of the Birkhoff invariants and the coefficients of the arc of points where those orbits start (as we shall see, this arc is made of points on the second obstacle which bounce perpendicularly on the third obstacle after one iteration of the dynamics). As mentioned above (see Theorem A.1), the Marked Lyapunov Spectrum (for palindromic orbits) is a  $\mathcal{MLS}$ -invariant. The key observation is stated in Lemma 4.21: we show that for each integer  $n \geq 0$ , the Lyapunov exponent of  $h_n$  admits a series expansion;

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<sup>6</sup> An analogous construction can be found in [BDKL].

more precisely, it holds

$$(1.6) \quad 2\lambda^n \cosh(2(n+1)\text{LE}(h_n)) = \sum_{p=0}^{+\infty} \sum_{q=0}^p L_{q,p} n^q \lambda^{np},$$

for some sequence  $(L_{q,p})_{\substack{p=0,\dots,+\infty \\ q=0,\dots,p}}$ , and where  $\lambda \in (0, 1)$  is the contracting eigenvalue of  $D\mathcal{F}^2$  at the two-periodic orbit (12). In particular, each coefficient  $L_{q,p}$  is a  $\mathcal{MLS}$ -invariant, and by restricting ourselves to  $q = 0, 1, 2$ , this gives enough information to recover the Birkhoff coefficients. Note that the expansion (1.6) obtained for the Lyapunov exponents of palindromic orbits in the horseshoe associated to the homoclinic orbit  $h_\infty$  can be seen as a hyperbolic analogue of the Marvizi–Melrose expansion (see for instance [MM, (1.11)] on p.3) for the maximum lengths of periodic orbits with a certain rotation number and period (the integer  $n$  being related to the period in either case). Let us also mention that similar expansions were studied in [FY] for a different purpose; moreover, the relation between coefficients in Birkhoff normal forms and spectral properties of the dynamics has been explored as well in different settings in several works, see e.g. [Be, Sib1].

One technical issue comes from the fact that the orbits  $(h_n)_n$  bounce on the third obstacle, thus there are additional terms which come from a certain gluing map  $\mathcal{G}$  taking this last bounce into account, and we need to find a way to recover this data as well. The idea is to leverage on the “triangular” structure of the coefficients: at each step, there are certain additive constants associated with some terms that we already know, as well as new coefficients that we want to recover. Then, we derive a linear system in the new coefficients, and show that it is invertible under a suitable twist condition (non-vanishing of the first Birkhoff invariant). By induction, modulo some “homoclinic parameter”  $\xi_\infty \in \mathbb{R}$ , we can thus recover the Birkhoff invariants, as well as some information on the third obstacle associated to the differential of the gluing map  $\mathcal{G}$ .

More precisely, we consider some arc  $\Gamma_\infty \ni (0, \xi_\infty)$  which is the image in Birkhoff coordinates of some small arc of points associated with a perpendicular bounce on the third scatterer (see also Figure 4). This arc is the graph of some analytic function  $\gamma$  that is determined by the gluing map  $\mathcal{G} = (\mathcal{G}^+, \mathcal{G}^-)$ , i.e., for  $\xi$  small,  $\eta = \xi_\infty + \gamma(\xi)$  satisfies the implicit equation

$$\mathcal{G}^+(\xi, \eta)\mathcal{G}^-(\xi, \eta) = \xi\eta.$$

We show that from the sequence  $(L_{q,p})_{\substack{p=0,\dots,+\infty \\ q=0,1,2}}$ , up to the parameter  $\xi_\infty$ , it is possible to recover the value of the Birkhoff invariants, as well as the function  $\gamma$  and the differential  $D\mathcal{G}|_{T\Gamma_\infty}$ .

Note that homotheties preserve Lyapunov exponents, hence by only considering the Marked Lyapunov Spectrum, one cannot expect to recover the scale of the billiard table: the missing parameter  $\xi_\infty$  can indeed be interpreted as a *scaling factor*. In Subsection 5.3, we prove that its value is a  $\mathcal{MLS}$ -invariant: we show (see Proposition 5.1) that for some  $\mathcal{L}^\infty \in \mathbb{R}$ , the quantity

$$(1.7) \quad \mathcal{L}(h_n) - (n+1)\mathcal{L}(12) - \mathcal{L}^\infty$$

decays exponentially fast as  $n$  goes to infinity (see Section 2 for the notation), and can be expanded as a series of the same form as the one obtained in (1.6) – and also similar to the expansion obtained in [MM]. The first order term in this expansion

is a  $\mathcal{MLS}$ -invariant and can be written in terms of  $\xi_\infty^2$  and of a certain quadratic form. It can be then shown (see the proof of Corollary 5.3) that the latter is a  $\mathcal{MLS}$ -invariant, thus  $\xi_\infty$  is a  $\mathcal{MLS}$ -invariant too.

It is worth emphasizing that all results proved up to Subsection 5.3 – in particular, the  $\mathcal{MLS}$ -determination of the Birkhoff Normal Form and of the gluing map  $\mathcal{G}$ , up to the homoclinic parameter  $\xi_\infty$  – do not require any symmetry assumption. Indeed, in our approach, axial symmetries are needed only to reconstruct the geometry from the Birkhoff Normal Form and the map  $\mathcal{G}$ . Furthermore, although we focus on Birkhoff Normal Forms associated to 2-periodic orbits in this paper, the procedure we describe is more general. More precisely, under a generic condition (non-vanishing of the first Birkhoff invariant of each periodic orbit), our construction can be adapted to show that the Birkhoff Normal Form of any (palindromic) periodic orbit is a  $\mathcal{MLS}$ -invariant.

Although we restrict ourselves to a specific class of dispersing billiards in the present work, similar computations can be carried out for other types of billiards. The general framework where this can be done is when the billiard under consideration possesses a hyperbolic periodic orbit with a transverse homoclinic intersection. If we stay away from singularities (tangential collisions), it is well-known that this intersection generates a horseshoe; we may then construct a sequence  $(h_n)_n$  of periodic orbits within this hyperbolic set with a specific combinatorics, and study the asymptotics of their Lyapunov exponents as  $n \rightarrow +\infty$ . The same computations as those explained in the present paper can be performed to show that the Birkhoff invariants of the periodic orbit can be reconstructed from the Lyapunov exponents of the orbits  $(h_n)_n$ . Yet, what is especially convenient for the class of open dispersing billiards considered here is that the symbolic coding which is used to select the orbits  $(h_n)_n$  from the horseshoe has a precise *geometric* meaning (namely, the sequence of obstacles corresponding to the consecutive bounces in the orbit  $h_n$ ), which makes it easier to relate the symbolic coding to the marking used in our definition of the Marked Length Spectrum.<sup>7</sup>

Let us now consider the case of symmetric billiard tables  $\mathcal{D} \subset \mathbf{B}_{\text{sym}}$ . In this case, we can introduce some flat wall between  $\mathcal{O}_1$  and  $\mathcal{O}_2$ , and “fold” the table in order to virtually create a fixed point of the billiard dynamics. For some auxiliary billiard table  $\mathcal{D}^*$  one of whose obstacles is now flat, we can extract enough information from  $\mathcal{MLS}(\mathcal{D})$  to reconstruct the Birkhoff Normal Form  $N^*$  of the square  $T^* := (\mathcal{F}^*)^2$  of the new billiard map  $\mathcal{F}^* = \mathcal{F}^*(\mathcal{D}^*)$  near the new 2-periodic orbit. Besides, by the construction of  $N^*$  (see e.g. [Bi, Ste, Mos]), and *due to the symmetry of  $\mathcal{O}_1, \mathcal{O}_2$ , the jets of  $N^*$  and  $T^*$  are in one-to-one correspondence, by some invertible triangular system*. Furthermore, as Colin de Verdière [CdV3] already observed in the elliptic setting, there is a bijective correspondence between the jet of  $T^*$  and the jet of the graphs of  $\mathcal{O}_1, \mathcal{O}_2$ , which can thus be reconstructed.

In order to recover the geometry of the last obstacle, we analyze the information that comes from the gluing map that we were mentioning previously. We can extract from this map some “averaged” information between the first two obstacles  $\mathcal{O}_1, \mathcal{O}_2$  and the third obstacle  $\mathcal{O}_3$ , and since the geometry of  $\mathcal{O}_1, \mathcal{O}_2$  is known, we can

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<sup>7</sup> For instance, convex billiards may exhibit a hyperbolic set associated to a hyperbolic periodic orbit; in this case, similar computations can be performed to show that the Birkhoff invariants can be determined from the variation of the Lyapunov exponents of certain periodic orbits  $(h_n)_n$  in the horseshoe, but it is less natural to “mark” the orbits  $(h_n)_n$  by some geometric information.

also reconstruct the local geometry of  $\mathcal{O}_3$  near a certain homoclinic point. This determines the obstacle  $\mathcal{O}_3$  entirely, by analyticity.

**1.3. Organization of the paper.** We use the notations introduced in Subsection 1.2. The proof of the Main Theorem follows different steps:

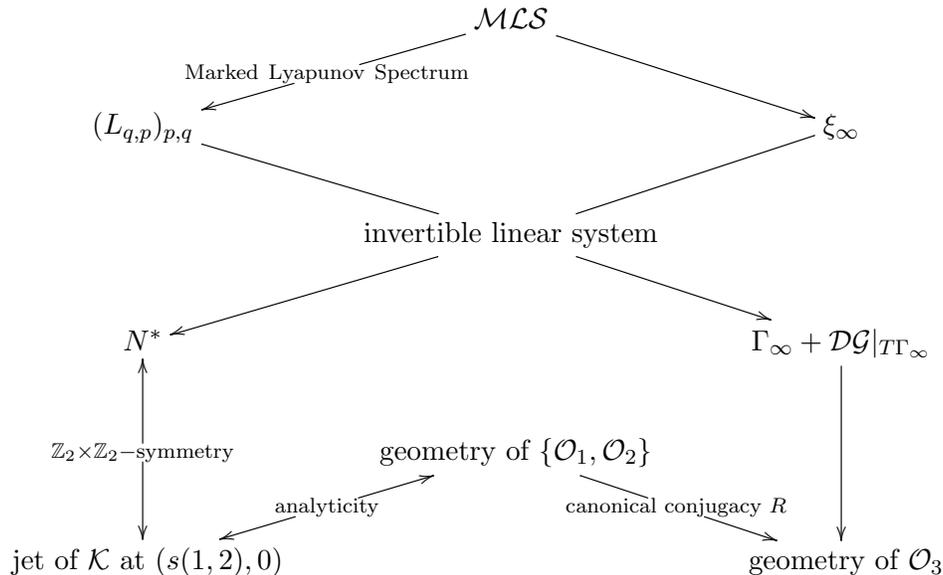
- \* **Step 1:** existence of a *canonical* (which respects the time reversal symmetry) change of coordinates under some non-degeneracy condition;
- \* **Step 2:** extension of the system of coordinates and expression of the palindromic orbits  $(h_n)_n$  in those coordinates;
- \* **Step 3:** definition of the gluing map  $\mathcal{G}$ ;
- \* **Step 4:** asymptotic expansion (in  $n$ ) of the Lyapunov exponent of  $h_n$ ;
- \* **Step 5:** extracting a triangular system from the Lyapunov expansion in terms of “scaled” Birkhoff coefficients and gluing terms;
- \* **Step 6:** invertibility of this system under some non-degeneracy condition (proof by induction: compute the  $i^{\text{th}}$  order terms using the previous ones);
- \* **Step 7:**  $\mathcal{MLS}$ -determination of the missing “scale” parameter  $\xi_\infty$ ;
- \* **Step 8:** Step 5 + Step 6  $\Rightarrow$  the Birkhoff Normal Form and the differential of the gluing map  $\mathcal{G}$  are  $\mathcal{MLS}$ -invariants;
- \* **Step 9:** determination of the geometry from the Birkhoff invariants + the gluing map  $\mathcal{G}$  in the case of symmetric billiard tables which satisfy a non-degeneracy condition ( $\star$ ), Lemma 6.6.

Those steps are detailed respectively in:

(1) Section 2; (2) Section 3; (3) Subsection 4.1; (4)-(5) Subsection 4.2; (6) Subsection 4.3; (7) Subsections 5.1-5.2; (8) Subsection 5.3; (9) Section 6.

The central technical part of the proof is Steps 4-5; an outline of the computations carried out there is given after Remark 4.9 (see also Remark 4.17). Let us also emphasize formula (4.8) which follows from the palindromic symmetry of the orbits  $(h_n)_n$  and on which the induction is based.

Moreover, the scheme of the proof can be summarized as follows:



**1.4. Previous results.** In this paper we will use and further develop many of the ideas that appeared in the paper [BDKL], joint with P. Bálint. Vaughn Osterman found out that [BDKL] contains a gap as-published; the gap affects the main result of the paper, but a number of useful technical results can be still recovered from the paper. We list the most important such results in this section; the reader will find in Appendix A self-contained proofs of them.

**Theorem 1.10.** *The Marked Lyapunov Spectrum (for palindromic orbits) is determined by the Marked Length Spectrum (see (1.3) and (1.5) for the definitions).*

In particular the above theorem allows to conclude the following result.

**Corollary 1.11.** *Assuming that  $\mathcal{O}_2$  is the mirror image of  $\mathcal{O}_1$ , then the (common) radius of curvature  $R$  at the bouncing points of the (12) periodic orbit is a  $\mathcal{MLS}$ -invariant.*

## 2. THE BIRKHOFF NORMAL FORM IN A NEIGHBORHOOD OF PERIOD TWO ORBITS

Let us fix a billiard table  $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^3 \mathcal{O}_i \in \mathbf{B}$  and study the local dynamics near 2-periodic orbits. We focus on the 2-periodic orbit  $\sigma = (12)$  (recall that our symmetry assumption holds for those two scatterera); it has two perpendicular bounces on the first and the second obstacles. Let us denote by  $x(0) = (s(0), 0)$  and  $x(1) = (s(1), 0)$  the  $(s, r)$  coordinates (recall the definition at the beginning of Section 1) of the points in this orbit, where  $s(0)$ , (resp.  $s(1)$ ) is the position of the point on the first (resp. second) obstacle. We extend this notation by periodicity by setting  $x(k) := x(k \bmod 2)$ , for  $k \in \mathbb{Z}$ . We let  $\tau := (32)$  and, given any integer  $n \geq 1$ , set

$$h_n = h_n(\sigma, \tau) := (\tau \sigma^n) = (32 \underbrace{1212 \dots 12}_{2n}).$$

The word  $h_n$  encodes a periodic orbit of period  $2n + 2$ ; observe that all such orbit are palindromic.

Let  $\mathcal{F} = \mathcal{F}(\mathcal{D}): x_0 = (s_0, r_0) \mapsto x_1 = (s_1, r_1)$  be the billiard map. In such coordinates,  $\mathcal{F}$  is exact symplectic, with generating function

$$(2.1) \quad h(s, s') := \|\Upsilon(s) - \Upsilon(s')\|,$$

where  $\|\Upsilon(s) - \Upsilon(s')\|$  is the Euclidean length of the line segment between the two points identified by parameters  $s$  and  $s'$ . In other words, we have

$$(2.2) \quad dh(s_0, s_1) = -r_0 ds_0 + r_1 ds_1,$$

i.e.,  $\partial_1 h(s_0, s_1) = -r_0$ , and  $\partial_2 h(s_0, s_1) = r_1$ .

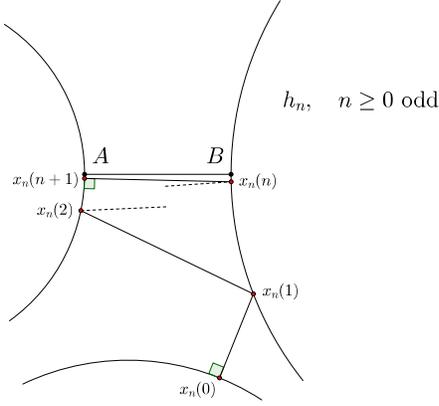


FIGURE 1. Trace of the orbits  $h_n$  on the billiard table when  $n \geq 0$  is odd.

Let us denote by  $(x_n(k) = (s_n(k), r_n(k)))_{k=0, \dots, 2n+1}$  the coordinates of the points in the orbit  $h_n$ , where  $x_n(0) = (s_n(0), 0)$  is the only collision on the third obstacle, and  $x_n(n+1) = (s_n(n+1), 0)$  is the point of the orbit which is closest to the periodic orbit  $\sigma = (12)$  (see Figure 2). Again, thanks to the  $2n + 2$ -periodicity of  $h_n$ , we can extend those coordinates to any  $k \in \mathbb{Z}$ . By the palindromic symmetry, for any  $k \in \{0, \dots, n+1\}$ , it holds

$$(2.3) \quad x_n(2n+2-k) = \mathcal{I}(x_n(k)),$$

where  $\mathcal{I}(s, r) := (s, -r)$ . Recall that for a periodic orbit encoded by a finite word  $\zeta$ , we denote its length by  $\mathcal{L}(\zeta)$ . We have

$$\mathcal{L}(h_n) - (n+1)\mathcal{L}(\sigma) = 2 \sum_{k=0}^n \left( h(s_n(k), s_n(k+1)) - h(s(k), s(k+1)) \right).$$

Let  $h_\infty = h_\infty(\sigma, \tau)$  be the homoclinic trajectory encoded by the infinite word  $(\sigma^\infty \tau \sigma^\infty) = (\dots 21212321212 \dots)$ . We denote by  $(x_\infty(k))_{k \in \mathbb{Z}}$  its coordinates, with  $x_\infty(k) = (s_\infty(k), r_\infty(k))$ , for  $k \in \mathbb{Z}$ . We label them in such a way that  $x_\infty(0)$  is associated with the unique bounce on the third obstacle, and  $r_\infty(k)r_n(k) \geq 0$  for all  $k \in \mathbb{Z}$ . Since  $h_\infty$  is homoclinic to  $\sigma$ , the quantity  $h(s_\infty(k), s_\infty(k+1)) - h(s(k), s(k+1))$

1)) decays exponentially fast as  $k \rightarrow +\infty$  (see also Proposition 2.1 below), and the following limit remains well defined:

$$(2.4) \quad \mathcal{L}^\infty = \mathcal{L}^\infty(\sigma) = \lim_{n \rightarrow \infty} (\mathcal{L}(h_n) - (n+1)\mathcal{L}(\sigma)) \\ = 2 \sum_{k=0}^{+\infty} \left( h(s_\infty(k), s_\infty(k+1)) - h(s(k), s(k+1)) \right).$$

Let us now write

$$\mathcal{L}(h_n) - (n+1)\mathcal{L}(\sigma) - \mathcal{L}^\infty = \Sigma_n^1 + \Sigma_n^2,$$

where

$$(2.5a) \quad \Sigma_n^1 := 2 \sum_{k=0}^n \left( h(s_n(k), s_n(k+1)) - h(s_\infty(k), s_\infty(k+1)) \right),$$

$$(2.5b) \quad \Sigma_n^2 := 2 \sum_{k=n+1}^{+\infty} \left( h(s(k), s(k+1)) - h(s_\infty(k), s_\infty(k+1)) \right).$$

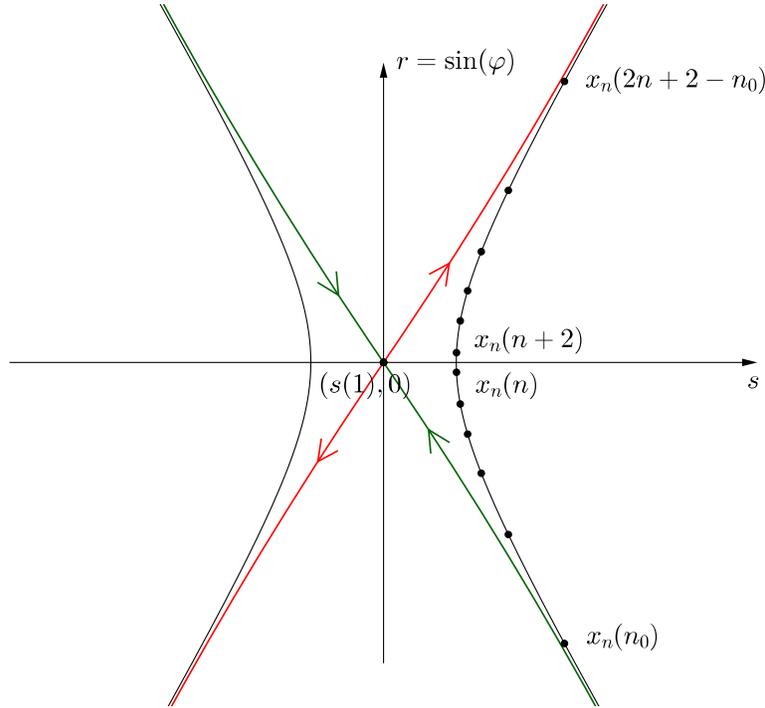


FIGURE 2.  $(s, r)$ -representation of the points in  $h_n$  near the orbit (12).

The point  $x(1) = (s(1), 0)$  is a saddle fixed point of  $\mathcal{F}^2$  with eigenvalues  $\lambda < 1 < \lambda^{-1}$ . The following proposition was shown in [BDKL], it will also be proved later as Proposition 4.6.

**Proposition 2.1.** *There exists an integer  $n_0 \geq 1$  such that*

$$\|x_\infty(k) - x(k)\| = O(\lambda^{\frac{k}{2}}), \quad \text{for all } k \in \mathbb{N}, \\ \|x_n(k) - x_\infty(k)\| = O(\lambda^{n-\frac{k}{2}}), \quad \text{for all } n \geq n_0 \text{ and } k \in \{0, \dots, n+1\}.$$

The first estimate comes from the fact that the points in the homoclinic orbit are on the stable manifold of the point  $x(1)$ . Moreover, as  $n \rightarrow \infty$ , everything happens in a neighborhood of the unstable and stable manifolds of the periodic orbit  $\sigma$ . Indeed, for the first half of the orbit  $h_n$ , i.e., for  $k \in \{0, \dots, n+1\}$ , the second estimate above tells us that the points  $x_n(k)$  shadow closely the associated points  $x_\infty(k)$  in the homoclinic orbit  $h_\infty$ , and thus, stay close to the stable manifold of  $x(1)$ . On the other hand, for the second half of  $h_n$ , i.e., for  $k \in \{n+1, \dots, 2n+2\}$ , then by the palindromic symmetry (2.3), the points  $x_n(k)$  shadow closely the points  $\mathcal{I}(x_\infty(k))$ , and thus, stay close to the unstable manifold of  $x(1)$ .

Let us consider the case where  $n$  is odd, i.e.,  $n = 2m - 1$  for some integer  $m \geq 1$ , and let us study the dynamics of  $T := \mathcal{F}^2$ . Here, the period of  $h_n = h_{2m-1}$  is equal to  $2n + 2 = 4m$ . For simplicity, we assume in the following that  $s(1) = 0$ .

By (2.2), the map  $\mathcal{F}$  is symplectic for the form  $ds \wedge dr$ , where  $r := \sin(\varphi)$ . It follows that  $T$  is symplectic too, i.e.,  $T^*(ds \wedge dr) = ds \wedge dr$ . Then, by [Mos] (see also [Ste]), there exists a neighborhood  $\mathcal{U}$  of  $(0, 0)$  in the  $(s, r)$ -plane, and an analytic symplectic change of coordinates

$$R: \begin{cases} \mathcal{U} & \rightarrow \mathbb{R}^2, \\ (s, r) & \mapsto (\xi, \eta) \end{cases}$$

with  $R^*(d\xi \wedge d\eta) = ds \wedge dr$ , which conjugates  $T$  to its Birkhoff Normal Form  $N = R \circ T \circ R^{-1}$ :

$$N = N_\Delta: (\xi, \eta) \mapsto (\Delta(\xi\eta) \cdot \xi, \Delta(\xi\eta)^{-1} \cdot \eta),$$

where  $\Delta$  is an analytic function

$$\Delta: z \mapsto \lambda + \sum_{k=1}^{+\infty} a_k z^k.$$

The numbers  $(a_k)_{k \geq 1}$  are called the *Birkhoff invariants* (or *Birkhoff coefficients*) of  $T$  at  $(0, 0)$ .

In the following, given two open sets  $\mathcal{V}, \mathcal{V}' \subset \mathbb{R}^2$ , we will denote by  $\text{Sympl}^\omega(\mathcal{V}, \mathcal{V}')$  the set of real analytic symplectomorphisms from  $\mathcal{V}$  to  $\mathcal{V}'$  which preserve the form  $d\xi \wedge d\eta$ .

**2.1. Canonical choice of the conjugacy map  $R$ .** The map  $R$  mentioned in the previous section is not uniquely defined; in this section we proceed to identify a canonical choice of coordinates which respects additional symmetries of the periodic orbit.

The following lemma ensures that the map  $N$  is well defined in a neighborhood of the vertical and horizontal axes.

**Lemma 2.2.** *For all  $(\xi, \eta) \in R(\mathcal{U})$ , and for each  $k \in \mathbb{Z}$ , the point  $(\Delta(\xi\eta)^{-k}\xi, \Delta(\xi\eta)^k\eta)$  is in the domain of definition of  $N$ . In particular,  $N$  can be extended to each point in the orbit of  $(\xi, \eta)$ .*

*Proof.* Let  $k \in \mathbb{Z}$ ; clearly,  $N$  is well defined at a point  $(\Delta(\xi\eta)^k\xi, \Delta(\xi\eta)^{-k}\eta)$  if and only if  $\Delta$  is well defined at the point  $(\Delta(\xi\eta)^k\xi) \cdot (\Delta(\xi\eta)^{-k}\eta) = \xi\eta$ , which is true, provided that  $(\xi, \eta) \in R(\mathcal{U})$ .  $\square$

As a consequence of this observation, there exists some  $\epsilon_0 > 0$  such that the Birkhoff Normal Form  $N$  is well defined on the neighborhood  $\mathcal{V}_{\epsilon_0} = \{(\xi, \eta) \in \mathbb{R}^2 : |\xi\eta| < \epsilon_0\}$  of the coordinate axes  $\{\xi = 0\}$  and  $\{\eta = 0\}$ . In particular,  $N$  preserves  $\mathcal{V}_{\epsilon_0}$  and acts merely as a translation along hyperbolas..

Next, we study the symmetries of Birkhoff Normal Forms.

**Definition 2.3** (Centralizer of  $N$ ). *Let  $0 < \epsilon_1 \leq \epsilon_0$ ; we define the symplectic centralizer  $\mathcal{C}_N = \mathcal{C}_N(\epsilon_1)$  of  $N$  as the set of maps  $F \in \text{Symp}^{\omega}(\mathcal{V}_{\epsilon_1}, \mathcal{V}_{\epsilon_0})$ , such that*

$$F \circ N|_{\mathcal{V}_{\epsilon_1}} = N \circ F|_{\mathcal{V}_{\epsilon_1}}.$$

*Notice that, since  $N$  preserves  $\mathcal{V}_{\epsilon}$ , for any  $\epsilon \leq \epsilon_0$ , we can iterate the above expression and obtain*

$$(2.6) \quad F \circ N^j|_{\mathcal{V}_{\epsilon_1}} = N^j \circ F|_{\mathcal{V}_{\epsilon_1}}, \quad \text{for all } j \in \mathbb{Z}.$$

**Remark 2.4.** *Observe that an analytic symplectic map  $F$  is defined in a neighborhood  $\mathcal{V}$  of  $(0, 0)$  and commutes with  $N$ , then it can always be analytically and symplectically extended to  $\tilde{\mathcal{V}} = \bigcup_{k \in \mathbb{Z}} N^k \mathcal{V}$  using  $N$  as follows. For  $x \in \tilde{\mathcal{V}}$ , let  $k$  be so that  $N^k x \in \mathcal{V}$ , then  $F(x) = N^{-k} \circ F \circ N^k$ . It is then immediate to show that this is an analytic, symplectic extension and it commutes with  $N$  on  $\tilde{\mathcal{V}}$ .*

**Lemma 2.5.** *Let  $N$  be so that the Birkhoff invariants  $(a_k)_{k \geq 1}$  are not all equal to zero. Then, provided that  $\epsilon_1 > 0$  is chosen sufficiently small (depending on the sequence  $(a_k)_{k \geq 1}$ ), the centralizer  $\mathcal{C}_N = \mathcal{C}_N(\epsilon_1)$  of  $N$  is given by the set of maps of the form*

$$(2.7) \quad F: (\xi, \eta) \mapsto (\tilde{\Delta}(\xi\eta)\xi, \tilde{\Delta}(\xi\eta)^{-1}\eta), \quad \text{for some } \tilde{\Delta} \in C^{\omega}((-\epsilon_1, \epsilon_1), \mathbb{R}^*).$$

*Proof.* Clearly, any map of the above form commutes with  $N$ . The fact that  $\mathcal{C}_N$  consists only of such maps follows from the fact that any map in  $\mathcal{C}_N$  has to respect the symmetries of  $N$ ; in particular, it has to map hyperbolas to hyperbolas, and preserve the rate of contraction/expansion along each of them.

More precisely, let us fix a sufficiently small  $\epsilon_1 > 0$  to be chosen later and introduce the shorthand notation  $\mathcal{V} = \mathcal{V}_{\epsilon_1}$ . Take  $F \in \mathcal{C}_N$ , and denote  $F: (\xi, \eta) \mapsto (u(\xi, \eta), v(\xi, \eta))$ .

We first show that  $F$  must necessarily fix the axes  $\{\xi = 0\}$  and  $\{\eta = 0\}$ , hence the origin  $(0, 0)$ . For any  $j \in \mathbb{Z}$ , by (2.6), and after projection on the two coordinates, for each  $(\xi, \eta) \in \mathcal{V}$ , we get

$$\begin{aligned} u(\Delta(\xi\eta)^j \xi, \Delta(\xi\eta)^{-j} \eta) &= \Delta(u(\xi, \eta)v(\xi, \eta))^j u(\xi, \eta), \\ v(\Delta(\xi\eta)^j \xi, \Delta(\xi\eta)^{-j} \eta) &= \Delta(u(\xi, \eta)v(\xi, \eta))^{-j} v(\xi, \eta). \end{aligned}$$

For  $\eta = 0$ , we have  $\Delta(\xi\eta) = \Delta(0) = \lambda$ , so that

$$\Delta(u(\xi, 0)v(\xi, 0))^j v(\lambda^j \xi, 0) = v(\xi, 0).$$

By letting  $j \rightarrow +\infty$ , we deduce that  $v(\xi, 0) = 0$ . Similarly,  $u(0, \eta) = 0$  for all  $\eta \in \mathbb{R}$ . In particular,  $F$  fixes the coordinate axes  $\{\xi = 0\}$  and  $\{\eta = 0\}$ , and therefore  $F(0, 0) = (0, 0)$ .

Let us now write

$$u(\xi, \eta) = \sum_{k, \ell \geq 0} u_{k, \ell} \xi^k \eta^{\ell}.$$

Fix  $(\xi, \eta) \in \mathcal{V} \setminus \{(0, 0)\}$ . The above equation yields:

$$\begin{aligned} \Delta^j(u(\xi, \eta)v(\xi, \eta))u(\xi, \eta) &= u(\Delta^j(\xi\eta)\xi, \Delta^{-j}(\xi\eta)\eta) \\ &= \sum_{k, \ell \geq 0} u_{k, \ell} \Delta^{j(k-\ell)}(\xi\eta) \xi^k \eta^\ell. \end{aligned}$$

The left hand side goes to zero as  $j$  goes to infinity, hence  $u_{k, \ell} = 0$  for  $\ell \geq k$ . We deduce that for  $j \gg 1$ ,<sup>8</sup>

$$\Delta^j(u(\xi, \eta)v(\xi, \eta))u(\xi, \eta) = \Delta^j(\xi\eta) \cdot \eta^{-1} \sum_{k \geq 1} u_{k, k-1}(\xi\eta)^k + o(\lambda^j),$$

and thus, as  $j \rightarrow +\infty$ ,

$$\left( \frac{\Delta(u(\xi, \eta)v(\xi, \eta))}{\Delta(\xi\eta)} \right)^j = \frac{\eta^{-1} \sum_{k \geq 1} u_{k, k-1}(\xi\eta)^k}{u(\xi, \eta)} + o(1).$$

Since the right hand side does not depend on  $j$ , we obtain

$$(2.8) \quad \Delta(u(\xi, \eta)v(\xi, \eta)) = \Delta(\xi\eta),$$

and also

$$(2.9) \quad u(\xi, \eta) = \eta^{-1} \sum_{k \geq 1} u_{k, k-1}(\xi\eta)^k.$$

Let  $c_1$  be the analytic function  $z \mapsto \sum_{k \geq 1} u_{k, k-1} z^k$ , so that  $u(\xi, \eta)\eta = c_1(\xi\eta)$ . Similarly, we have  $v(\xi, \eta)\xi =: c_2(\xi\eta)$  for some analytic function  $c_2$ . For any  $z \neq 0$ , we set  $c(z) := \frac{c_1(z)c_2(z)}{z}$ . Then, for each  $(\xi, \eta) \in \mathcal{V} \setminus \{(0, 0)\}$ , it holds

$$u(\xi, \eta)v(\xi, \eta) = \frac{c_1(\xi\eta)c_2(\xi\eta)}{\xi\eta} = c(\xi\eta).$$

Therefore, for  $|c_b|$  sufficiently small,  $F$  maps the hyperbola  $\{\xi\eta = c_b\}$  to the hyperbola  $\{uv = c_\sharp\}$ , with  $c_\sharp := c(c_b)$ . By (2.8), we also have  $\Delta(uv) = \Delta(\xi\eta)$ , i.e.,  $\Delta(c(\xi\eta)) = \Delta(\xi\eta)$ . Note that  $\lim_{\xi\eta \rightarrow 0} c(\xi\eta) = \lim_{\xi\eta \rightarrow 0} u(\xi, \eta)v(\xi, \eta) = 0$ .

By assumption, the Birkhoff invariants  $(a_k)_{k \geq 1}$  are not all equal to zero. Let  $k_0 \geq 1$  be the smallest positive integer such that  $a_{k_0} \neq 0$ . Then,  $\Delta(z) - \lambda = a_{k_0} z^{k_0} + o(|z|^{k_0})$  for  $|z| \ll 1$ , and thus, assuming that  $\epsilon_1 > 0$  was chosen sufficiently small,  $\Delta|_{(0, \epsilon_1)}$  is strictly monotonic. It follows from the previous discussion that for all  $(\xi, \eta) \in \mathcal{V}$ ,

$$\Delta(c(\xi\eta)) - \lambda = \sum_{k \geq k_0} a_k (c(\xi\eta))^k = \sum_{k \geq k_0} a_k (\xi\eta)^k = \Delta(\xi\eta) - \lambda,$$

and then,  $c(\xi\eta) = \xi\eta$ , by the strict monotonicity of  $\Delta|_{(0, \epsilon_1)}$ . In other words, since  $F$  maps hyperbolas to hyperbolas, the local non-degeneracy of  $\Delta$  together with (2.8) compel  $F$  to fix each hyperbola near the origin, i.e.,

$$c(\xi\eta) = u(\xi, \eta)v(\xi, \eta) = \xi\eta.$$

For any  $(\xi, \eta) \in \mathcal{V}$  such that  $\xi\eta \neq 0$ , let us set

$$\tilde{\Delta}(\xi\eta) := \frac{u(\xi, \eta)}{\xi} = \frac{c(\xi\eta)}{v(\xi, \eta)\xi} = \frac{\xi\eta}{v(\xi, \eta)\xi} = \frac{\eta}{v(\xi, \eta)}.$$

<sup>8</sup> By symplecticity,  $u_{1,0}v_{0,1} = 1$  hence the right hand side is different from zero for  $\xi\eta \ll 1$ .

For any  $(\xi, \eta) \in \mathcal{V} \setminus \{(0, 0)\}$ , we also have  $\tilde{\Delta}(\xi\eta) = \frac{u(\xi, \eta)}{\xi} = \sum_{j \geq 0} u_{j+1, j}(\xi\eta)^j$ , thus we set  $\tilde{\Delta}(0) := \lim_{(\xi, \eta) \rightarrow (0, 0)} \frac{u(\xi, \eta)}{\xi} = u_{1, 0} = v_{0, 1}^{-1}$ . We conclude that  $\tilde{\Delta} \in C^\omega((-\epsilon_1, \epsilon_1), \mathbb{R}^*)$ , and for each  $(\xi, \eta) \in \mathcal{V}$ :

$$F(\xi, \eta) = (u(\xi, \eta), v(\xi, \eta)) = (\tilde{\Delta}(\xi\eta)\xi, \tilde{\Delta}(\xi\eta)^{-1}\eta). \quad \square$$

**Remark 2.6.** *In  $(s, r)$ -coordinates, the horizontal axis  $\{r = 0\} = \{\varphi = 0\}$  plays a special role, because of the reflection symmetry of the billiard map:*

$$\mathcal{F}(s, r) = (s', r') \iff \mathcal{F}(s', -r') = (s, -r).$$

*This time-reserval symmetry also exchanges the stable and unstable spaces.*

*In  $(\xi, \eta)$ -coordinates, the stable space is the horizontal axis  $\{\eta = 0\}$ , while the unstable space is the vertical axis  $\{\xi = 0\}$ . Moreover, 2-periodic points are on the axis of symmetry  $\{r = 0\}$  – and more generally, all the points associated to perpendicular bounces in palindromic orbits – hence their stable and unstable manifolds are symmetric with respect to  $\{r = 0\}$ . It is thus natural to require the new axis of symmetry to be  $\{\xi = \eta\}$ . By the previous study, under some non-degeneracy condition, maps in the centralizer of  $N$  translate points along hyperbolas  $\{\xi\eta = \text{cst}\}$ , hence typically, they do not preserve the axis  $\{\xi = \eta\}$ . As a consequence, there is a canonical choice for the conjugacy map  $R$  defined above, which preserves this symmetry.*

In the following, we assume that the Birkhoff invariants  $(a_k)_{k \geq 1}$  are not all equal to zero, and that the neighborhood  $\mathcal{U}$  in the definition of the change of coordinates  $R$  introduced at the beginning of Section 2 is sufficiently small such that the neighborhood  $\mathcal{V} := R(\mathcal{U}) \subset \mathbb{R}^2$  of  $(0, 0)$  satisfies  $\mathcal{V} \subset \mathcal{V}_{\epsilon_1}$ , where  $\epsilon_1$  satisfies the conclusion of Lemma 2.5.

**Corollary 2.7.** *Assume that the Birkhoff invariants  $(a_k)_{k \geq 1}$  are not all equal to zero. Let  $\mathcal{U}^* = \mathcal{U} \cap \mathcal{F}^{-2}\mathcal{U}$ . Then, there exists a unique map  $R_0 \in \text{Symp}^\omega(\mathcal{U}, \mathbb{R}^2)$  such that*

$$R_0 \circ \mathcal{F}^2|_{\mathcal{U}^*} = N \circ R_0|_{\mathcal{U}^*}, \quad \text{and} \quad R_0(\{(s, 0) \mid s \geq 0\}) \subset \{(\xi, \xi), \xi \geq 0\}.$$

*Proof.* Let us start by showing the uniqueness of  $R_0$ . Let  $R, \tilde{R}$  be two such maps. Then  $R^{-1} \circ N \circ R = \tilde{R}^{-1} \circ N \circ \tilde{R} = \mathcal{F}^2$ , hence, letting  $F := \tilde{R} \circ R^{-1}$ , it holds

$$(2.10) \quad F \circ N = N \circ F.$$

By construction of  $R$  and  $\tilde{R}$ , the map  $F$  fixes the horizontal and vertical axes  $\{\xi = 0\}$  and  $\{\eta = 0\}$ ; as noted in Remark 2.4, it can be extended to a neighborhood  $\mathcal{V}_\epsilon$  for some  $\epsilon < \epsilon_1$ . By (2.10),  $F \in \mathcal{C}_N(\epsilon)$  (recall Definition 2.3). By Lemma 2.5, the centralizer  $\mathcal{C}_N(\epsilon)$  is reduced to the set of maps which translate points along hyperbolas  $\{\xi\eta = \text{cst}\}$ , and then

$$F(\xi, \eta) = (\tilde{\Delta}(\xi\eta)\xi, \tilde{\Delta}(\xi\eta)^{-1}\eta),$$

for some real analytic map  $\tilde{\Delta} \in C^\omega((-\epsilon, \epsilon), \mathbb{R}^*)$ ,  $\epsilon > 0$ . Since both  $R$  and  $\tilde{R}$  fix the positive axis  $\{(\xi, \xi), \xi \geq 0\}$ , so does  $N_{\tilde{\Delta}}$ , and then,

$$(\tilde{\Delta}(\xi^2)\xi, \tilde{\Delta}(\xi^2)^{-1}\xi) = (\tilde{\xi}(\xi), \tilde{\xi}(\xi)), \quad \forall \xi \in \mathbb{R},$$

for some function  $\tilde{\xi}: \mathbb{R} \rightarrow \mathbb{R}$ . By taking the product of the two coordinates, we deduce that  $\xi^2 = (\tilde{\xi}(\xi))^2$ , and then  $\tilde{\xi}(\xi) = \xi$ , since  $\xi, \tilde{\xi} \geq 0$ . We deduce that  $\tilde{\Delta} \equiv 1$ ,

and then  $F = \tilde{R} \circ R^{-1} = \text{id}$ , which concludes the proof of uniqueness.

To show the existence of such a map  $R_0$ , let us fix an analytic symplectomorphism  $R(s, r) = (\xi(s, r), \eta(s, r))$  such that  $R \circ T \circ R^{-1} = N$ . After possibly composing  $R$  with  $-\text{id}$ ,<sup>9</sup> we may assume that  $\xi(s, 0), \eta(s, 0) \geq 0$  for all  $s \geq 0$ . Let  $R^{-1}: (\xi, \eta) \mapsto (\mathcal{S}(\xi, \eta), \mathcal{R}(\xi, \eta))$ , and let  $\pi: (\xi, \eta) \mapsto (\pi_1(\xi\eta), \pi_2(\xi\eta))$  be the projection along hyperbolas  $\{\xi\eta = \text{cst}\}$  onto the set  $\{\mathcal{R}(\xi, \eta) = 0\}$ . We denote by  $\theta \in [0, \frac{\pi}{2}]$  the angle between the positive parts of the horizontal axis and of the unstable space of  $T$ . Since the coordinate  $\eta$  vanishes only on the stable space  $\{r = \tan(\theta)s\}$ , we may define  $\delta(\xi\eta) := \sqrt{\frac{\pi_2(\xi\eta)}{\pi_1(\xi\eta)}}$ , and we set  $N_\delta: (\xi, \eta) \mapsto (\delta(\xi\eta)\xi, \delta^{-1}(\xi\eta)\eta)$ . Clearly,  $N_\delta \in \text{Symp}^\omega(\mathbb{R}^2, \mathbb{R}^2)$ , and  $N_\delta(\{\mathcal{R}(\xi, \eta) = 0, \xi, \eta \geq 0\}) = \{(\tilde{\xi}, \tilde{\xi}), \tilde{\xi} \geq 0\}$ . Then, the map  $R_0: (s, r) \mapsto N_\delta \circ R(s, r)$  satisfies the required conditions:

$$R_0(s, 0) = (\delta(\xi(s, 0)\eta(s, 0))\xi(s, 0), \delta(\xi(s, 0)\eta(s, 0))^{-1}\eta(s, 0)) \in \{(\tilde{\xi}, \tilde{\xi}), \tilde{\xi} \geq 0\},$$

$$\text{and } R_0 T R_0^{-1} = N_\delta R T R^{-1} N_\delta^{-1} = N_\delta N N_\delta^{-1} = N. \quad \square$$

We call *Birkhoff coordinates* the coordinates  $(\xi, \eta)$  obtained via the change of coordinates  $R_0$ .

**2.2. The time reversal involution in Birkhoff coordinates.** By the time reversal property, the map  $\mathcal{I}: (s, r) \mapsto (s, -r)$  conjugates the billard map  $\mathcal{F}$  to its inverse  $\mathcal{F}^{-1}$ , and thus,  $\mathcal{I} \circ T \circ \mathcal{I} = T^{-1}$ . Assume that the Birkhoff invariants  $(a_k)_{k \geq 1}$  are not all equal to zero, and let  $R_0: \mathcal{U} \rightarrow \mathbb{R}^2$  be the canonical symplectic change of coordinates given by Lemma 2.7. Since  $R_0 \circ T \circ R_0^{-1} = N$ , we get

$$(R_0 \circ \mathcal{I} \circ R_0^{-1}) \circ N \circ (R_0 \circ \mathcal{I} \circ R_0^{-1}) = R_0 \circ T^{-1} \circ R_0^{-1} = N^{-1}.$$

Set  $\mathcal{I}^* := R_0 \circ \mathcal{I} \circ R_0^{-1}$ . We thus have

$$(2.11) \quad \mathcal{I}^* \circ N \circ \mathcal{I}^* = N^{-1}.$$

**Lemma 2.8.** *The map  $\mathcal{I}^*$  is the reflection along the bisectrix  $\{\xi = \eta\}$ :*

$$(2.12) \quad \mathcal{I}^* = \mathcal{I}_0: (\xi, \eta) \mapsto (\eta, \xi).$$

*Proof.* Let us write  $\mathcal{I}^*(\xi, \eta) = (u, v)$ , with  $u = u(\xi, \eta)$  and  $v = v(\xi, \eta)$ . For every  $(\xi, \eta) \in \mathbb{R}^2$ , we have

$$(2.13) \quad u(\xi, 0) = 0, \quad v(0, \eta) = 0.$$

In other words,  $\mathcal{I}^*$  maps the horizontal axis  $\{\eta = 0\} = \{(\xi, 0) : \xi \in \mathbb{R}\}$  to the vertical axis  $\{\xi = 0\} = \{(0, \eta) : \eta \in \mathbb{R}\}$ , and vice versa. Indeed it follows from the definition of the map  $N$  that  $\{\eta = 0\}$  is the stable manifold of  $(0, 0)$ , since  $N^j(\xi, 0) = (\lambda^j \xi, 0)$ , for  $j \geq 0$ , and similarly,  $\{\xi = 0\}$  is the unstable manifold of the origin. Moreover, (2.11) implies that  $N$  exchanges the stable manifold with the unstable manifold: given  $p \in \mathbb{R}^2$  such that  $\lim_{j \rightarrow +\infty} N^j(p) = (0, 0)$ , then its image  $p^* := \mathcal{I}^*(p)$  satisfies

$$\lim_{j \rightarrow +\infty} N^{-j}(p^*) = \mathcal{I}^*(\lim_{j \rightarrow +\infty} N^j(p)) = \mathcal{I}^*(0, 0) = (0, 0).$$

Here, we have used that  $\mathcal{I}^*(0, 0) = (0, 0)$  (by (2.13)).

<sup>9</sup> By symplecticity, we do not need to consider the reflections  $(\xi, \eta) \mapsto (\xi, -\eta)$  or  $(\xi, \eta) \mapsto (-\xi, \eta)$ .

Moreover, by (2.11), we know that  $\mathcal{I}^* \circ N^{-1} = N \circ \mathcal{I}^*$ . Therefore, given any  $(\xi, \eta) \in \mathbb{R}^2$ , we obtain

$$\mathcal{I}^*(\Delta(\xi\eta)^{-1}\xi, \Delta(\xi\eta)\eta) = (\Delta(u(\xi, \eta)v(\xi, \eta))u(\xi, \eta), \Delta(u(\xi, \eta)v(\xi, \eta))^{-1}v(\xi, \eta)).$$

In particular, by considering the projection on the first coordinate, we get

$$(2.14) \quad u(\Delta(\xi\eta)^{-1}\xi, \Delta(\xi\eta)\eta) = \Delta(u(\xi, \eta)v(\xi, \eta))u(\xi, \eta).$$

For  $\xi = 0$ , by the power series expansion of  $\Delta$ , and by (2.13), we have  $\Delta(\xi\eta) = \Delta(0 \cdot \eta) = \Delta(0) = \lambda$  and  $\Delta(u(\xi, \eta)v(\xi, \eta)) = \Delta(u(0, \eta)v(0, \eta)) = \Delta(u(0, \eta) \cdot 0) = \Delta(0) = \lambda$ . We deduce from (2.14) that for any  $\eta \in \mathbb{R}$ ,

$$u(0, \lambda\eta) = \lambda u(0, \eta).$$

By considering the power series expansion  $u(\xi, \eta) = \sum_{k, \ell \geq 0} u_{k, \ell} \xi^k \eta^\ell$ , this relation implies that  $u_{0, \ell} = 0$  for all  $\ell \neq 1$ , and then,

$$u(0, \eta) = u_{0, 1} \eta.$$

Besides, for any  $(\xi, \eta) \in \mathbb{R}^2$ , and any  $j \geq 0$ , we have  $\mathcal{I}^* \circ N^j = N^{-j} \circ \mathcal{I}^*$ . Similarly, by projecting on the first coordinate, we obtain

$$(2.15) \quad u(\Delta(\xi\eta)^j \xi, \Delta(\xi\eta)^{-j} \eta) = \Delta(u(\xi, \eta)v(\xi, \eta))^{-j} u(\xi, \eta).$$

For any  $j \geq 0$ , we have

$$\begin{aligned} u(\xi, \eta) &= \Delta(u(\xi, \eta)v(\xi, \eta))^j \cdot u(\Delta(\xi\eta)^j \xi, \Delta(\xi\eta)^{-j} \eta) \\ &= \left( \frac{\Delta(u(\xi, \eta)v(\xi, \eta))}{\Delta(\xi\eta)} \right)^j \cdot \left( u_{0, 1} \eta + \Delta(\xi\eta)^j \sum_{k=1}^{+\infty} \sum_{\ell=0}^{+\infty} u_{k, \ell} \Delta(\xi\eta)^{j(k-\ell)} \cdot \xi^k \eta^\ell \right), \end{aligned}$$

where we have used that  $u(0, \Delta(\xi\eta)^{-j} \eta) = u_{0, 1} \Delta(\xi\eta)^{-j} \eta$ . Since the left hand side is bounded independently of  $j$ , then, arguing as in Lemma 2.5, we get  $\Delta(u(\xi, \eta)v(\xi, \eta)) = \Delta(\xi\eta)$ , and

$$u(\xi, \eta) = u_{0, 1} \eta.$$

Similarly, there exists  $v_{1, 0} \in \mathbb{R}$  such that  $v(\xi, \eta) = v_{1, 0} \xi$ . Since  $\mathcal{I}^*$  is anti-symplectic ( $R_0$  is symplectic and  $\mathcal{I}$  is anti-symplectic), we have

$$d\xi \wedge d\eta = dv \wedge du = (u_{0, 1} v_{1, 0}) d\xi \wedge d\eta,$$

and then  $u_{0, 1}, v_{1, 0} \in \mathbb{R}^*$ , and  $v_{1, 0} = u_{0, 1}^{-1}$ . Besides,  $R_0^{-1} = (S, \Phi)$  maps  $\{(\xi, \xi), \xi \geq 0\}$  to  $\{(s, 0), s \geq 0\}$ , hence for any  $\xi \geq 0$ , we have

$$(u_{0, 1} \xi, u_{0, 1}^{-1} \xi) = \mathcal{I}^*(\xi, \xi) = R_0 \circ \mathcal{I}(S(\xi, \xi), 0) = R_0(S(\xi, \xi), 0) \in \{(\xi, \xi), \xi \geq 0\},$$

and then  $u_{0, 1} = v_{1, 0} = 1$ . We conclude that

$$\mathcal{I}^*(\xi, \eta) = (\eta, \xi). \quad \square$$

**Remark 2.9.** Note that (2.12) can also be obtained as follows: by (2.11), both  $\mathcal{I}^* = R_0 \mathcal{I} R_0^{-1}$  and  $\mathcal{I}_0$  conjugate  $N$  with  $N^{-1}$ , hence  $\mathcal{I}^* \circ \mathcal{I}_0^{-1}$  is in the centralizer of  $N$ . By Lemma 2.5 and since  $\mathcal{I}^*, \mathcal{I}_0$  preserve the bisectrix  $\{\xi = \eta\}$  (as  $R_0$  does), we conclude that  $\mathcal{I}^* = \mathcal{I}_0$ .

### 3. EXTENSION OF THE BIRKHOFF COORDINATES ALONG THE SEPARATRICES

Let us fix a billiard table  $\mathcal{D} \in \mathbf{B}$ . In this section, we consider the Birkhoff Normal Form  $N$  introduced above for the 2-periodic (12) and we assume that the Birkhoff invariants  $(a_k)_{k \geq 1}$  are not all equal to zero. We denote by  $R_0: \mathcal{U} \rightarrow \mathbb{R}^2$  the canonical symplectic change of coordinates given by Lemma 2.7 and we set  $\mathcal{V} := R_0(\mathcal{U})$ . We will also use the notation introduced at the beginning of Section 2.

Up to this point, the model for the dynamics of  $T$  given by its Birkhoff Normal Form  $N$  only accounts for the dynamics in a neighborhood of the 2-periodic orbit. In this section, we explain how to extend this model in such a way that it also describes the global dynamics of the palindromic orbits  $(h_n)_{n \geq 1}$  introduced earlier. In the  $(\xi, \eta)$ -coordinates, the only non-wandering point of the map  $N$  is the origin  $(0, 0)$ ; to describe recurrence properties of the dynamics of  $T$ , we explain a *gluing* construction for some points in this model, for which we have more information due to additional symmetries. This is, in particular, the case for the palindromic orbits  $(h_n)_{n \geq 1}$ , which have two symmetries, and for which we have a good control on the gluing map. Moreover, for  $n$  large enough, those orbits always stay in a neighborhood of the separatrices, and the local dynamics of  $N$  near the fixed point is sufficient to describe them, based on the relation  $NR_0 = R_0T$  which can be used to extend the system of coordinates by the dynamics. Although this relation is only true locally (some points escape in the billiard dynamics, so the map  $T$  is not everywhere defined), it is sufficient for our purpose, which consists in determining explicitly a link between the Birkhoff invariants and the Lyapunov exponents of the palindromic orbits. The extension of the coordinates to a neighborhood of the separatrices that we describe in the following can be seen as a hyperbolic analogue of the local coordinates in a neighborhood of the boundary given by the Birkhoff Normal Form in the elliptic setting, which was used, for instance, in [CdV3].

After possibly replacing  $\mathcal{U}$  with  $\mathcal{U} \cap \mathcal{I}(\mathcal{U})$ , where  $\mathcal{I}: (s, r) \mapsto (s, -r)$ , we can assume that the neighborhood  $\mathcal{U}$  is symmetric with respect to the axis  $\{r = 0\}$ . For any sufficiently large odd integer  $n = 2m - 1$ , and after a certain time, Proposition 2.1 implies that the iterates under  $T = \mathcal{F}^2$  of the point  $x_n(1)$  in the palindromic orbit  $h_n$  are contained in the neighborhood  $\mathcal{U}$ . More precisely, there exists  $m_0 \geq 0$  such that if  $n = 2m - 1 \geq n_0 := 2m_0 - 1$ , we have  $x_n(2k + 1) \in \mathcal{U}$ , for all  $k \in \{m_0, m_0 + 1, \dots, 2m - m_0 - 1\}$ . We denote by  $(\xi_n(2k + 1), \eta_n(2k + 1))$  the coordinates of the point  $R_0(x_n(2k + 1))$ . The contraction rate  $\Delta$  is constant along this orbit segment: for any integer  $k \in \{m_0, m_0 + 1, \dots, 2m - m_0 - 1\}$ , we have

$$\Delta(\xi_n(2k + 1)\eta_n(2k + 1)) = \Delta(\xi_n(2m - 1)\eta_n(2m - 1)) =: \Delta_n.$$

By Lemma 2.2, the map  $N$  is well defined in a neighborhood of the coordinate axes. In particular, due to the relations  $R_0 = NR_0T^{-1}$  and  $R_0 = N^{-1}R_0T$ , it is possible to extend the system of coordinates given by  $R_0$  to a neighborhood of the separatrices as follows.

Let  $\mathcal{NE}^{-1}$  be the set of all parameters  $(s, r) \in \mathcal{M}$  in the collision space such that  $T^{-1}(s, r)$  is well defined, i.e., such that both  $\mathcal{F}^{-1}(s, r)$  and  $\mathcal{F}^{-2}(s, r)$  are well defined. For any  $(s, r) \in T^{-1}(\mathcal{U} \cap \mathcal{NE}^{-1})$ , we have  $T(s, r) \in \mathcal{U}$ , thus we can set  $R_{-1}(s, r) := N^{-1}R_0T(s, r)$ . By induction, for each integer  $\ell \geq 2$ , we define

$$\mathcal{NE}^{-\ell} := \{(s, r) \in \mathcal{NE}^{-1} : T^{-1}(s, r) \in \mathcal{NE}^{-(\ell-1)}\},$$

and for each  $(s, r) \in \mathcal{U}^{-\ell} := T^{-\ell}(\mathcal{U} \cap \mathcal{N}\mathcal{E}^{-\ell})$ , we set

$$R_{-\ell}(s, r) := N^{-1}R_{-(\ell-1)}T(s, r) = N^{-\ell}R_0T^\ell(s, r).$$

On  $\mathcal{U}^{-(\ell-1)} \cap \mathcal{U}^{-\ell}$ , it holds  $N^{-1}R_{-(\ell-1)}T = R_{-(\ell-1)}$ , hence  $R_{-\ell}$  coincides with  $R_{-(\ell-1)}$  on this set. We define  $R_-$  as the map obtained in this way by extending the conjugacy  $R_0$  to a neighborhood of the arc of the stable manifold between the points  $(0, 0)$  and  $x_\infty(1)$ . More precisely,  $R_-$  is defined on  $\mathcal{U}^- := \bigcup_{\ell=0}^{m_0} \mathcal{U}^{-\ell}$  as follows: for any  $\ell \in \{0, \dots, m_0\}$  and  $(s, r) \in \mathcal{U}^{-\ell} \setminus \bigcup_{k=0}^{\ell-1} \mathcal{U}^{-k}$ , we set  $R_-(s, r) := R_{-\ell}(s, r)$ . In a symmetric way, we define  $\mathcal{U}^+$  and extend  $R_0$  to a map  $R_+$  defined on a neighborhood of the arc of the unstable manifold between the points  $(0, 0)$  and  $x_\infty(-1)$ .

By the above remark, for any integer  $n = 2m - 1 \geq n_0$ , every point  $h_n(k)$  labeled with some odd integer  $k$  belongs to the set  $\mathcal{U}^+ \cup \mathcal{U}^-$ , thus it has an image either by  $R_+$  or  $R_-$ . We let  $R$  be the map defined on  $\mathcal{U}^+ \cup \mathcal{U}^-$  by  $R|_{\mathcal{U}^\pm} := R_\pm$ . For each  $k \in \{0, \dots, m - m_0 - 1\}$ , we have  $R(x_n(2m \pm (2k + 1))) = R_0(x_n(2m \pm (2k + 1)))$ , while for  $k \in \{m - m_0, \dots, m - 1\}$ , the point  $R(x_n(2m \pm (2k + 1))) = R_{\pm(m-k)}(x_n(2m \pm (2k + 1)))$  is well defined.

Moreover, for some neighborhood  $\mathcal{U}_n \subset \mathcal{U}$  of the point  $x_n(2m) = x_n(n + 1)$ , it follows from the above definitions that

$$(3.1) \quad R \circ T^{\pm k}|_{\mathcal{U}_n} = N^{\pm k} \circ R|_{\mathcal{U}_n} = N^{\pm k} \circ R_0|_{\mathcal{U}_n}, \quad \forall k \in \{0, \dots, m - 1\}.$$

More generally, for each  $x \in \mathcal{U}^+ \cup \mathcal{U}^-$  and each integer  $k$  such that  $x, \dots, T^k(x) \in \mathcal{U}^+ \cup \mathcal{U}^-$ , we have

$$(3.2) \quad R \circ T^k(x) = N^k \circ R(x).$$

The next lemma says that after the extension, the image of the time reversal involution is still given by the map  $\mathcal{I}_0: (\xi, \eta) \mapsto (\eta, \xi)$ :

**Lemma 3.1.** *The extended system of coordinates  $R$  satisfies*

$$(3.3) \quad R \circ \mathcal{I} \circ R^{-1} = \mathcal{I}_0.$$

*Proof.* It follows directly from (3.2) and Lemma 2.8.  $\square$

In particular, by (3.2), for each  $n = 2m - 1$  with  $m \geq m_0$ , and for  $k \in \{0, \dots, n\}$ , it holds

$$R(x_n(2k + 1)) = (\xi_n(2k + 1), \eta_n(2k + 1)) := ((\Delta_n^{m-1-k})^{-1}\xi_n(n), \Delta_n^{m-1-k}\eta_n(n)).$$

Let us abbreviate

$$R(x_n(1)) = (\xi_n, \eta_n) := ((\Delta_n^{m-1})^{-1}\xi_n(n), \Delta_n^{m-1}\eta_n(n)).$$

**Lemma 3.2.** *It holds*

$$\eta_n = \Delta_n^n \xi_n.$$

*Proof.* By (3.3), we have  $R(x_n(-1)) = R \circ \mathcal{I}(x_n(1)) = \mathcal{I}_0 \circ R(x_n(1)) = \mathcal{I}_0(\xi_n, \eta_n) = (\eta_n, \xi_n)$ . Therefore,

$$(\eta_n, \xi_n) = R(x_n(-1)) = R(x_n(2n + 1)) = R \circ T^n(x_n(1)) = N^n(\xi_n, \eta_n),$$

which gives the required identity.  $\square$

Then, we have

$$(3.4) \quad (\xi_n(2k+1), \eta_n(2k+1)) = (\Delta_n^k \xi_n, \Delta_n^{-k} \eta_n) = \xi_n(\Delta_n^k, \Delta_n^{n-k}), \quad \forall k \in \{0, \dots, n\}.$$

In the same way as above, we can extend our system of coordinates such that the images of the forward iterates of the point  $x_\infty(1)$  in the homoclinic orbit  $h_\infty$  are

$$R(x_\infty(2k+1)) = R_-(x_\infty(2k+1)) = (\xi_\infty(2k+1), 0) = (\lambda^k \xi_\infty, 0), \quad \forall k = 0, 1, \dots$$

for some  $\xi_\infty \in \mathbb{R} \setminus \{0\}$  (the second coordinate vanishes since we are on the stable manifold  $\{\eta = 0\}$  of the origin). Recall that  $\mathcal{I}: (s, r) \mapsto (s, -r)$ . By (2.3), for all  $n = 2m-1 \geq n_0$ ,  $k \geq 0$ , we have  $x_n(-2k-1) = x_n(2n+2-(2k+1)) = \mathcal{I}(x_n(2k+1))$ . Thus, we extend analogously the coordinates in the past, such that the preimages of  $x_\infty(-1)$  have coordinates  $R(x_\infty(-2k-1)) = R_+(x_\infty(-2k-1))$ , i.e.,

$$R(x_\infty(-2k-1)) = (0, \xi_\infty(-2k-1)) = (0, \lambda^k \xi_\infty), \quad \forall k = 1, 2, \dots$$

**Remark 3.3.** *In the previous construction, we stop the extension after the time  $\pm m_0$  where we reach a neighborhood of the point  $x_\infty(\pm 1)$ . Indeed, after that time, in the initial  $(s, r)$ -collision space, the neighborhoods of the separatrices start to overlap; in particular, they both contain a neighborhood of the point  $x_\infty(0)$  on the third obstacle. Besides, the point of this construction is to study the dynamics of the map  $T$  through its Birkhoff Normal Form  $N$ . Note that the latter only depends on the obstacles  $\mathcal{O}_1, \mathcal{O}_2$ . By analyticity, as long as the points bounce between the first two obstacles, it is legitimate to replace the billiard dynamics with that of  $N$ , but it does not carry any meaningful information once the points reach the third obstacle.*

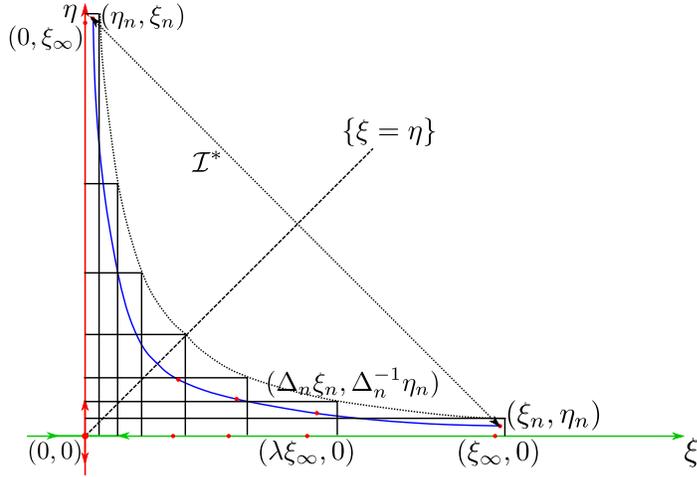


FIGURE 3.  $(\xi, \eta)$ -representation of the points in the palindromic orbit  $h_n$ .

#### 4. MARKED LYAPUNOV SPECTRUM AND BIRKHOFF INVARIANTS

**4.1. Preliminary estimates on the parameters.** Recall that  $T = \mathcal{F}^2$  where  $\mathcal{F}$  is the billiard map and that  $\mathcal{M}_i$  denotes the set of  $(s, r)$ -coordinates of collisions emanating from the  $i^{\text{th}}$  scatterer.

We let  $\mathcal{U} \subset \mathbb{R}^2$  be a small neighborhood of  $(0, 0)$  as defined in Section 2 and denote  $\mathcal{V} = R_0\mathcal{U} \subset \mathbb{R}^2$ ; recall that by construction of  $R_0$  For any  $x = (s, 0) \in \mathcal{U} \cap \{r = 0\}$ , we have  $R_0(x) \in \mathcal{V} \cap \{\xi = \eta\}$ .

Let  $\mathcal{O}_\infty \subset \mathcal{M}_2$  be a small neighborhood of the point  $x_\infty(-1)$ . We denote by  $\Omega_\infty := R(\mathcal{O}_\infty)$  the image of  $\mathcal{O}_\infty$  in Birkhoff coordinates, and we let

$$\mathcal{G} := R \circ T \circ R^{-1}|_{\Omega_\infty} = R_- \circ T \circ R_+^{-1}|_{\Omega_\infty}$$

be the gluing map between  $R_+$  and  $R_-$ . It satisfies the time reversal property

$$(4.1) \quad \mathcal{G}^{-1} = \mathcal{I}_0 \circ \mathcal{G} \circ \mathcal{I}_0, \quad \mathcal{I}_0: (\xi, \eta) \mapsto (\eta, \xi).$$

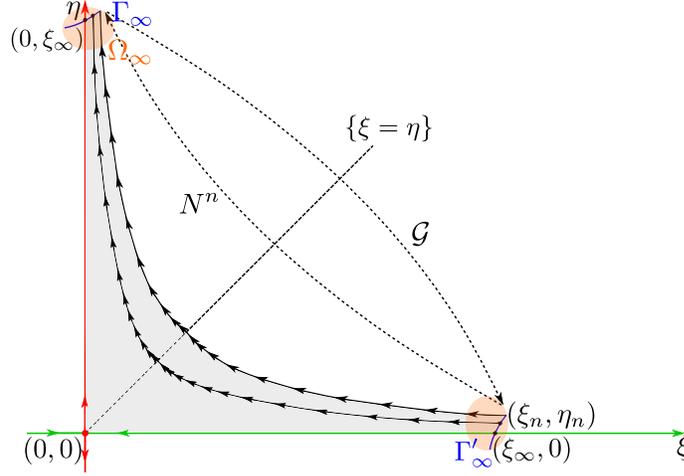


FIGURE 4. Gluing map between the extensions  $R_+, R_-$ .

Let  $\mathcal{A}_\infty := \mathcal{O}_\infty \cap \mathcal{F}^{-1}(\{r = 0\})$  be the curve in  $\mathcal{O}_\infty$  containing  $x_\infty(-1)$  and made of points whose image under the billiard map  $\mathcal{F}$  is associated to an orthogonal collision on  $\mathcal{O}_3$ ; let  $\Gamma_\infty := R(\mathcal{A}_\infty) \subset \Omega_\infty$  be the image of  $\mathcal{A}_\infty$  in Birkhoff coordinates, and set  $\Gamma'_\infty := \mathcal{G}(\Gamma_\infty) \subset \mathcal{G}(\Omega_\infty)$ .

**Lemma 4.1.** *For any  $x = (s, r) \in \mathcal{A}_\infty$ , we have  $T(s, r) = \mathcal{I}(x) = (s, -r)$ , and  $DT_{(s,r)} \in \text{SL}(2, \mathbb{R})$ . Analogously, for any  $(\xi, \eta) \in \Gamma_\infty$ , it holds  $\mathcal{G}(\xi, \eta) = \mathcal{I}_0(\xi, \eta) = (\eta, \xi)$ , and  $D\mathcal{G}_{(\xi,\eta)} \in \text{SL}(2, \mathbb{R})$ . In particular, we have  $\Gamma'_\infty = \mathcal{I}_0(\Gamma_\infty)$ .*

*Proof.* Let  $x = (s, r) \in \mathcal{A}_\infty$ . The point  $\mathcal{F}(x) \in \{r = 0\}$  is invariant under the involution  $\mathcal{I}: (s, r) \mapsto (s, -r)$ , hence, using Remark 2.6,

$$T(x) = \mathcal{F}(\mathcal{F}(x)) = \mathcal{F} \circ \mathcal{I}(\mathcal{F}(x)) = \mathcal{I} \circ \mathcal{F}^{-1}(\mathcal{F}(x)) = \mathcal{I}(x).$$

Moreover, by (1.2), we have  $\det DT_x = \det D\mathcal{F}_{(s,r)}^2 = 1$ . By (3.3),  $R \circ \mathcal{I} = \mathcal{I}_0 \circ R$ , hence for any  $(\xi, \eta) = R(x) \in \Gamma_\infty$ , we also have

$$\mathcal{G}(\xi, \eta) = RTR^{-1}(R(x)) = RT(x) = R \circ \mathcal{I}(x) = \mathcal{I}_0 \circ R(x) = \mathcal{I}_0(\xi, \eta),$$

and  $\det D\mathcal{G}_{(\xi,\eta)} = \det DT_x = 1$ .  $\square$

As in Section 3, for any large integer  $n \geq n_0$ , we let  $(\xi_n, \eta_n) := R_-(x_n(1))$ ,  $\Delta_n := \Delta(\xi_n, \eta_n)$ , and we let  $(\xi_\infty, 0) := R_-(x_\infty(1)) = \lim_{n \rightarrow +\infty} (\xi_n, \eta_n)$ .

We let  $\gamma: \xi \mapsto \sum_{j=1}^{\infty} \gamma_j \xi^j$  be the analytic function such that  $\Gamma_\infty$  is the graph of  $\xi_\infty + \gamma(\cdot)$ , i.e., for any  $(\xi, \eta) \in \Gamma_\infty$ , we have  $\eta = \xi_\infty + \gamma(\xi)$ . As we have seen in Lemma 4.1,  $\Gamma'_\infty = \mathcal{I}_0(\Gamma_\infty)$ , hence for any  $(\xi, \eta) \in \Gamma'_\infty$ , we also have  $\xi = \xi_\infty + \gamma(\eta)$ .

**Lemma 4.2.** *For each integer  $n \geq n_0$ , it holds*

$$R(x_n(-1)) = (\eta_n, \xi_n), \quad R(x_\infty(-1)) = (0, \xi_\infty), \quad \xi_n = \xi_\infty + \gamma(\eta_n).$$

*In particular,  $(\eta_n, \xi_n), (0, \xi_\infty) \in \Gamma_\infty$ , while  $(\xi_n, \eta_n), (\xi_\infty, 0) \in \Gamma'_\infty$ .*

*Proof.* Let  $n \geq n_0$ . We have  $x_n(0) = \mathcal{F}(x_n(-1)) \in \{r = 0\}$ , i.e.,  $x_n(-1) \in \mathcal{A}_\infty$ , and  $R(x_n(-1)) \in \Gamma_\infty$ . It follows from Lemma 4.1 that  $(\xi_n, \eta_n) = R(T(x_n(-1))) = \mathcal{G}(R(x_n(-1))) = \mathcal{I}_0(R(x_n(-1)))$ , which gives the first identity. Besides,  $(\eta_n, \xi_n) \in \Gamma_\infty$ , and  $(\xi_n, \eta_n) = R(x_n(1)) \in \Gamma'_\infty$ , so that  $\xi_n = \xi_\infty + \gamma(\eta_n)$ .

Similarly,  $x_\infty(-1) \in \mathcal{A}_\infty$ ,  $R(x_\infty(-1)) \in \Gamma_\infty$ , and by Lemma 4.1, we have

$$R(x_\infty(-1)) = \mathcal{G}^{-1}(\xi_\infty, 0) = \mathcal{I}_0(\xi_\infty, 0) = (0, \xi_\infty). \quad \square$$

Let us denote by  $\mathcal{G}^\pm$  the coordinate functions of  $\mathcal{G}$ , i.e.,  $\mathcal{G}: (\xi, \eta) \mapsto (\mathcal{G}^+(\xi, \eta), \mathcal{G}^-(\xi, \eta))$ . Since  $\mathcal{G}(0, \xi_\infty) = (\xi_\infty, 0)$ , for any  $(\xi, \eta) \in \Omega_\infty$ , we may write

$$\mathcal{G}(\xi, \eta) = (\mathcal{G}^+(\xi, \eta), \mathcal{G}^-(\xi, \eta)) = (\xi_\infty + G^+(\xi, \eta - \xi_\infty), G^-(\xi, \eta - \xi_\infty)),$$

for two analytic functions  $G^\pm: (\xi, \eta) \mapsto \sum_{j+k \geq 1} G_{j,k}^\pm \xi^j \eta^k$ . Note that by the time reversal property (4.1), for any  $(\xi, \eta) \in \mathcal{G}(\Omega_\infty)$ , we have

$$(4.2) \quad \mathcal{G}^{-1}(\xi, \eta) = (G^-(\eta, \xi - \xi_\infty), \xi_\infty + G^+(\eta, \xi - \xi_\infty)).$$

As a consequence of Lemma 4.1, we get, for  $|\eta|$  sufficiently small:

$$(4.3) \quad G^-(\eta, \gamma(\eta)) = \eta, \quad G^+(\eta, \gamma(\eta)) = \gamma(\eta).$$

For  $i = 1, 2$ , we set  $G_i^\pm: \eta \mapsto \partial_i G^\pm(\eta, \gamma(\eta))$ .

**Lemma 4.3.** *The following relations hold:*

$$\begin{aligned} G_1^- &= -G_2^+ = 1 - \gamma' G_2^-, \\ G_1^+ &= \gamma'(2 - \gamma' G_2^-). \end{aligned}$$

*Proof.* By differentiating (4.3), for  $|\eta|$  sufficiently small, we obtain

$$\begin{aligned} \partial_1 G^-(\eta, \gamma(\eta)) + \gamma'(\eta) \partial_2 G^-(\eta, \gamma(\eta)) &= G_1^-(\eta) + \gamma'(\eta) G_2^-(\eta) = 1, \\ \partial_1 G^+(\eta, \gamma(\eta)) + \gamma'(\eta) \partial_2 G^+(\eta, \gamma(\eta)) &= G_1^+(\eta) + \gamma'(\eta) G_2^+(\eta) = \gamma'(\eta). \end{aligned}$$

Now, by Lemma 4.1, the differential  $D\mathcal{G}_{(\eta, \xi_\infty + \gamma(\eta))}$  of the gluing map is in  $\text{SL}(2, \mathbb{R})$ . We have

$$D\mathcal{G}_{(\eta, \xi_\infty + \gamma(\eta))} = \begin{pmatrix} \partial_1 G^+(\eta, \gamma(\eta)) & \partial_2 G^+(\eta, \gamma(\eta)) \\ \partial_1 G^-(\eta, \gamma(\eta)) & \partial_2 G^-(\eta, \gamma(\eta)) \end{pmatrix} = \begin{pmatrix} G_1^+(\eta) & G_2^+(\eta) \\ G_1^-(\eta) & G_2^-(\eta) \end{pmatrix},$$

and thus,

$$G_1^-(\eta) G_2^+(\eta) = G_1^+(\eta) G_2^-(\eta) - 1.$$

We deduce from the relations obtained previously that

$$\begin{aligned} (G_1^+(\eta) - \gamma'(\eta))(\gamma'(\eta) G_2^-(\eta) - 1) &= \gamma'(\eta) G_1^-(\eta) G_2^+(\eta) \\ &= \gamma'(\eta) (G_1^+(\eta) G_2^-(\eta) - 1), \end{aligned}$$

which yields

$$G_1^+ = \gamma'(2 - \gamma' G_2^-).$$

Combining this with the relations obtained above, we conclude that  $G_1^- = 1 - \gamma' G_2^-$  and  $G_2^+ = 1 - (\gamma')^{-1} G_1^+ = \gamma' G_2^- - 1 = -G_1^-$ .  $\square$

The above lemma leads us to define the analytic function

$$g: \eta \mapsto \sum_{k=0}^{+\infty} g_k \eta^k := G_2^-(\eta) = \partial_2 G^-(\eta, \gamma(\eta)),$$

so that for  $|\eta|$  sufficiently small, we have

$$(4.4) \quad D\mathcal{G}_{(\eta, \xi_\infty + \gamma(\eta))} = \begin{pmatrix} \gamma'(\eta)(2 - \gamma'(\eta)g(\eta)) & \gamma'(\eta)g(\eta) - 1 \\ 1 - \gamma'(\eta)g(\eta) & g(\eta) \end{pmatrix}.$$

**Remark 4.4.** Let  $\mathcal{W}_\infty^+ \subset \Omega_\infty$  be the image in Birkhoff coordinates of the arc of the stable manifold of  $(0, 0)$  containing the homoclinic point  $(0, \xi_\infty)$ , and let  $\mathcal{W}_\infty^- \subset \mathcal{G}(\Omega_\infty)$  be the arc of the unstable manifold of  $(0, 0)$  containing  $(\xi_\infty, 0)$ . We write  $\mathcal{W}_\infty^+ = \{(\eta, \xi_\infty + w(\eta)) : |\eta| \text{ small}\}$ , for some analytic function  $w$ . By the time reversal symmetry,  $T \circ R_+^{-1}(\mathcal{W}_\infty^+) = \mathcal{I}(T^{-1} \circ R_-^{-1}(\mathcal{W}_\infty^-))$ , hence

$$\mathcal{W}_\infty^- = \mathcal{I}_0(\mathcal{W}_\infty^+),$$

i.e.,  $\mathcal{W}_\infty^- = \{(\xi_\infty + w(\eta), \eta) : |\eta| \text{ small}\} \subset \mathcal{G}(\Omega_\infty)$ . By (4.2) and the fact that the gluing map  $\mathcal{G}$  preserves the invariant subspaces of the saddle fixed point  $(0, 0)$ , we can write analogous relations between  $G^+$  and  $G^-$ , but involving the function  $w$  instead of  $\gamma$ , i.e., for  $|\eta|$  sufficiently small, it holds

$$G^-(\eta, w(\eta)) = 0, \quad G^+(0, \eta) = w(G^-(0, \eta)).$$

Let us denote by  $\sum_{j=1}^{\infty} \gamma_j \xi^j$ , resp.  $\sum_{j=1}^{\infty} w_j \xi^j$  the expansion of  $\gamma$ , resp.  $w$ . Differentiating  $G^-(\eta, w(\eta)) = 0$  and evaluating at 0, we get  $G_1^-(0) = -w_1 G_2^-(0)$ . On the other hand, the identities in Lemma 4.3 yield  $G_1^-(0) = 1 - \gamma_1 G_2^-(0)$ . In particular,

$$(4.5) \quad g_0 = G_2^-(0) = (\gamma_1 - w_1)^{-1}.$$

The arc  $\mathcal{W}_\infty^+$  of the stable manifold is transverse to the unstable manifold of  $N$  at the homoclinic point  $(0, \xi_\infty)$ , which is vertical in those coordinates, hence  $w_1 \neq \infty$ . Besides,  $\mathcal{A}_\infty \subset T^{-1}(\{r = 0\})$  is the image under  $T^{-1}$  of some arc on the third scatterer, and then, its image  $\Gamma_\infty$  under  $R$  is also transverse to the unstable manifold of  $N$  at  $(0, \xi_\infty)$ , i.e.,  $\gamma_1 \neq \infty$ . Since the gluing map  $\mathcal{G} = R \circ T \circ R^{-1}|_{\Omega_\infty}$  is defined dynamically, we deduce that

$$\Gamma'_\infty = \mathcal{G}(\Gamma_\infty) = \mathcal{I}_0(\Gamma_\infty) = \{(\xi_\infty + \gamma(\eta), \eta) : |\eta| \text{ small}\}$$

$$\text{and } \mathcal{W}_\infty^- = \mathcal{G}(\{\xi = 0\}) = \mathcal{I}_0(\mathcal{W}_\infty^+) = \{(\xi_\infty + w(\eta), \eta) : |\eta| \text{ small}\}$$

are still transverse at  $(\xi_\infty, 0)$ , i.e.,  $\gamma_1 \neq w_1$ ; as  $\gamma_1, w_1 \neq \infty$ , and by (4.5), we deduce

$$(4.6) \quad |g_0| = |G_2^-(0)| = |\gamma_1 - w_1|^{-1} \in (0, +\infty).$$

Let us recall that for any integer  $n \geq n_0$ , we let  $\Delta_n := \Delta(\zeta_n)$ , with  $\zeta_n := \xi_n \eta_n$ . Let us also recall that by Lemma 3.2, it holds

$$(4.7) \quad \eta_n = \Delta_n^n \xi_n.$$

**Lemma 4.5.** For any integer  $n \geq n_0$ , we have

$$(4.8) \quad \eta_n = \Delta(\eta_n(\xi_\infty + \gamma(\eta_n)))^n (\xi_\infty + \gamma(\eta_n)).$$

*Proof.* Let  $n \geq n_0$ . By (4.7),  $\eta_n = \Delta_n^n \xi_n$ , and by definition,  $\Delta_n = \Delta(\zeta_n) = \Delta(\xi_n \eta_n)$ ; moreover, by Lemma 4.2, we have  $\xi_n = \xi_\infty + \gamma(\eta_n)$ , which concludes the proof.  $\square$

In other words, Lemma 4.5 tells us that for each integer  $n \geq n_0$ , the coordinates  $(\xi_n, \eta_n) = (\xi_\infty + \gamma(\eta_n), \eta_n)$  of the image under  $R$  of the periodic point  $x_n(1)$  are defined implicitly in terms of the coefficients of  $\Delta$  and  $\gamma$ , according to the previous equation.

Let us now give the proof of Proposition 2.1 stated earlier in the paper<sup>10</sup>.

**Proposition 4.6.** *For  $n \geq n_0$ , it holds*

$$\begin{aligned} \|x_\infty(k) - x(k)\| &= O(\lambda^{\frac{k}{2}}), \quad \text{for all } k \in \mathbb{N}, \\ \|x_n(k) - x_\infty(k)\| &= O(\lambda^{n-\frac{k}{2}}), \quad \text{for all } k \in \{0, \dots, n+1\}. \end{aligned}$$

*Proof.* The first point follows from the fact that for each  $k \geq 0$ ,

$$R(x_\infty(2k+1)) = (\lambda^k \xi_\infty, 0),$$

and when  $k \gg 1$ ,  $x_\infty(2k) - x(2k) = D\mathcal{F}_{x_\infty(2k-1)}(x_\infty(2k-1) - x(2k-1)) + H.O.T.$

Let us deal with the second point. As  $n \rightarrow +\infty$ , the symbolic codings of the points  $x_n(1)$  and  $x_\infty(1)$  match on longer and longer chunks, hence, by expansiveness,  $\lim_{n \rightarrow +\infty} x_n(1) = x_\infty(1)$ . We deduce that  $R(x_n(1)) = R(\xi_n, \eta_n) \rightarrow_n R(x_\infty(1))$ , i.e.,

$$\lim_{n \rightarrow +\infty} \xi_n = \xi_\infty, \quad \lim_{n \rightarrow +\infty} \eta_n = 0,$$

and  $\Delta_n = \Delta(\xi_n \eta_n) = \lambda + O(\xi_n \eta_n) = \lambda + o(1)$ . Then, by Lemma 3.2,  $\eta_n = \lambda^n \xi_\infty + o(\lambda^n)$ , and thus,  $\Delta_n = \Delta(\xi_n \eta_n) = \lambda + O(\lambda^n)$ . By (3.4), we have

$$(\xi_n(2k+1), \eta_n(2k+1)) = \xi_n(\Delta_n^k, \Delta_n^{n-k}),$$

for all  $k \in \{0, \dots, n\}$ , hence, for  $k \in \{0, \dots, \frac{n+1}{2}\}$ ,

$$R(x_n(2k+1)) - R(x_\infty(2k+1)) = (\Delta_n^k \xi_n - \lambda^k \xi_\infty, \Delta_n^{n-k} \xi_n).$$

We have seen that  $|\Delta_n - \lambda| = O(\lambda^n)$ , and by Lemma 4.2,

$$|\xi_n - \xi_\infty| = |\gamma(\eta_n)| = O(\lambda^n),$$

thus for any for  $k \in \{0, \dots, \frac{n+1}{2}\}$ ,  $|\Delta_n^k \xi_n - \lambda^k \xi_\infty| = O(n\lambda^n) = O(\lambda^{n-k})$ , while  $|\Delta_n^{n-k} \xi_n| = O(\lambda^{n-k})$ , hence

$$\|x_n(2k+1) - x_\infty(2k+1)\| = O(\|(\Delta_n^k \xi_n - \lambda^k \xi_\infty, \Delta_n^{n-k} \xi_n)\|) = O(\lambda^{n-k}).$$

The estimate for even indices follows from the estimate for odd indices and the fact that  $x_n(2k) - x_\infty(2k) = D\mathcal{F}_{x_\infty(2k-1)}(x_n(2k-1) - x_\infty(2k-1)) + H.O.T.$   $\square$

## 4.2. Lyapunov exponents and asymptotic expansions of the parameters.

In this part, we use the same notation as in the previous subsection, and show the relation between the above formulas and the Marked Lyapunov Spectrum of the billiard table.

**Remark 4.7.** *Let us make a few comments and introduce some notation.*

- (1) *Given  $\xi_\infty \in \mathbb{R}$  and the pair of functions  $(\gamma, g)$ , then by (4.4), it is possible to reconstruct the restriction of the gluing map  $\mathcal{G}|_{\Gamma_\infty}$ . Conversely, given  $\xi_\infty \in \mathbb{R}$  and the coordinate functions  $(\mathcal{G}^+, \mathcal{G}^-)$  of  $\mathcal{G}$ , then the function  $\gamma$  can be recovered. Indeed, the gluing map  $\mathcal{G}$  is dynamically defined, hence it*

<sup>10</sup> This proposition can be proved using several coordinate systems; in [BDKL] it was proved using a linearization near the periodic point; here it will be proved using the Birkhoff Normal Form.

maps some unstable cone at  $(0, \xi_\infty)$  into some unstable cone at  $(\xi_\infty, 0)$ . In particular, for  $\xi$  small,  $\eta = \xi_\infty + \gamma(\xi)$  is determined by the implicit equation

$$\mathcal{G}^+(\xi, \eta)\mathcal{G}^-(\xi, \eta) = \xi\eta.$$

- (2) The homoclinic parameter  $\xi_\infty \in \mathbb{R}$  can be regarded as a scaling factor: we will show in Subsection 5.2 how its value is determined by the Marked Length Spectrum. For any integer  $j \geq 0$ , we introduce scaled coefficients

$$(4.9) \quad \bar{a}_j := \lambda^{-1}a_j\xi_\infty^{2j}, \quad \bar{\gamma}_j := \gamma_j\xi_\infty^{j-1}, \quad \text{and} \quad \bar{g}_j := g_j\xi_\infty^j,$$

with  $a_0 := \lambda$  and  $\gamma_0 := \xi_\infty$ . Note that  $\bar{a}_0 = \bar{\gamma}_0 = 1$  and  $\bar{g}_0 = g_0$ .

In the following, for any integer  $n \geq n_0$ , we also let

$$(4.10) \quad \bar{\eta}_n := (\xi_\infty\lambda^n)^{-1}\eta_n, \quad \text{and} \quad \bar{\zeta}_n := (\xi_\infty^2\lambda^n)^{-1}\zeta_n = (\xi_\infty^2\lambda^n)^{-1}\xi_n\eta_n.$$

- (3) Given the homoclinic parameter  $\xi_\infty \in \mathbb{R}$ , the Birkhoff Normal Form  $N$  and the gluing map  $\mathcal{G}$ , we will find an explicit expression for the parameters  $(\xi_n, \eta_n)_n$  of the periodic orbits  $(h_n)_{n \geq 0}$  and of their Lyapunov exponent. More precisely, under the assumption that the first Birkhoff invariant  $a_1$  does not vanish, we show in Lemma 4.21 and Corollary 4.22 that there is a one-to-one correspondence between the sequence of Lyapunov exponents  $(\text{LE}(h_n))_{n \geq 0}$  and the coefficients  $(\bar{a}_j, \bar{\gamma}_j, \bar{g}_j)_{j=0}^\infty$ .

By the above remark, if we know the value of  $\xi_\infty$  and of the scaled coefficients  $\{\bar{a}_j\}_j$ ,  $\{\bar{\gamma}_j\}_j$  and  $\{\bar{g}_j\}_j$ , then it is possible to reconstruct  $\{a_j\}_j$ ,  $\{\gamma_j\}_j$  and  $\{g_j\}_j$ . In order to ease our notation, we henceforth assume that  $\xi_\infty = 1$  in the rest of this section.

The next lemma tells us how the Lyapunov exponent of the associated orbit can be expressed in terms of the new coordinates.

**Lemma 4.8.** *For each integer  $n \geq n_0$ , we let*

$$\Delta'_n := \Delta'(\zeta_n)\zeta_n = \sum_{k=1}^{+\infty} ka_k\zeta_n^k.$$

Then, the Lyapunov exponent of the periodic orbit  $h_n$  satisfies

$$2 \cosh(2(n+1)\text{LE}(h_n)) = \lambda^{-n}\text{I}_n + \text{II}_n + \lambda^n\text{III}_n,$$

where

$$\begin{aligned} \text{I}_n &:= \lambda^n \Delta_n^{-n} (1 - n \Delta'_n \Delta_n^{-1}) g(\eta_n), \\ \text{II}_n &:= 2n \Delta'_n \Delta_n^{-1} (1 - \gamma'(\eta_n) g(\eta_n)), \\ \text{III}_n &:= \lambda^{-n} \Delta_n^n (1 + n \Delta'_n \Delta_n^{-1}) \gamma'(\eta_n) (2 - \gamma'(\eta_n) g(\eta_n)). \end{aligned}$$

Let us recall that here,  $\Delta_n = \Delta(\zeta_n)$ , and  $\Delta(0) = \lambda$ .

*Proof.* By the  $(2n+2)$ -periodicity of  $h_n$ , we have  $T^{n+1}(x_n(-1)) = x_n(-1)$ , and since  $D_{x_n(-1)}T^{n+1} \in \text{SL}(2, \mathbb{R})$ , we obtain

$$\begin{aligned} 2 \cosh(2(n+1)\text{LE}(h_n)) &= \text{tr}(DT_{x_n(-1)}^{n+1}) = \text{tr}(D(T^n R^{-1} \circ RTR^{-1} \circ R)_{x_n(-1)}) \\ &= \text{tr}(D(RT^n R^{-1})_{(\xi_n, \eta_n)} \cdot D(RTR^{-1})_{(\eta_n, \xi_n)}) = \text{tr}(DN_{(\xi_n, \eta_n)}^n \cdot D\mathcal{G}_{(\eta_n, \xi_n)}). \end{aligned}$$

By Lemma 4.5, we have

$$DN_{(\xi_n, \eta_n)}^n = \begin{pmatrix} \Delta_n^n & 0 \\ 0 & \Delta_n^{-n} \end{pmatrix} + n\Delta_n' \Delta_n^{-1} \begin{pmatrix} \Delta_n^n & 1 \\ -1 & -\Delta_n^{-n} \end{pmatrix},$$

with  $\Delta_n' := \Delta'(\zeta_n)\zeta_n$ , and then, it follows from (4.4) that

$$\begin{aligned} & 2 \cosh(2(n+1)\text{LE}(h_n)) \\ &= \text{tr} \left( \begin{pmatrix} \Delta_n^n & 0 \\ 0 & \Delta_n^{-n} \end{pmatrix} \cdot \begin{pmatrix} G_1^+(\eta_n) & G_2^+(\eta_n) \\ G_1^-(\eta_n) & G_2^-(\eta_n) \end{pmatrix} \right) \\ &+ n\Delta_n' \Delta_n^{-1} \text{tr} \left( \begin{pmatrix} \Delta_n^n & 1 \\ -1 & -\Delta_n^{-n} \end{pmatrix} \cdot \begin{pmatrix} G_1^+(\eta_n) & G_2^+(\eta_n) \\ G_1^-(\eta_n) & G_2^-(\eta_n) \end{pmatrix} \right) \\ &= \Delta_n^{-n} g(\eta_n) + \Delta_n^n \gamma'(\eta_n) (2 - \gamma'(\eta_n) g(\eta_n)) \\ &- n\Delta_n' \Delta_n^{-1} [\Delta_n^{-n} g(\eta_n) + 2(\gamma'(\eta_n) g(\eta_n) - 1) - \Delta_n^n \gamma'(\eta_n) (2 - \gamma'(\eta_n) g(\eta_n))] \\ &= \Delta_n^{-n} (1 - n\Delta_n' \Delta_n^{-1}) g(\eta_n) + 2n\Delta_n' \Delta_n^{-1} (1 - \gamma'(\eta_n) g(\eta_n)) \\ &+ \Delta_n^n (1 + n\Delta_n' \Delta_n^{-1}) \gamma'(\eta_n) (2 - \gamma'(\eta_n) g(\eta_n)). \quad \square \end{aligned}$$

**Remark 4.9.** *Our choice for the definitions of  $I_n$ ,  $\Pi_n$ ,  $\text{III}_n$  will become clearer in the following. Roughly speaking, we write them in this way so that their expansions begin with the “same weight”, i.e., are 0-triangular in the sense of Definition 4.18.*

In the following, as explained in Remark 4.7, we derive asymptotic expansions with respect to  $n$  of the parameters  $\eta_n$  and of the other symbols which appear in the expression of the Lyapunov exponent  $\text{LE}(h_n)$  obtained in Lemma 4.8. In Lemma 4.10, we compute the first terms in these expansions. In Lemma 4.11, we study their general structure, and show that they can be expressed as certain series mixing polynomials and exponentials in  $n$ , and whose coefficients are “homogeneous” combinations of the gluing terms and of Birkhoff coefficients. In Lemma 4.16, we compute the value of the coefficients of the different terms in the expansions of the parameters; each time, we focus on the terms with the largest index, as we see them for the first time, while the previous terms appear as additive constants. These estimates will later be used (see e.g. Lemma 4.21) to show that the expression involving the Lyapunov exponents  $\text{LE}(h_n)$  in Lemma 4.8 admits an asymptotic expansion as  $n$  goes to  $+\infty$ , of the form  $\sum_{q,p \geq 0} L_{q,p} n^q (\lambda^n)^p$ , where the coefficients  $(L_{q,p})_{q,p}$  depend on the constants  $\{\bar{a}_i\}_{i \geq 1}$  and  $\{\bar{\gamma}_i\}_{i \geq 1}$ . In particular, as  $\lambda \in (0, 1)$ , each of these terms decays at a different speed with respect to  $n$ , hence the coefficients  $(L_{q,p})_{q,p}$  are determined by the collection of Lyapunov exponents  $(\text{LE}(h_n))_{n \geq 1}$ .

Due to the different roles that the various coefficients  $\{\bar{a}_i\}_{i \geq 1}$ ,  $\{\bar{\gamma}_i\}_{i \geq 1}$  of  $\Delta$  and  $\gamma$  play (see e.g. the formula given by Lemma 4.5), we hope to be able to distinguish between them in the estimates; in particular, we expect Birkhoff coefficients to “weigh more” than the gluing terms, since the periodic orbits  $h_n$  spend much more time in a neighborhood of the saddle than in the gluing region. The way we actually show this is by carefully analyzing the dependence of the quantities  $(L_{q,p})_{q,p}$  on the coefficients  $\{\bar{a}_i\}_{i \geq 1}$ ,  $\{\bar{\gamma}_i\}_{i \geq 1}$ : in a first time, we compute inductively the expansion of  $\eta_n$  in terms of  $\{\bar{a}_i\}_{i \geq 1}$ ,  $\{\bar{\gamma}_i\}_{i \geq 1}$  thanks to the formula given by Lemma 4.5. Next, we compute the expansions of the other expressions which appear in the formula given by Lemma 4.8. Finally, we show that, provided a certain non-degeneracy condition is satisfied, the system of equations given by the coefficients  $(L_{q,p})_{q,p}$  can be inverted,

i.e., the coefficients  $\{\bar{a}_i\}_{i \geq 1}$  and  $\{\bar{\gamma}_i\}_{i \geq 1}$  can be reconstructed from these data (see e.g. Corollary 4.22 for more details).

**Lemma 4.10.** *With the notation introduced in (4.9)–(4.10), it holds:*

$$\begin{aligned} \bar{\eta}_n &= 1 + [n\bar{a}_1 + \bar{\gamma}_1]\lambda^n + \left[ n^2 \frac{3\bar{a}_1^2}{2} + n \left( -\frac{\bar{a}_1^2}{2} + 4\bar{a}_1\bar{\gamma}_1 + \bar{a}_2 \right) + (\bar{\gamma}_1^2 + \bar{\gamma}_2) \right] \lambda^{2n} \\ &\quad + O(n^3 \lambda^{3n}). \end{aligned}$$

By the fact that  $\Delta_n = \Delta(\zeta_n)$  and  $\Delta'_n := \Delta'(\zeta_n)\zeta_n$ , the previous estimates give

$$\begin{aligned} \lambda^n \Delta_n^{-n} &= 1 - n\bar{a}_1 \lambda^n - \left[ n^2 \frac{\bar{a}_1^2}{2} + n(2\bar{a}_1\bar{\gamma}_1 + \bar{a}_2 - \frac{\bar{a}_1^2}{2}) \right] \lambda^{2n} + O(n^3 \lambda^{3n}), \\ 1 - n\Delta'_n \Delta_n^{-1} &= 1 - n\bar{a}_1 \lambda^n - [n^2 \bar{a}_1^2 + n(2\bar{a}_1\bar{\gamma}_1 + 2\bar{a}_2 - \bar{a}_1^2)] \lambda^{2n} + O(n^3 \lambda^{3n}), \end{aligned}$$

and

$$\lambda^n \Delta_n^{-n} (1 - n\Delta'_n \Delta_n^{-1}) = 1 - 2n\bar{a}_1 \lambda^n - \left[ n^2 \frac{\bar{a}_1^2}{2} + n(4\bar{a}_1\bar{\gamma}_1 + 3\bar{a}_2 - \frac{3\bar{a}_1^2}{2}) \right] \lambda^{2n} + O(n^3 \lambda^{3n}).$$

In particular,

$$(4.11) \quad 2 \cosh(2(n+1)\text{LE}(h_n)) = \lambda^{-n} g_0 - 2ng_0 \bar{a}_1 + O(1),$$

hence the coefficients  $g_0$  and  $\bar{a}_1$  are determined by the Marked Length Spectrum.

*Proof.* Let  $n \geq n_0$ . By Lemma 4.5, and since we assume that  $\xi_\infty = 1$ , we have

$$\begin{aligned} \eta_n &= \Delta(\eta_n(1 + \gamma(\eta_n)))^n (1 + \gamma(\eta_n)) \\ &= \left( \sum_{j=0}^3 a_j \eta_n^j \cdot \left( \sum_{k=0}^3 \gamma_k \eta_n^k \right)^j + O(\eta_n^4) \right)^n \cdot \left( \sum_{\ell=0}^3 \gamma_\ell \eta_n^\ell + O(\eta_n^4) \right), \end{aligned}$$

where recall that  $\gamma_0 = \xi_\infty = 1$ ; this yields the expansion

$$\begin{aligned} \eta_n &= \lambda^n \left( 1 + [n\lambda^{-1}a_1 + \gamma_1]\eta_n + \left[ n^2 \frac{(\lambda^{-1}a_1)^2}{2} \right. \right. \\ &\quad \left. \left. + n \left( -\frac{(\lambda^{-1}a_1)^2}{2} + 2\lambda^{-1}a_1\gamma_1 + \lambda^{-1}a_2 \right) + \gamma_2 \right] \eta_n^2 + O(n^3 \eta_n^3) \right). \end{aligned}$$

By considering first order terms, we obtain  $\eta_n = \lambda^n + O(n\lambda^{2n})$ . Plugging this back into the previous equation, we deduce that

$$\eta_n = \lambda^n + [n\bar{a}_1 + \bar{\gamma}_1]\lambda^{2n} + O(n^2 \lambda^{3n}).$$

We thus obtain

$$\begin{aligned} \bar{\eta}_n &= \lambda^{-n} \left( \lambda^n + [n\bar{a}_1 + \bar{\gamma}_1]\lambda^{2n} + \left[ (n\bar{a}_1 + \bar{\gamma}_1)^2 + n^2 \frac{\bar{a}_1^2}{2} \right. \right. \\ &\quad \left. \left. + n \left( -\frac{\bar{a}_1^2}{2} + 2\bar{a}_1\bar{\gamma}_1 + \bar{a}_2 \right) + \bar{\gamma}_2 \right] \lambda^{3n} + O(n^3 \lambda^{4n}) \right) \\ &= 1 + [n\bar{a}_1 + \bar{\gamma}_1]\lambda^n + \left[ n^2 \frac{3\bar{a}_1^2}{2} + n \left( -\frac{\bar{a}_1^2}{2} + 4\bar{a}_1\bar{\gamma}_1 + \bar{a}_2 \right) + (\bar{\gamma}_1^2 + \bar{\gamma}_2) \right] \lambda^{2n} + O(n^3 \lambda^{3n}). \end{aligned}$$

To obtain the expansions of  $\Delta_n^{\pm n}$ , we argue as follows: by definition, we have  $\Delta_n = \Delta(\zeta_n)$ , with  $\zeta_n = \eta_n(1 + \gamma(\eta_n)) = \lambda^n \bar{\zeta}_n$  as in (4.10), so that

$$\begin{aligned} \bar{\zeta}_n &= \bar{\eta}_n + \bar{\gamma}_1 \lambda^n \bar{\eta}_n^2 + \bar{\gamma}_2 \lambda^{2n} \bar{\eta}_n^3 + O(n^3 \lambda^{3n}) \\ &= 1 + [n\bar{a}_1 + 2\bar{\gamma}_1]\lambda^n + \left[ n^2 \frac{3\bar{a}_1^2}{2} + n \left( -\frac{\bar{a}_1^2}{2} + 6\bar{a}_1\bar{\gamma}_1 + \bar{a}_2 \right) + (3\bar{\gamma}_1^2 + 2\bar{\gamma}_2) \right] \lambda^{2n} + O(n^3 \lambda^{3n}). \end{aligned}$$

To conclude, it suffices to expand the following expressions:

$$\Delta_n^{-n} = (\Delta^{-1}(\zeta_n))^n = \lambda^{-n} (1 - \bar{a}_1 \lambda^n \bar{\zeta}_n + (\bar{a}_1^2 - \bar{a}_2) \lambda^{2n} \bar{\zeta}_n^2)^n + O(n^3 \lambda^{2n}),$$

and

$$\begin{aligned} 1 - n \Delta_n' \Delta_n^{-1} &= 1 - n \Delta'(\zeta_n) \zeta_n \cdot \Delta^{-1}(\zeta_n) \\ &= 1 - n \lambda^n \bar{\zeta}_n (\bar{a}_1 + 2\bar{a}_2 \lambda^n \bar{\zeta}_n) \cdot (1 - \bar{a}_1 \lambda^n \bar{\zeta}_n) + O(n^3 \lambda^{3n}) \\ &= 1 - n \lambda^n (\bar{a}_1 \bar{\zeta}_n + (2\bar{a}_2 - \bar{a}_1^2) \lambda^n \bar{\zeta}_n^2) + O(n^3 \lambda^{3n}). \end{aligned}$$

The previous estimates and the expression obtained in Lemma 4.8 yield

$$2 \cosh(2(n+1)\text{LE}(h_n)) = \lambda^{-n} g_0 - 2n g_0 \bar{a}_1 + O(1).$$

Indeed, the other expressions in the formula given by Lemma 4.8 are bounded, since  $\Delta_n^{\pm n} = O(\lambda^{\pm n})$ , while  $\eta_n = O(\lambda^n)$ .

By Theorem 1.10, the quantities on the left hand side can be computed, thus we can recover the value of  $g_0$  and  $\bar{a}_1$  by separating terms growing at different speeds:

$$\begin{aligned} g_0 &= \lim_{n \rightarrow +\infty} 2 \cosh(2(n+1)\text{LE}(h_n)) \lambda^n, \\ \bar{a}_1 &= \lim_{n \rightarrow +\infty} \frac{1}{2n} (\lambda^{-n} - 2g_0^{-1} \cosh(2(n+1)\text{LE}(h_n))). \end{aligned}$$

Indeed, recall that by Remark 4.4, the coefficient  $g_0$  does not vanish.  $\square$

More generally, we prove the following lemma.

**Lemma 4.11.** *There exists  $n_0 > 0$  so that for any integer  $i \geq 1$ , there exists a sequence  $(P_k^{(i)})_{k \geq 1}$  of polynomials such that for any integer  $n \geq n_0$ :*

$$\bar{\eta}_n^i = 1 + \sum_{k=1}^{+\infty} P_k^{(i)}(n) \lambda^{nk},$$

where for each  $k \geq 1$ , the polynomial  $P_k^{(i)}(X) = \sum_{j=0}^k \mu_{j,k}^{(i)} X^j$  has degree  $k$ . For simplicity, we abbreviate  $P_k^{(1)} = P_k$  and  $\mu_{j,k}^{(1)} = \mu_{j,k}$  in the following.<sup>11</sup>

Similarly, there exist three sequences  $(Q_k^{\pm})_{k \geq 1}$ ,  $(R_k)_{k \geq 1}$  of polynomials such that

$$\begin{aligned} \lambda^{\mp n} \Delta_n^{\pm n} &= 1 + \sum_{k=1}^{+\infty} Q_k^{\pm}(n) \lambda^{nk}, \\ 1 - n \Delta_n' \Delta_n^{-1} &= 1 + \sum_{k=1}^{+\infty} R_k(n) \lambda^{nk}, \end{aligned}$$

where for each  $k \geq 1$ , the polynomials  $Q_k^{\pm}(X) = \sum_{j=0}^k \nu_{j,k}^{\pm} X^j$  and  $R_k(X) = \sum_{j=0}^k \rho_{j,k}^{\pm} X^j$  have degree  $k$ .

In particular, by Lemma 4.10, it holds

$$(4.12) \quad \begin{cases} \mu_{0,0}^{(i)} = 1, & \mu_{1,1}^{(i)} = i \bar{a}_1, & \mu_{2,2}^{(i)} = \frac{i(i+2)}{2} \bar{a}_1^2, \\ \nu_{0,0}^- = 1, & \nu_{1,1}^- = -\bar{a}_1, & \nu_{2,2}^- = -\frac{\bar{a}_1^2}{2}, \\ \rho_{0,0} = 1, & \rho_{1,1} = -\bar{a}_1, & \rho_{2,2} = -\bar{a}_1^2. \end{cases}$$

<sup>11</sup> Note that it is sufficient to show the result for  $i = 1$ , as such expansions are stable by taking powers. This is what we are going to do in the following proof.

*Proof.* Let us first consider  $\bar{\eta}_n$ ,  $n \geq n_0$ . We will prove by induction on  $\ell \geq 0$  that  $\bar{\eta}_n = \bar{\eta}_{n,\ell} + O(n^{\ell+1}\lambda^{n(\ell+1)})$ , where

$$(4.13) \quad \bar{\eta}_{n,\ell} = 1 + \sum_{k=1}^{\ell} P_k(n)\lambda^{nk},$$

for certain polynomials  $P_1, P_2, \dots, P_\ell$  satisfying the above properties.

It is clear for  $\ell = 0$ . Assume that it holds for  $\ell - 1 \geq 0$ . To show the result for  $\ell$ , we use the formula given by Lemma 4.5:

$$(4.14) \quad \begin{aligned} \bar{\eta}_n &= \lambda^{-n} \Delta(\eta_n(1 + \gamma(\eta_n)))^n (1 + \gamma(\eta_n)) \\ &= \left(1 + \sum_{p=1}^{\ell} \lambda^{-1} a_p \eta_n^p (1 + \gamma(\eta_n))^p + O(\eta_n^{\ell+1})\right)^n (1 + \gamma(\eta_n)) \\ &= \sum_{r=0}^n \binom{n}{r} \left(\sum_{p=1}^{\ell} \lambda^{-1} a_p \eta_n^p \left(1 + \sum_{q=1}^{\ell} \gamma_q \eta_n^q\right)^p + O(\eta_n^{\ell+1})\right)^r \left(1 + \sum_{s=1}^{\ell} \gamma_s \eta_n^s + O(\eta_n^{\ell+1})\right) \\ &= \sum_{r=0}^{\ell} \binom{n}{r} \left(\sum_{p=1}^{\ell} \bar{a}_p \lambda^{np} \bar{\eta}_{n,\ell-1}^p \left(1 + \sum_{q=1}^{\ell} \bar{\gamma}_q \lambda^{nq} \bar{\eta}_{n,\ell-1}^q\right)^p\right)^r \left(1 + \sum_{s=1}^{\ell} \bar{\gamma}_s \lambda^{ns} \bar{\eta}_{n,\ell-1}^s\right) \\ &\quad + O(n^{\ell+1}\lambda^{n(\ell+1)}). \end{aligned}$$

Indeed, it is sufficient to consider the  $(\ell - 1)$ -expansion  $\bar{\eta}_{n,\ell-1} = 1 + \sum_{k=1}^{\ell-1} P_k(n)\lambda^{nk}$  of  $\bar{\eta}_n$  obtained previously to go from  $\ell - 1$  to  $\ell$ , as the summations indices  $p, q, s$  are all at least equal to one, and hence each term  $\bar{\eta}_n^*$  in the above expression is multiplied by a factor  $\lambda^{nk}$ , with  $k \geq 1$ . Moreover, we can restrict ourselves to indices  $r, p, q, s \in \{0, \dots, \ell\}$ , since for  $r, p, q, s \geq \ell + 1$ , the associated terms are of order  $O(n^{\ell+1}\lambda^{n(\ell+1)})$ .

We claim that the degree of the polynomial in  $n$  associated to the factor  $\lambda^{n\ell}$  is at most  $\ell$ . Indeed, the expansion of the previous expression is a combination of powers of  $\bar{\eta}_{n,\ell-1}$  (which are themselves combinations of polynomials in  $n$  multiplied by powers of  $\lambda^n$ , where the degree of the polynomial is at most equal to the exponent of  $\lambda^n$ ) multiplied by binomial coefficients  $\binom{n}{*}$  and powers of  $\bar{a}_* \lambda^{n*}$  or  $\bar{\gamma}_* \lambda^{n*}$ . Besides the degree of the polynomial in  $n$  associated to binomial coefficients is always less than or equal to the exponent of  $\lambda^n$  (for each  $r \geq 1$ , the coefficient  $\binom{n}{r}$  gives a polynomial of degree  $r$  in  $n$ , while the second factor  $\left(\sum_{p=1}^{\ell} \bar{a}_p \lambda^{np} \dots\right)^r$  in formula (4.14) above already contributes to a power of  $\lambda^n$  with an exponent  $rp \geq r$ ).

We conclude that the new expansion will be of the same form as before, i.e., for some polynomial  $P_\ell$  of degree at most  $\ell$ , we have

$$\bar{\eta}_n = 1 + \sum_{k=1}^{\ell} P_k(n)\lambda^{nk} + O(n^{\ell+1}\lambda^{n(\ell+1)}).$$

Let us now consider the expansion of  $\Delta_n^{\pm n}$ . We first remark that  $\Delta(z)^{\pm} = \lambda^{\pm} + \sum_{k=1}^{+\infty} a_k^{\pm} z^k$ , where for each  $k \geq 1$ ,  $a_k^{\pm} = a_k$ , and

$$a_k^{-} = -\lambda^{-2} a_k - \lambda^{-1} (a_1 a_{k-1}^{-} + \dots + a_{k-1} a_1^{-}).$$

Similarly, for  $i \geq 1$ , let  $\bar{a}_i^{\pm} := \lambda^{\mp} a_i^{\pm} \xi_{\infty}^{2i} = \lambda^{\mp} a_i^{\pm}$ . As a result, for each  $k \geq 0$ , it holds

$$(4.15) \quad \bar{a}_k^{-} = -\bar{a}_k - (\bar{a}_1 \bar{a}_{k-1}^{-} + \dots + \bar{a}_{k-1} \bar{a}_1^{-}).$$

Thus, for any integer  $\ell \geq 0$ , we obtain

$$\begin{aligned} \Delta_n^{\pm n} &= (\Delta(\eta_n(1 + \gamma(\eta_n)))^{\pm 1})^n \\ &= \lambda^{\pm n} \left( 1 + \sum_{p=1}^{\ell} \lambda^{\mp 1} a_p^{\pm} \eta_n^p (1 + \gamma(\eta_n))^p + O(\eta_n^{\ell+1}) \right)^n \\ &= \lambda^{\pm n} \left( 1 + \sum_{r=1}^{\ell} \binom{n}{r} \left( \sum_{p=1}^{\ell} \bar{a}_p^{\pm} \lambda^{np} \bar{\eta}_n^p \left( 1 + \sum_{q=1}^{\ell} \bar{\gamma}_q \lambda^{nq} \bar{\eta}_n^q \right)^p \right)^r \right) + O(n^{\ell+1} \lambda^{n(\ell+1\pm 1)}). \end{aligned}$$

The form of the expansions of  $\Delta_n^{\pm n}$  and  $1 - n\Delta'_n \Delta_n^{-1}$  follows from the expression of  $\bar{\eta}_n$  obtained previously, since  $\Delta_n = \Delta(\zeta_n)$  and  $\Delta'_n = \Delta'(\zeta_n)\zeta_n$ , with  $\zeta_n = \lambda^n \bar{\eta}_n (1 + \gamma(\lambda^n \bar{\eta}_n))$ .  $\square$

**Remark 4.12.** *On a formal level, we see that  $\bar{\eta}_n$ ,  $\lambda^{\mp n} \Delta_n^{\pm n}$  and  $1 - n\Delta'_n \Delta_n^{-1}$  can be expressed as (formal) series in  $\lambda^n$  with coefficients in the ring of polynomials in  $n$ . Moreover the coefficient of order  $k$  is a polynomial of degree  $k$ . Let us call balanced those formal series with coefficients in the ring of polynomials in  $n$  with the property that the coefficient of order  $k$  is a polynomial of degree at most  $k$ . Observe that such series are closed under sum and product; moreover they are also closed under composition with an analytic function. We conclude that the quantities  $\mathbf{I}_n$ ,  $\mathbf{II}_n$ ,  $\mathbf{III}_n$  introduced in Lemma 4.8 are also balanced series. Let us also note that expansions of a similar type were studied earlier in the paper [FY] for a different purpose.*

**Lemma 4.13.** *For any integers  $k \geq 1$ ,  $j \in \{0, \dots, k\}$ , and  $\ell \geq 1$ , the coefficients  $\mu_{j,k}^{(\ell)}$ ,  $\nu_{j,k}^{\pm}$ ,  $\rho_{j,k}$  are “homogeneous” expressions in the parameters  $\{\bar{a}_i\}_i$ ,  $\{\bar{\gamma}_i\}_i$ :*

$$*_{j,k} = *_{j,k}(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_{k-j+1}, \bar{\gamma}_1, \bar{\gamma}_2, \dots, \bar{\gamma}_{k-j}), \quad *_{j,k} = \mu_{j,k}^{(\ell)}, \nu_{j,k}^{\pm}, \rho_{j,k},$$

where  $*_{j,k}$  is a linear combination of terms of the form

$$(4.16) \quad \bar{a}_1^{p_1} \bar{a}_2^{p_2} \dots \bar{a}_{k-j+1}^{p_{k-j+1}} \bar{\gamma}_1^{q_1} \bar{\gamma}_2^{q_2} \dots \bar{\gamma}_{k-j}^{q_{k-j}},$$

with

$$\text{i) } \sum_{i=1}^{k-j} p_i \geq j, \quad \text{ii) } \sum_{i=1}^{k-j+1} i p_i + \sum_{i=1}^{k-j} i q_i = k.$$

*Proof.* Let us study how the coefficients  $\{\mu_{j,k}\}_{j,k}$  in the expansion of  $\bar{\eta}_n$  depend on the parameters  $\{\bar{a}_i\}_i$ ,  $\{\bar{\gamma}_i\}_i$ . The “homogeneous” structure of the expansion of the coefficients  $\{\mu_{j,k}^{(\ell)}\}_{j,k}$ ,  $\{\nu_{j,k}^{\pm}\}_{j,k}$  and  $\{\rho_{j,k}\}_{j,k}$  is shown in the same way as for the coefficients  $\{\mu_{j,k}\}_{j,k}$ .

For any integers  $\ell \geq 1$  and  $n \geq n_0$ , recall the equation for  $\bar{\eta}_n$  obtained in the proof of Lemma 4.11:

$$\bar{\eta}_n = \sum_{r=0}^{\ell} \binom{n}{r} \left( \sum_{p=1}^{\ell} \bar{a}_p \lambda^{np} \bar{\eta}_n^p \left( 1 + \sum_{q=1}^{\ell} \bar{\gamma}_q \lambda^{nq} \bar{\eta}_n^q \right)^p \right)^r \left( 1 + \sum_{s=1}^{\ell} \bar{\gamma}_s \lambda^{ns} \bar{\eta}_n^s \right) + O(n^{\ell+1} \lambda^{n(\ell+2)}).$$

The expansion of this expression is a combination of terms as in (4.16). In particular, for any integer  $\ell \geq 1$ , we see that  $\bar{a}_\ell$  first appears with the weight  $n\lambda^{n\ell}$  (for  $r = 1$  and  $p = \ell$  with the above notation), while  $\bar{\gamma}_\ell$  first appears with the weight  $\lambda^{n\ell}$  (for  $r = 0$  and  $s = \ell$  with the above notation).

More generally, the ‘‘homogeneity’’ property ii) above is essentially due to the fact that in the previous expansion,  $\bar{a}_p$  always comes together with the weight  $\lambda^{np}$ , while  $\bar{\gamma}_q$  always comes together with the weight  $\lambda^{nq}$ .  $\square$

**Remark 4.14.** *Note that increasing the exponent of  $n$  in the expansion of  $\bar{\eta}_n$  corresponds to taking derivatives of the function  $\Delta$  in formula (4.8). In terms of the above expansion, those derivatives are associated to certain binomial coefficients, and each time we increase the exponent of  $n$  by one, the exponent of  $\lambda^n$  is increased by at least one too, depending on the weight of the coefficient  $\bar{a}_p$  associated to this derivative. Together with the previous remark on the first appearance of  $\bar{a}_\ell$ ,  $\bar{\gamma}_\ell$ , this explains the constraint in (4.16) and ii) on the coefficients which can enter the expression associated to a specific weight, and why they depend on the difference  $k - j$  between the exponent  $j$  of  $n^j$  and the exponent  $k$  of  $\lambda^{nk}$  (the coefficients  $\bar{a}_{k-j+1}^{p_{k-j+1}}$  and  $\bar{\gamma}_{k-j}^{q_{k-j}}$  are obtained when all the derivatives we take are associated to  $\bar{a}_1$ ).*

*Besides, the reason why we have an inequality and not an equality in point i) is because the binomial coefficients  $\binom{n}{r}$  are not homogeneous polynomials in  $n$ .*

**Remark 4.15.** *The reason why the respective weights of  $\bar{a}_\ell$  and  $\bar{\gamma}_\ell$  on their first appearance differ by a factor  $n$  is due to the fact that any orbit under consideration spends much more time ( $n$  steps) in the region where we have Birkhoff coordinates, while the gluing term associated to the coefficient  $\bar{\gamma}_\ell$  accounts for a bounded number of steps in the orbit (i.e., the number of such steps does not depend on  $n$ ).*

**Lemma 4.16.** *With the notation introduced in Lemma 4.11, for every  $k \geq 1$ , there exist constants  $c_{0,k}, c_{j,k+j}, c_{j,k+j}^\pm, c'_{j,k+j} \in \mathbb{R}$ ,  $j = 1, 2$ , with*

$$\begin{aligned} c_{0,k} &= c_{0,k}(\bar{\gamma}_1, \bar{\gamma}_2, \dots, \bar{\gamma}_{k-1}), \\ *_{j,k+j} &= *_{j,k+j}(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_k, \bar{\gamma}_1, \bar{\gamma}_2, \dots, \bar{\gamma}_{k-1}), \quad * = c, c^\pm, c', \end{aligned}$$

such that

$$(4.17a) \quad \begin{cases} \mu_{0,k} &= \bar{\gamma}_k + c_{0,k}, \\ \mu_{1,k+1} &= (k+3)\bar{a}_1 \cdot \bar{\gamma}_k + \bar{a}_{k+1} + c_{1,k+1}, \\ \mu_{2,k+2} &= \frac{(k+3)(k+5)}{2}\bar{a}_1^2 \cdot \bar{\gamma}_k + (k+3)\bar{a}_1 \cdot \bar{a}_{k+1} + c_{2,k+2}; \end{cases}$$

$$(4.17b) \quad \begin{cases} \nu_{0,k}^\pm &= 0, \\ \nu_{1,k+1}^\pm &= \pm(2\bar{a}_1 \cdot \bar{\gamma}_k + \bar{a}_{k+1}) + c_{1,k+1}^\pm, \\ \nu_{2,k+2}^\pm &= \pm(k+2 \pm 1)\bar{a}_1(2\bar{a}_1 \cdot \bar{\gamma}_k + \bar{a}_{k+1}) + c_{2,k+2}^\pm; \end{cases}$$

$$(4.17c) \quad \begin{cases} \rho_{0,k} &= 0, \\ \rho_{1,k+1} &= -2\bar{a}_1 \cdot \bar{\gamma}_k - (k+1)\bar{a}_{k+1} + c'_{1,k+1}, \\ \rho_{2,k+2} &= -2(k+2)\bar{a}_1^2 \cdot \bar{\gamma}_k - ((k+1)^2 + 1)\bar{a}_1 \cdot \bar{a}_{k+1} + c'_{2,k+2}. \end{cases}$$

**Remark 4.17.** Before giving the details of the proof, let us explain how the computations are carried out. We first focus on  $\bar{\eta}_n$  and study the coefficients of the expansion given by Lemma 4.11; the expressions of  $\lambda^{\mp n} \Delta_n^{\pm n}$  and  $1 - n\Delta'_n \Delta_n^{-1}$  follow from that of  $\bar{\eta}_n$ , as they are obtained by evaluating the functions  $\Delta, \Delta', \gamma, \dots$  at the point  $\bar{\eta}_n$ . Note that the equation in Lemma 4.5 can be rewritten as

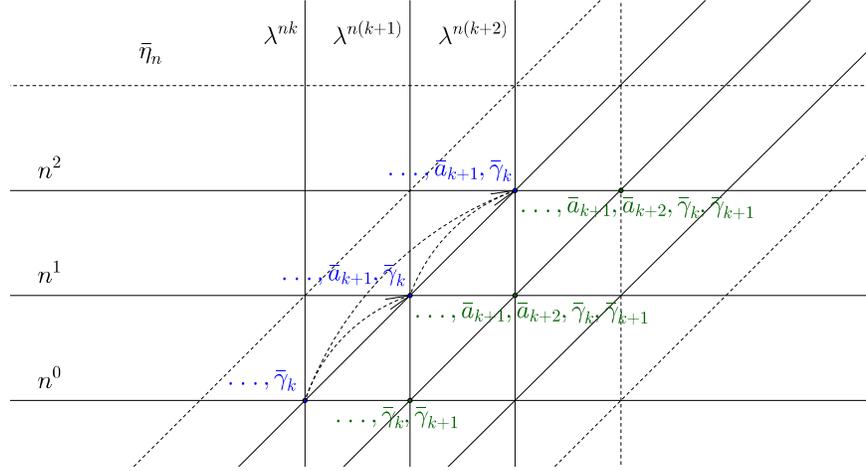
$$(4.18) \quad \bar{\eta}_n = \bar{\Delta}(\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n))^n \bar{\gamma}(\lambda^n \bar{\eta}_n),$$

denoting

$$\bar{\Delta}: z \mapsto 1 + \sum_{j=1}^{+\infty} \bar{a}_j z^j, \quad \text{and} \quad \bar{\gamma}: z \mapsto 1 + \gamma(z) = 1 + \sum_{j=1}^{+\infty} \bar{\gamma}_j z^j.$$

Fix an integer  $k \geq 1$ . Based on the implicit equation (4.18) satisfied by  $\bar{\eta}_n$ , we determine inductively the coefficients of  $\bar{a}_{k+1}$  and  $\bar{\gamma}_k$  in the expression of  $\bar{\eta}_n$ . More precisely, in order to explicit the dependence of  $\bar{\eta}_n$  on  $\bar{a}_{k+1}$ , resp.  $\bar{\gamma}_k$ , we differentiate (4.18) with respect to  $\bar{a}_{k+1}$ , resp.  $\bar{\gamma}_k$ , and plug the expansion we already have in the right hand side. The presence of the extra factor  $\lambda^n$  acts as a shift, i.e., it “propagates” the information we have one step further. Besides, in order to avoid seeing new unknown quantities  $\bar{a}_{k+2}, \bar{\gamma}_{k+1}, \dots$ , we only consider the terms in the expansion of  $\bar{\eta}_n$  which are aligned along the line of slope 1 based at the points where  $\bar{a}_{k+1}$  and  $\bar{\gamma}_k$  first appear (see Fig. 5). At each step, the expressions we obtain in the derivative of (4.18) are combinations of terms of two kinds:

- terms of the form  $n^A \bar{a}_1^B \bar{a}_{k+1} \lambda^{nC} \bar{\eta}_n^D$  or  $n^A \bar{a}_1^B \bar{\gamma}_k \lambda^{nC} \bar{\eta}_n^D$  with  $C$  large; in this case, we use the expressions of the first coefficients of  $\bar{\eta}_n^D$  given by (4.12);
- terms of the form  $n^A \bar{a}_1^B \lambda^{nC} \bar{\eta}_n^D$  with  $C$  small; in this case, based on the expansion computed previously, we identify the coefficients of  $\bar{a}_{k+1}$  and  $\bar{\gamma}_k$  in the expression of  $\bar{\eta}_n^D$  in order to go one step further in the expansion.


 FIGURE 5. Coefficients in the series expansion of  $\bar{\eta}_n$ .

**Definition 4.18.** For any integer  $k \geq 0$ , a formal series  $\mathcal{S}$  is called  $k$ -triangular if

- (1) it is of the form  $\mathcal{S} = \sum_{p-q \geq k, q \geq 0} s_{q,p} n^q \lambda^{np}$ ;
- (2) its principal part  $\mathbb{P}(\mathcal{S}) := \sum_{p-q=k} s_{q,p} n^q \lambda^{np}$  is non-zero, i.e.,  $\mathbb{P}(\mathcal{S}) \neq 0$ .

In other words,  $\mathcal{S}$  is the product of  $\lambda^{nk}$  and of a balanced series with non-zero principal part.

Let us state some basic properties of triangular series which will be useful in the following.

**Remark 4.19.** Let  $\mathcal{S}_0$  be a  $k$ -triangular series, for some integer  $k \geq 0$ .

- for any  $k$ -triangular series  $\mathcal{S}_1$  such that  $\mathbb{P}(\mathcal{S}_0) + \mathbb{P}(\mathcal{S}_1) \neq 0$ , the series  $\mathcal{S}_0 + \mathcal{S}_1$  is  $k$ -triangular, and  $\mathbb{P}(\mathcal{S}_0 + \mathcal{S}_1) = \mathbb{P}(\mathcal{S}_0) + \mathbb{P}(\mathcal{S}_1)$ .
- For any integer  $\ell \geq 0$  and any  $\ell$ -triangular series  $\mathcal{S}_2$  such that  $\mathbb{P}(\mathcal{S}_0)\mathbb{P}(\mathcal{S}_2) \neq 0$ , the series  $\mathcal{S}_0\mathcal{S}_2$  is  $(k + \ell)$ -triangular and it holds

$$\mathbb{P}(\mathcal{S}_0\mathcal{S}_2) = \mathbb{P}(\mathcal{S}_0)\mathbb{P}(\mathcal{S}_2).$$

In particular, if  $\mathcal{S}$  is a balanced series and if  $\omega: z \mapsto \sum_{j=0}^{+\infty} \omega_j z^j$  is an analytic function with  $\omega_0 \neq 0$ , then  $\omega(\lambda^n \mathcal{S})$  is a 0-triangular series, and  $\mathbb{P}(\omega(\mathcal{S})) = \omega_0$ .

*Proof of Lemma 4.16.* Let  $k \geq 1$ .

PROOF OF (4.17a): since  $\partial_{\bar{a}_{k+1}} \bar{\Delta}: z \mapsto z^{k+1}$ , differentiating (4.18) with respect to  $\bar{a}_{k+1}$ , we thus get

$$\begin{aligned} \partial_{\bar{a}_{k+1}} \bar{\eta}_n &= n \bar{\Delta} (\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n))^{n-1} \left[ \lambda^{n(k+1)} \bar{\eta}_n^{k+1} (\bar{\gamma}(\lambda^n \bar{\eta}_n))^{k+2} + \lambda^n \partial_{\bar{a}_{k+1}} \bar{\eta}_n [\bar{\gamma}(\lambda^n \bar{\eta}_n) + \right. \\ &\quad \left. + \lambda^n \bar{\eta}_n \bar{\gamma}'(\lambda^n \bar{\eta}_n)] \bar{\Delta}'(\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n)) \bar{\gamma}(\lambda^n \bar{\eta}_n) \right] + \partial_{\bar{a}_{k+1}} \bar{\eta}_n \cdot \bar{\Delta} (\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n))^n \lambda^n \bar{\gamma}'(\lambda^n \bar{\eta}_n). \end{aligned}$$

By the fact that  $\bar{\eta}_n$  is a 0-triangular series, it follows from this expression that  $\partial_{\bar{a}_{k+1}}\bar{\eta}_n$  is a  $k$ -triangular series with leading term

$$(4.19) \quad \partial_{\bar{a}_{k+1}}\bar{\eta}_n = n\lambda^{n(k+1)} + \dots$$

Besides, the terms  $\gamma(\lambda^n\bar{\eta}_n)$  and  $\lambda^n\gamma'(\lambda^n\bar{\eta}_n)$  are 1-triangular and are multiplied by  $k$ -triangular terms in the previous expression, hence by Remark 4.19 they do not contribute to the principal part of  $\partial_{\bar{a}_{k+1}}\bar{\eta}_n$ . For the same reason, the only term in the expansion of  $\bar{\Delta}'(\lambda^n\bar{\eta}_n\bar{\gamma}(\lambda^n\bar{\eta}_n))$  which contributes to the principal part of  $\partial_{\bar{a}_{k+1}}\bar{\eta}_n$  is the constant term  $\bar{a}_1$ . By (4.12) and (4.19), we thus get

$$\begin{aligned} \mathbb{P}(\partial_{\bar{a}_{k+1}}\bar{\eta}_n) &= \mathbb{P}\left(\bar{\Delta}(\lambda^n\bar{\eta}_n)^{n-1}\left[n\lambda^{n(k+1)}\bar{\eta}_n^{k+1} + n\lambda^n\partial_{\bar{a}_{k+1}}\bar{\eta}_n\bar{a}_1\right]\right) \\ &= (1 + n\bar{a}_1\lambda^n)\left[n\lambda^{n(k+1)}(1 + (k+1)n\bar{a}_1\lambda^n) + n^2\bar{a}_1\lambda^{n(k+2)}\right] + O(n^3\lambda^{n(k+3)}) \\ &= n\lambda^{n(k+1)} + (k+3)n^2\bar{a}_1\lambda^{n(k+2)} + O(n^3\lambda^{n(k+3)}). \end{aligned}$$

Similarly,  $\partial_{\bar{\gamma}_k}\bar{\gamma} = \partial_{\bar{\gamma}_k}\gamma : z \mapsto z^k$ , hence

$$\begin{aligned} \partial_{\bar{\gamma}_k}\bar{\eta}_n &= \bar{\Delta}(\lambda^n\bar{\eta}_n\bar{\gamma}(\lambda^n\bar{\eta}_n))^n\left[\lambda^{nk}\bar{\eta}_n^k + \lambda^n\partial_{\bar{\gamma}_k}\bar{\eta}_n\gamma'(\lambda^n\bar{\eta}_n)\right] + n\bar{\Delta}'(\lambda^n\bar{\eta}_n\bar{\gamma}(\lambda^n\bar{\eta}_n)) \cdot \\ &\quad \bar{\Delta}(\lambda^n\bar{\eta}_n\bar{\gamma}(\lambda^n\bar{\eta}_n))^{n-1}\left[\lambda^n\partial_{\bar{\gamma}_k}\bar{\eta}_n[\bar{\gamma}(\lambda^n\bar{\eta}_n) + \lambda^n\bar{\eta}_n\gamma'(\lambda^n\bar{\eta}_n)] + \lambda^{n(k+1)}\bar{\eta}_n^{k+1}\right]\bar{\gamma}(\lambda^n\bar{\eta}_n). \end{aligned}$$

By the fact that  $\bar{\eta}_n$  is a balanced series, it follows from this expression that  $\partial_{\bar{\gamma}_k}\bar{\eta}_n$  is a  $k$ -triangular series with leading term

$$\partial_{\bar{\gamma}_k}\bar{\eta}_n = \lambda^{nk} + \dots$$

As previously, the terms  $\gamma(\lambda^n\bar{\eta}_n)$  and  $\lambda^n\gamma'(\lambda^n\bar{\eta}_n)$  do not contribute to the principal part of  $\partial_{\bar{\gamma}_k}\bar{\eta}_n$ , as they are 1-triangular and are multiplied by  $k$ -triangular terms in the previous expression, and  $\bar{\Delta}'(\lambda^n\bar{\eta}_n\bar{\gamma}(\lambda^n\bar{\eta}_n))$  can be replaced with  $\bar{a}_1$ . Together with (4.12), the previous expansion thus yields

$$\begin{aligned} \mathbb{P}(\partial_{\bar{\gamma}_k}\bar{\eta}_n) &= \mathbb{P}\left(\bar{\Delta}(\lambda^n\bar{\eta}_n)^n\lambda^{nk}\bar{\eta}_n^k + n\bar{a}_1\lambda^n\bar{\Delta}(\lambda^n\bar{\eta}_n)^{n-1}\left[\partial_{\bar{\gamma}_k}\bar{\eta}_n + \lambda^{nk}\bar{\eta}_n^{k+1}\right]\right) \\ &= (1 + n\bar{a}_1\lambda^n)\left[\lambda^{nk}(1 + kn\bar{a}_1\lambda^n) + n\bar{a}_1\lambda^n \cdot 2\lambda^{nk}\right] + O(n^2\lambda^{n(k+2)}) \\ &= \lambda^{nk} + (k+3)n\bar{a}_1\lambda^{n(k+1)} + O(n^2\lambda^{n(k+2)}). \end{aligned}$$

Plugging this back in the expression of  $\mathbb{P}(\partial_{\bar{\gamma}_k}\bar{\eta}_n)$ , and going one step further in the expansion of  $\bar{\Delta}(\lambda^n\bar{\eta}_n)^n$ , we can compute the third term in the principal part of  $\partial_{\bar{\gamma}_k}\bar{\eta}_n$  (we do not give the details here as it will not be needed in the following).

PROOF OF (4.17b): the derivation of the coefficients  $\{\nu_{j,k}^\pm\}_{j,k}$  is done in a similar way. Yet, unlike  $\bar{\eta}_n$ , now there is no implicit equation anymore, thus we can differentiate  $\lambda^{\mp n}\Delta_n^{\pm n}$  directly and use the expressions of  $\{\mu_{j,k}\}_{j,k}$  obtained above. We note that

$$\lambda^{\mp n}\Delta_n^{\pm n} = \bar{\Delta}(\lambda^n\bar{\eta}_n\bar{\gamma}(\lambda^n\bar{\eta}_n))^{\pm n}.$$

We will detail the calculations only for  $\pm = -$  which is the case we will need in the following. The case where  $\pm = +$  is analogous. By the above formula, we thus get

$$\begin{aligned} \partial_{\bar{a}_{k+1}}\lambda^n\Delta_n^{-n} &= -n\bar{\Delta}(\lambda^n\bar{\eta}_n\bar{\gamma}(\lambda^n\bar{\eta}_n))^{-(n+1)}\left[\lambda^{n(k+1)}\bar{\eta}_n^{k+1}(\bar{\gamma}(\lambda^n\bar{\eta}_n))^{k+1} + \right. \\ &\quad \left. + \lambda^n\partial_{\bar{a}_{k+1}}\bar{\eta}_n[\bar{\gamma}(\lambda^n\bar{\eta}_n) + \lambda^n\bar{\eta}_n\gamma'(\lambda^n\bar{\eta}_n)]\bar{\Delta}'(\lambda^n\bar{\eta}_n\bar{\gamma}(\lambda^n\bar{\eta}_n))\right]. \end{aligned}$$

We see that the associated series is also  $k$ -triangular. For the same reason as before, the terms  $\gamma(\lambda^n \bar{\eta}_n)$  and  $\lambda^n \gamma'(\lambda^n \bar{\eta}_n)$  need not be considered, and the only term of  $\bar{\Delta}'(\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n))$  which contributes to the principal part of  $\partial_{\bar{a}_{k+1}} \lambda^n \Delta_n^{-n}$  is  $\bar{a}_1$ . Replacing  $\partial_{\bar{a}_{k+1}} \bar{\eta}_n$  with the value computed previously, and by (4.12), we thus obtain

$$\begin{aligned} \mathbb{P}(\partial_{\bar{a}_{k+1}} \lambda^n \Delta_n^{-n}) &= -\mathbb{P}\left(\bar{\Delta}(\lambda^n \bar{\eta}_n)^{-(n+1)} \left[ n \lambda^{n(k+1)} \bar{\eta}_n^{k+1} + n \bar{a}_1 \lambda^n \partial_{\bar{a}_{k+1}} \bar{\eta}_n \right]\right) \\ &= -(1 - n \bar{a}_1 \lambda^n) \left[ n \lambda^{n(k+1)} (1 + (k+1) n \bar{a}_1 \lambda^n) + n^2 \bar{a}_1 \lambda^{n(k+2)} \right] + O(n^3 \lambda^{n(k+3)}) \\ &= -n \lambda^{n(k+1)} - (k+1) n \bar{a}_1 \lambda^{n(k+2)} + O(n^3 \lambda^{n(k+3)}). \end{aligned}$$

Similarly,

$$\begin{aligned} \partial_{\bar{\gamma}_k} \lambda^n \Delta_n^{-n} &= -n \bar{\Delta}'(\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n)) \bar{\Delta}(\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n))^{-(n+1)} \left[ \lambda^{n(k+1)} \bar{\eta}_n^{k+1} + \right. \\ &\quad \left. + \lambda^n \partial_{\bar{\gamma}_k} \bar{\eta}_n [\bar{\gamma}(\lambda^n \bar{\eta}_n) + \lambda^n \bar{\eta}_n \gamma'(\lambda^n \bar{\eta}_n)] \right]. \end{aligned}$$

Arguing as before, and replacing  $\partial_{\bar{\gamma}_k} \bar{\eta}_n$  with the value computed previously, we get

$$\begin{aligned} \mathbb{P}(\partial_{\bar{\gamma}_k} \lambda^n \Delta_n^{-n}) &= -\mathbb{P}\left(n \bar{a}_1 \bar{\Delta}(\lambda^n \bar{\eta}_n)^{-(n+1)} \left[ \lambda^{n(k+1)} \bar{\eta}_n^{k+1} + \lambda^n \partial_{\bar{\gamma}_k} \bar{\eta}_n \right]\right) \\ &= -n \bar{a}_1 \lambda^n (1 - n \bar{a}_1 \lambda^n) \left[ \lambda^{nk} (1 + (k+1) n \bar{a}_1 \lambda^n) + \lambda^{nk} + (k+3) n \bar{a}_1 \lambda^{n(k+1)} \right] + O(n^3 \lambda^{n(k+3)}) \\ &= -2n \lambda^{n(k+1)} - 2(k+1) n^2 \bar{a}_1 \lambda^{n(k+2)} + O(n^3 \lambda^{n(k+3)}). \end{aligned}$$

PROOF OF (4.17c): let us now deal with the coefficients  $\{\rho_{j,k}\}_{j,k}$ . The computations are carried out in the same way as for the coefficients  $\{\nu_{j,k}^-\}_{j,k}$ . Set  $\bar{D}: z \mapsto \sum_{j=1}^{+\infty} j \bar{a}_j z^j$ . It holds

$$1 - n \Delta_n' \Delta_n^{-1} = 1 - n \bar{D}(\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n)) \bar{\Delta}(\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n))^{-1}.$$

We have  $\partial_{\bar{a}_{k+1}} \bar{D}: z \mapsto (k+1)z^{k+1}$ . It follows that

$$\begin{aligned} \partial_{\bar{a}_{k+1}} (1 - n \Delta_n' \Delta_n^{-1}) &= -n \lambda^{n(k+1)} \bar{\eta}_n^{k+1} (\bar{\gamma}(\lambda^n \bar{\eta}_n))^{k+1} \bar{\Delta}(\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n))^{-1} \left[ (k+1) - \right. \\ &\quad \left. - \Delta_n' \Delta_n^{-1} \right] - n \lambda^n \partial_{\bar{a}_{k+1}} \bar{\eta}_n [\bar{\gamma}(\lambda^n \bar{\eta}_n) + \lambda^n \bar{\eta}_n \gamma'(\lambda^n \bar{\eta}_n)] \bar{\Delta}(\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n))^{-1} \\ &\quad \cdot \left[ \bar{D}'(\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n)) - \bar{\Delta}'(\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n)) \Delta_n' \Delta_n^{-1} \right]. \end{aligned}$$

In the previous expression,  $[(k+1) - \Delta_n' \Delta_n^{-1}]$  is a 0-triangular series whose principal part is reduced to  $k+1$ , as  $\Delta_n' \Delta_n^{-1}$  is 1-triangular. Similarly,  $\bar{D}'(\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n)) - \bar{\Delta}'(\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n)) \Delta_n' \Delta_n^{-1}$  is 0-triangular, and its principal part is equal to  $\bar{a}_1$ . We see that  $\partial_{\bar{a}_{k+1}} (1 - n \Delta_n' \Delta_n^{-1})$  is  $k$ -triangular, and as before, the terms  $\gamma(\lambda^n \bar{\eta}_n)$  and  $\lambda^n \gamma'(\lambda^n \bar{\eta}_n)$  do not contribute to the principal part of  $\partial_{\bar{a}_{k+1}} (1 - n \Delta_n' \Delta_n^{-1})$ . We thus obtain

$$\begin{aligned} &\mathbb{P}\left(\partial_{\bar{a}_{k+1}} (1 - n \Delta_n' \Delta_n^{-1})\right) \\ &= -\mathbb{P}\left(\bar{\Delta}(\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n))^{-1} \left[ n \lambda^{n(k+1)} \bar{\eta}_n^{k+1} (k+1) + n \bar{a}_1 \lambda^n \partial_{\bar{a}_{k+1}} \bar{\eta}_n \right]\right) \\ &= -n \lambda^{n(k+1)} (1 + (k+1) n \bar{a}_1 \lambda^n) (k+1) - n \bar{a}_1 \lambda^n \cdot n \lambda^{n(k+1)} + O(n^3 \lambda^{n(k+3)}) \\ &= -(k+1) n \lambda^{n(k+1)} - [(k+1)^2 + 1] n^2 \bar{a}_1 \lambda^{n(k+2)} + O(n^3 \lambda^{n(k+3)}). \end{aligned}$$

Finally, we have

$$\begin{aligned} \partial_{\bar{\gamma}_k}(1 - n\Delta'_n\Delta_n^{-1}) &= -n \left[ \lambda^n \partial_{\bar{\gamma}_k} \bar{\eta}_n [\bar{\gamma}(\lambda^n \bar{\eta}_n) + \lambda^n \bar{\eta}_n \gamma'(\lambda^n \bar{\eta}_n)] + \lambda^{n(k+1)} \bar{\eta}_n^{k+1} \right] \\ &\cdot \bar{\Delta}(\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n))^{-1} \cdot \left[ \bar{D}'(\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n)) - \bar{\Delta}'(\lambda^n \bar{\eta}_n \bar{\gamma}(\lambda^n \bar{\eta}_n)) \Delta'_n \Delta_n^{-1} \right]. \end{aligned}$$

Arguing as above, we deduce that

$$\begin{aligned} \mathbb{P}\left(\partial_{\bar{\gamma}_k}(1 - n\Delta'_n\Delta_n^{-1})\right) &= -\mathbb{P}\left(n\bar{a}_1(\lambda^n \partial_{\bar{\gamma}_k} \bar{\eta}_n + \lambda^{n(k+1)} \bar{\eta}_n^{k+1})\right) \\ &= -n\bar{a}_1 \lambda^{n(k+1)} - (k+3)n^2 \bar{a}_1^2 \lambda^{n(k+2)} - n\bar{a}_1 \lambda^{n(k+1)} - (k+1)n^2 \bar{a}_1^2 \lambda^{n(k+2)} + O(n^3 \lambda^{n(k+3)}) \\ &= -2n\bar{a}_1 \lambda^{n(k+1)} - 2(k+2)n^2 \bar{a}_1^2 \lambda^{n(k+2)} + O(n^3 \lambda^{n(k+3)}). \end{aligned}$$

□

We reported the above computations since they could be useful in some further developments of this work. In the current section we will in fact only rely upon some specific combinations, which occur in the term denoted with  $I_n$  and we collect in the following corollary (recall that the quantities  $I_n$ ,  $\Pi_n$ ,  $\text{III}_n$  below are those which were introduced in Lemma 4.8).

**Corollary 4.20.** *The following holds:*

$$\lambda^n \Delta_n^{-n} (1 - n\Delta'_n \Delta_n^{-1}) = \sum_{p=0}^{\infty} \sum_{q=0}^p L_{q,p}^* n^q \lambda^{np},$$

and we have:

$$(4.20a) \quad L_{0,p}^* = c_{0,p}^*,$$

$$(4.20b) \quad L_{1,p+1}^* = -4\bar{a}_1 \bar{\gamma}_p - (p+2)\bar{a}_{p+1} + c_{1,p+1}^*,$$

$$(4.20c) \quad L_{2,p+2}^* = -2(2p+1)\bar{a}_1^2 \bar{\gamma}_p - (p+1)^2 \bar{a}_1 \bar{a}_{p+1} + c_{2,p+2}^*,$$

where  $c_{i,p+1}^*$  depend only on the coefficients  $\{\bar{\gamma}_\ell, \bar{a}_{\ell+1}\}_{0 \leq \ell < p}$ . Moreover

$$(4.21) \quad L_{0,0}^* = 1, \quad L_{1,1}^* = -2\bar{a}_1, \quad L_{2,2}^* = -\frac{1}{2}\bar{a}_1^2.$$

*Proof.* By definition of  $L_{q,p}^*$ , we gather that for any  $p \geq 0$  and  $0 \leq q \leq p$ :

$$L_{q,p}^* = \sum_{\substack{p'+p''=p \\ q'+q''=q}} \nu_{q',p'}^- \rho_{q'',p''}.$$

Observe that the contribution of terms for which both  $p' - q' < p$  and  $p'' - q'' < p$  can be absorbed in the terms  $c^*$ , since they do not depend on either  $\bar{a}_{p+1}$  nor  $\bar{\gamma}_p$  by Lemma 4.13.

In particular, if  $q = 0$ , then necessarily  $q' = q'' = 0$ , hence:

$$L_{0,p}^* = \nu_{0,p}^- \rho_{0,0} + \nu_{0,0}^- \rho_{0,p} + c_{0,p};$$

and using (4.12) and Lemma 4.16 we obtain (4.20a). If  $q = 1$  then either  $q' = 1$  and  $q'' = 0$  or  $q' = 0$  and  $q'' = 1$ . Observe that by Lemma 4.16, the coefficients  $\nu_{0,k}^-$  and  $\rho_{0,k}$  are 0 unless  $k = 0$ ; we conclude that:

$$L_{1,p+1}^* = \nu_{1,p+1}^- \rho_{0,0} + \nu_{0,0}^- \rho_{1,p+1} + c_{1,p+1},$$

which yields (4.20b). Finally, we consider the case  $q = 2$ ; in this case one could have  $q' = 0, 1, 2$  and correspondingly  $q'' = 2 - q'$ . This leads to:

$$L_{2,p+2}^* = \nu_{2,p+2}^- \rho_{0,0} + \nu_{1,p+1}^- \rho_{1,1} + \nu_{1,1}^- \rho_{1,p+1} + \nu_{0,0}^- \rho_{2,p+2},$$

which yields (4.20c). Equations (4.21) then follow from similar arguments, or directly from Lemma 4.10.  $\square$

**4.3. Determination of the scaled coefficients**  $\{\bar{g}_\ell, \bar{\gamma}_\ell, \bar{a}_\ell\}_{\ell \geq 0}$ . In this part, we keep the same notation and show how the above estimates can be employed to show that the scaled coefficients  $\{\bar{g}_\ell, \bar{\gamma}_\ell, \bar{a}_\ell\}_{\ell \geq 0}$  introduced in (4.9) are  $\mathcal{MLS}$ -invariants.

**Lemma 4.21.** *There exists a sequence of real numbers*

$$(L_{q,p})_{\substack{p=0, \dots, +\infty \\ q=0, \dots, p}}$$

such that for any integer  $n \geq n_0$ , we have the following expansion:

$$(4.22) \quad 2\lambda^n \cosh(2(n+1)\text{LE}(h_n)) = \sum_{p=0}^{+\infty} \sum_{q=0}^p L_{q,p} n^q \lambda^{np}.$$

Moreover, for any  $p \geq 1$ , the following linear relation holds:

$$(4.23) \quad W_p = A_p V_p + C_p,$$

where  $V_p, W_p, C_p \in \mathbb{R}^3$  are defined as:

$$(4.24) \quad V_p := \begin{pmatrix} \bar{g}_p \\ \bar{\gamma}_p \\ \bar{a}_{p+1} \end{pmatrix}, \quad W_p := \begin{pmatrix} L_{0,p} \\ L_{1,p+1} \\ L_{2,p+2} \end{pmatrix}, \quad C_p := \begin{pmatrix} C_{0,p} \\ C_{1,p+1} \\ C_{2,p+2} \end{pmatrix},$$

$A_p \in \mathfrak{M}_3(\mathbb{R})$  is given by:

$$A_p := \begin{pmatrix} 1 & 0 & 0 \\ (p-2)\bar{a}_1 & -4\bar{a}_1 g_0 & -(p+2)g_0 \\ \frac{p^2-2p-1}{2}\bar{a}_1^2 & -2(2p+1)\bar{a}_1^2 g_0 & -(p+1)^2 \bar{a}_1 g_0 \end{pmatrix},$$

and for  $i \in \{0, 1, 2\}$ , the constants  $C_{i,p+i} \in \mathbb{R}$  only depend on the coefficients<sup>12</sup>  $\{\bar{g}_\ell, \bar{\gamma}_\ell, \bar{a}_{\ell+1}\}_{0 \leq \ell < p}$ .

*Proof.* Let  $n \geq n_0$ . By Lemma 4.8, we have

$$2\lambda^n \cosh(2(n+1)\text{LE}(h_n)) = \text{I}_n + \lambda^n \text{II}_n + \lambda^{2n} \text{III}_n.$$

By Remark 4.12,  $\text{I}_n$ ,  $\text{II}_n$  and  $\text{III}_n$  are balanced series, i.e.:

$$\text{I}_n = \sum_{p=0}^{+\infty} \sum_{q=0}^p L_{q,p}^{\text{I}} n^q \lambda^{np}, \quad \text{II}_n = \sum_{p=0}^{+\infty} \sum_{q=0}^p L_{q,p}^{\text{II}} n^q \lambda^{np}, \quad \text{III}_n = \sum_{p=0}^{+\infty} \sum_{q=0}^p L_{q,p}^{\text{III}} n^q \lambda^{np},$$

and therefore also the left hand side of (4.22) is a balanced series, i.e., (4.22) holds. We thus need to show (4.23). Let us fix an integer  $p \geq 1$ . Observe that

$$(4.25) \quad L_{q,p} = L_{q,p}^{\text{I}} + L_{q,p-1}^{\text{II}} + L_{q,p-2}^{\text{III}};$$

moreover, by construction (see Lemma 4.13), we can also conclude that  $L_{q,p}^*$  only depend on  $\{\bar{a}_{i+1}, \bar{\gamma}_i, \bar{g}_i\}_{i=0, \dots, p-q}$  for  $*$  =  $\text{I}_n$ ,  $\text{II}_n$  and  $\text{III}_n$ . Hence, the contributions to  $L_{0,p}$ , (resp.  $L_{1,p+1}$ ,  $L_{2,p+2}$ ) of the last two terms in (4.25) do contain no  $\bar{g}_p$ ,  $\bar{\gamma}_p$  or

<sup>12</sup> Recall the notation introduced in (4.9).

$\bar{a}_{p+1}$ , and can thus be absorbed in  $C_{0,p}$  (resp.  $C_{1,p+1}$ ,  $C_{2,p+2}$ ). In order to show (4.23) it thus suffices to study the coefficients  $L_{q,p}^I$  of the balanced series:

$$I_n = \lambda^n \Delta_n^{-n} (1 - n \Delta_n' \Delta_n^{-1}) g(\lambda^n \bar{\eta}_n).$$

We begin to study the dependence of  $I_n$  on  $\bar{g}_p$  (i.e. the first column of  $A_p$ ). Thanks to Lemma 4.11, we can write:

$$g(\eta_n) = \sum_{\ell=0}^{+\infty} \bar{g}_\ell \lambda^{n\ell} \bar{\eta}_n^\ell = \sum_{\ell=0}^{+\infty} \bar{g}_\ell \lambda^{n\ell} \sum_{k=0}^{\infty} \sum_{j=0}^k \mu_{j,k}^{(\ell)} n^j \lambda^{nk}.$$

Observe that in the expansion of  $g$ , the coefficient  $\bar{g}_\ell$  is multiplied by  $\lambda^{n\ell} \bar{\eta}_n^\ell$ . Therefore, using Corollary 4.20, we obtain

$$\begin{aligned} I_n &= \left[ \sum_{p'=0}^{\infty} \sum_{q'=0}^{p'} L_{q',p'}^* n^{q'} \lambda^{np'} \right] \cdot \left[ \sum_{\ell=0}^{+\infty} \bar{g}_\ell \lambda^{n\ell} \sum_{k=0}^{\infty} \sum_{j=0}^k \mu_{j,k}^{(\ell)} n^j \lambda^{nk} \right] \\ &= \sum_{p=0}^{\infty} \sum_{p'+p''+p'''=p} \sum_{q=0}^{p'+p'''} \sum_{q'+q'''=q} L_{q',p'}^* \bar{g}_{p''} \mu_{q''',p'''}^{(p'')} n^q \lambda^{np}, \end{aligned}$$

which yields:

$$L_{q,p}^I = \sum_{p'+p''+p'''=p} \sum_{\substack{q'+q'''=q \\ 0 \leq q' \leq p' \\ 0 \leq q''' \leq p'''}} L_{q',p'}^* \bar{g}_{p''} \mu_{q''',p'''}^{(p'')}.$$

In order to extract the contribution of  $\bar{g}_p$  we thus need to set  $p'' = p$ ; we conclude that, for  $i \in \{0, 1, 2\}$ , the coefficient  $\bar{g}_p$  appears in  $L_{i,p+i}$  multiplied by a factor  $K_i^p = \sum_{r+s=i} \mu_{r,r}^{(p)} L_{s,s}^*$  and by (4.12) and Corollary 4.20 we can at last conclude:

$$K_0^p = 1, \quad K_1^p = (p-2)\bar{a}_1, \quad K_2^p = \frac{p^2 - 2p - 1}{2} \bar{a}_1^2.$$

We now proceed to study the second and third columns of  $A_p$ , which amounts to study the dependence on  $\bar{\gamma}$  and  $\bar{a}$ . This, in principle, entails more work than the previous task, since the coefficients  $\bar{\gamma}$  and  $\bar{a}$  show up in the expansions of each of the terms in  $I_n$ , and not just the last term. As a matter of fact, the last term does not contribute at all; in fact notice that, as before, we can write:

$$g(\eta_n) = \sum_{\ell=0}^{+\infty} \bar{g}_\ell \lambda^{n\ell} \sum_{k=0}^{\infty} \sum_{j=0}^k \mu_{j,k}^{(\ell)} n^j \lambda^{nk},$$

with the convention that  $\mu_{j,k}^{(0)} = 0$  for all  $(j,k) \neq (0,0)$ , and  $\mu_{0,0}^{(0)} = 1$ . As noted earlier (recall Lemma 4.13), the coefficients  $\bar{a}_{p+1}$  and  $\bar{\gamma}_p$  would only occur in the expression for  $\mu_{j,k}^{(\ell)}$  with  $k-j \geq p$ . If we consider  $L_{0,p}$  (resp.  $L_{1,p+1}$ ,  $L_{2,p+2}$ ), we thus must set  $k = p$  (resp.  $p+1$ ,  $p+2$ ); in turn this implies that  $\ell = 0$  (since  $\ell+k = p+i$ ). But then  $\mu_{j,k}^{(0)} = 0$  for any  $j,k$ . Thus it suffices to consider the expansion of

$$\tilde{I}_n = g_0 \cdot \lambda^n \Delta_n^{-n} (1 - n \Delta_n' \Delta_n^{-1})$$

and the statement follows from Corollary 4.20.  $\square$

We have seen in Lemma 4.10 that the values of  $g_0$  and  $\bar{a}_1$  are  $\mathcal{MLS}$ -invariants; moreover by Remark 4.4,  $g_0 \neq 0$ .

**Corollary 4.22.** *Under the assumption that the first Birkhoff coefficient does not vanish, i.e.,  $\bar{a}_1 \neq 0$ , then the coefficients  $\{\bar{g}_\ell, \bar{\gamma}_\ell, \bar{a}_\ell\}_{\ell \geq 0}$  are  $\mathcal{MLS}$ -invariants.*

*Proof.* By Theorem 1.10, the Marked Lyapunov Spectrum is a  $\mathcal{MLS}$ -invariant; in particular, for each  $n \geq n_0$ ,  $\text{LE}(h_n)$  is a  $\mathcal{MLS}$ -invariant. By Lemma 4.21, we also have

$$2\lambda^n \cosh(2(n+1)\text{LE}(h_n)) = \sum_{p=0}^{+\infty} \sum_{q=0}^p L_{q,p} n^q \lambda^{np}.$$

Notice that each term  $\{n^q \lambda^{np}\}_{\substack{p=0, \dots, +\infty \\ q=0, \dots, p}}$ , grows at a different rate as  $n \rightarrow +\infty$ ; hence, their associated weights can be determined inductively, i.e., each coefficient  $L_{q,p}$  is a  $\mathcal{MLS}$ -invariant. It thus suffices to prove that the coefficients  $L_{q,p}$  determine the coefficients  $\{\bar{g}_p, \bar{\gamma}_p, \bar{a}_{p+1}\}_{p \geq 0}$ .

As recalled above,  $g_0$  and  $\bar{a}_1$  are  $\mathcal{MLS}$ -invariants, and  $g_0 \neq 0$ , by transversality. We proceed by induction on  $p$ ; by Corollary 4.10,  $(\bar{g}_0, \bar{\gamma}_0, \bar{a}_1) = (g_0, 1, \bar{a}_1)$  is spectrally determined. Given  $p \geq 1$ , let us assume that the coefficients  $\{\bar{g}_\ell, \bar{\gamma}_\ell, \bar{a}_{\ell+1}\}_{0 \leq \ell < p}$  are known; we want to compute  $V_p$  (recall (4.24)). By Lemma 4.21, we have  $W_p = A_p V_p + C_p$  for some matrix  $A_p \in \mathfrak{M}_3(\mathbb{R})$ , which only depends on  $p$ ,  $\bar{a}_1$  and  $g_0$  and hence it is spectrally determined; moreover:

$$\det A_p = -2p\bar{a}_1^2 g_0^2.$$

In particular, under the assumption that  $\bar{a}_1 \neq 0$ , we have  $\det A_p \neq 0$ , since  $g_0 \neq 0$ , and  $p \geq 1$ . Therefore:

$$V_p = A_p^{-1}(W_p - C_p).$$

By Lemma 4.21, the vector  $C_p$  is determined by inductive hypothesis;  $W_p$  is obtained from the coefficients  $L_{q,p}$ , which are  $\mathcal{MLS}$ -invariants; we conclude that the vector  $V_p$  is a  $\mathcal{MLS}$ -invariant.  $\square$

**4.4. Change of Lyapunov exponents in the horseshoe.** In this subsection, we consider the more general case of a  $C^\infty$  billiard table  $\mathcal{D} = \mathbb{R}^2 \setminus \cup_{i=1}^m \mathcal{O}_i$  with  $m \geq 3$  obstacles that satisfies the non-eclipse condition. For any periodic orbit encoded by an admissible word  $\hat{\sigma}$  of length  $p \geq 2$ , the construction explained above can be suitably adapted, by considering a sequence  $(\hat{h}_n)_{n \geq 0}$  of periodic orbits accumulating some orbit  $\hat{h}_\infty$  homoclinic to  $\hat{\sigma}$ .

Thus, an analogue of Lemma 4.21 tells us that the local variation of the Lyapunov exponent in the horseshoe can be expressed in terms of three sets of data, which we denote analogously by  $\{\hat{g}_\ell, \hat{\gamma}_\ell, \hat{a}_\ell\}_{\ell \geq 0}$  (associated to the jets of the corresponding functions).

Let us now consider the (degenerate) situation in which Lyapunov exponents does not change at all, i.e.,  $\text{LE}(\hat{h}_n) = \hat{L} = -\frac{1}{p} \log(\hat{\lambda}) > 0$  for all  $n \geq 1$ , where  $\hat{\lambda} \in (0, 1)$  is the contracting eigenvalue of  $D\mathcal{F}^p$  at the points of  $\hat{\sigma}$ .<sup>13</sup> By (4.11), we deduce that  $\hat{g}_0 \neq 0$  and  $\hat{a}_1 = 0$  (let us also recall that the parameter  $\hat{\gamma}_0$  is equal to

<sup>13</sup> According to [BD, Proposition 7.13], this condition is necessary for the measure of maximal entropy  $\mu_*$  constructed in [BD] and the SRB measure  $\mu_{\text{SRB}}$  to coincide.

some homoclinic coefficient  $\hat{\xi}_\infty \neq 0$  as previously). In particular, if the Lyapunov Spectrum is reduced to a single point, then necessarily the first Birkhoff invariant of each periodic orbit has to vanish (which is an infinite codimension condition – arguing as in Lemma 6.6 below), and the coefficient  $\hat{g}_0$  depends very little on the homoclinic orbit  $\hat{h}_\infty$ , as it is prescribed by the length of the word  $\hat{h}_0$  (which is also non-generic, as  $\hat{g}_0$  may be changed by perturbing the geometry of the points in  $\hat{h}_\infty$  far from the orbit  $\hat{\sigma}$  – see Remark 4.4). This is elaborated in more detail as [DLVY, Theorem 5.1]

Under the same assumption, let us now consider the analogue of (4.22). In this case, the LHS is the sum of a constant and of a multiple of  $\hat{\lambda}^{2n}$ , which implies that the coefficients  $(\hat{L}_{q,p})_{q,p}$  on the RHS satisfy  $\hat{L}_{q,p} = 0$  for all  $p \geq 0$  and  $0 \leq q \leq p$  but  $(q,p) = (0,0)$  and  $(q,p) = (0,2)$ . We cannot argue directly as in Corollary 4.22 to recover the value of  $\{\hat{g}_\ell, \hat{\gamma}_\ell, \hat{a}_\ell\}_{\ell \geq 0}$ , as the computations above were carried out under the assumption that  $\hat{a}_1 \neq 0$ . Nevertheless, as explained in Remark 4.17, by considering other directions in the array of coefficients  $(\hat{L}_{q,p})_{p,q}$ , we may be able to derive many more constraints that those coefficients have to satisfy in this case. For instance, in the same way as the first order term in the variation of the Lyapunov exponent is controlled by  $\hat{g}_0, \hat{a}_1$ , the variation of  $\hat{a}_1$  is controlled by  $\hat{g}_1, \hat{\gamma}_1, \hat{a}_2 \dots$ , and as  $\hat{a}_1$  vanishes identically, it is reasonable to expect that  $\hat{a}_2 = 0$ ; by induction, it may then be possible to show that the Birkhoff invariants are all zero. This statement is elaborated as [DLVY, Corollary 5.4].

## 5. FURTHER ESTIMATES ON THE MARKED LENGTH SPECTRUM

**5.1. Basic facts about twist maps and generating functions.** Recall that for an angle  $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ , we denote  $r := \sin \varphi$ , so that the billiard map  $\mathcal{F}$  takes the form  $\mathcal{F}: (s, r) \mapsto (s', r')$ . For any point  $(s, r)$  whose iterate  $(s', r') = \mathcal{F}(s, r)$  is well defined, we let  $(s, s') := \Psi(s, r)$ . The billiard map  $\mathcal{F}$  is exact symplectic: for the one-form  $\omega_1 = \omega_1(s, r) := rds$ , it holds

$$(5.1) \quad \mathcal{F}^* \omega_1 - \omega_1 = dh(s, s'),$$

for the generating function  $h(s, s') := \|\Upsilon(s) - \Upsilon(s')\|$ , letting  $\Upsilon(\cdot)$  is the length-parametrization of  $\partial\mathcal{D}$ . Let  $T := \mathcal{F}^2: (s, r) \mapsto (s'', r'')$  be the square of the billiard map, and  $h^{(2)}(s, s'') = h(s, s') + h(s', s'')$  be the generating function for  $T$ , with  $s' = s'(s, s'')$  being determined implicitly by the condition  $\partial_2 h(s, s') + \partial_1 h(s', s'') = 0$ . We get

$$(5.2) \quad T^* \omega_1 - \omega_1 = dh^{(2)}(s, s'').$$

In the neighborhood  $\mathcal{O}_\infty$  of  $x_\infty(-1)$ , we have  $\mathcal{G}R = RT$ , while in a neighborhood of any other points of the homoclinic orbit  $h_\infty$ , we have  $NR = RT$ . Let us set  $\omega := (R^{-1})^* \omega_1$  and  $H := \Psi^* h$ . In either case, for  $\Phi = N$  or  $\mathcal{G}$ , we obtain

$$\Phi^* \omega - \omega = dS,$$

for the generating function

$$S := (R^{-1})^*(\mathcal{F}^* H + H).$$

**5.2. Estimates on the Marked Length Spectrum.** In this part, we go back to the length data and derive some asymptotic expansion of the lengths of the periodic orbits  $(h_n)_n$  as  $n \rightarrow +\infty$ . Recall the notation  $(\xi_n, \eta_n) := R_-(x_n(1))$  and  $\Delta_n := \Delta(\xi_n, \eta_n)$ . Recall also that  $(\xi_\infty, 0) = R_-(x_\infty(1))$ .

As in Lemma 3.2 and in (3.4),  $\eta_n = \Delta_n^n \xi_n$ , and for any  $k \in \{0, \dots, n\}$ , it holds

$$(5.3) \quad R_-(x_n(2k+1)) = (\xi_n(2k+1), \eta_n(2k+1)) = \xi_n(\Delta_n^k, \Delta_n^{n-k}).$$

**Proposition 5.1.** *For  $n \rightarrow +\infty$ , the following asymptotics hold:*

$$(5.4) \quad \mathcal{L}(h_n) - (n+1)\mathcal{L}(\sigma) - \mathcal{L}^\infty = -\xi_\infty^2 \operatorname{tr}(d^2 S_{(0,0)}) \frac{1}{\lambda^{-1} - \lambda} \lambda^n + o(\lambda^n),$$

with  $\operatorname{tr}(d^2 S_{(0,0)}) := \partial_{11} S_{(0,0)} + \partial_{22} S_{(0,0)}$ .

*Proof.* Let  $\Sigma_n^1, \Sigma_n^2$  be as in (2.5a) and (2.5b). We will compute the value of the sum  $\Sigma_n^1 + \Sigma_n^2 = \mathcal{L}(h_n) - (n+1)\mathcal{L}(\sigma) - \mathcal{L}^\infty$  in terms of the generating function  $S$  for the  $(\xi, \eta)$ -coordinates.

In the following, we only consider the case where  $n$  is odd, i.e.,  $n = 2m-1$  for some integer  $m \geq m_0$ ; in particular, the period of  $h_n = h_{2m-1}$  equals  $2n+2 = 4m$ . By the palindromic symmetry of the orbit  $h_n$ ,  $x_n(-k) = \mathcal{I}(x_n(k))$  for  $k = 0, \dots, n+1$ . Besides,  $H \circ \mathcal{I} = H$  and  $S \circ \mathcal{I}_0 = S$ , with  $\mathcal{I}_0: (\xi, \eta) \mapsto (\eta, \xi)$ . We have

$$(5.5) \quad \begin{aligned} \Sigma_n^1 &= 2 \sum_{k=0}^n \left( h(s_n(k), s_n(k+1)) - h(s_\infty(k), s_\infty(k+1)) \right) \\ &= -2 \left( h^{(n+1)}(s_\infty(0), s_\infty(n+1)) - h^{(n+1)}(s_n(0), s_n(n+1)) \right), \end{aligned}$$

letting  $h^{(n+1)} = h^{(n+1)}(s_0, s_{n+1})$  be the generating function for  $\mathcal{F}^{n+1}$  near the point  $(s_n(0), s_n(n+1))$ ; in other words, for  $(s_0, s_{n+1})$  close to  $(s_n(0), s_n(n+1))$ , we set

$$h^{(n+1)}(s_0, s_{n+1}) = h(s_0, s_1) + h(s_1, s_2) + \dots + h(s_n, s_{n+1}),$$

the intermediate parameters  $s_1 = s_1(s_0, s_{n+1}), \dots, s_n = s_n(s_0, s_{n+1})$  being determined by the condition  $\partial_2 h(s_{i-1}, s_i) + \partial_1 h(s_i, s_{i+1}) = 0$ , for  $i \in \{1, \dots, n\}$ .

Considering the Taylor expansion of (5.5), we obtain

$$\begin{aligned} \Sigma_n^1 &= -2dh^{(n+1)}(s_n(0), s_n(n+1)) \left[ (s_\infty(0), s_\infty(n+1)) - (s_n(0), s_n(n+1)) \right] \\ &\quad - d^2 h^{(n+1)}(s_n(0), s_n(n+1)) \left[ (s_\infty(0), s_\infty(n+1)) - (s_n(0), s_n(n+1)) \right]^2 + H.O.T. \end{aligned}$$

Since  $h^{(n+1)}$  is the generating function for  $\mathcal{F}^{n+1}$ , as in (5.1), we have

$$\mathcal{F}^{n+1} \omega_1 - \omega_1 = dh^{(n+1)}(s_0, s_{n+1}).$$

Therefore,  $dh^{(n+1)}(s_n(0), s_n(n+1)) = 0$ , since  $x_n(0), x_n(n+1) \in \{r = 0\}$  are associated to perpendicular bounces, and  $\omega_1 = rds$ . In particular, we only have to care about second order terms to obtain the leading term in the expansion (5.4).

Let us now express the sum  $\Sigma_n^1$  in  $(\xi, \eta)$ -coordinates:

$$\begin{aligned} \Sigma_n^1 &= 2 \sum_{k=0}^{m-1} \left( S(\xi_n(2k+1), \eta_n(2k+1)) - S(\xi_\infty(2k+1), 0) \right) \\ &= -2 \sum_{k=0}^{m-1} \left( S(\lambda^k \xi_\infty, 0) - S(\Delta_n^k \xi_n, \Delta_n^{n-k} \xi_n) \right). \end{aligned}$$

By the previous discussion, first order terms in the Taylor expansion vanish, thus

$$\Sigma_n^1 = - \sum_{|\beta|=2} \frac{2}{\beta!} \sum_{k=0}^{m-1} \partial^\beta S_{\xi_n(\Delta_n^k, \Delta_n^{n-k})} \cdot \left( \lambda^k \xi_\infty - \Delta_n^k \xi_n, -\Delta_n^{n-k} \xi_n \right)^\beta + H.O.T.,$$

where in the above expression, the sum is taken over the multi-indices  $\beta = (\beta_1, \beta_2) \in \{(2, 0), (1, 1), (0, 2)\}$ , denoting  $\partial^\beta := \partial^{\beta_1} \circ \partial^{\beta_2}$ ,  $\beta! := \beta_1! \beta_2!$ , and  $(v_1, v_2)^\beta := v_1^{\beta_1} v_2^{\beta_2}$ , for  $v_1, v_2 \in \mathbb{R}$ . By Lemma 4.10 and Lemma 4.2, we have

$$|\xi_n - \xi_\infty| = O(\lambda^n), \quad |\Delta_n - \lambda| = O(\lambda^n),$$

thus for any  $0 \leq k \leq m$ ,  $|\Delta_n^k \xi_n - \lambda^k \xi_\infty| = O(n\lambda^n)$ , while  $|\Delta_n^{n-k} \xi_n| \simeq \lambda^{n-k} \xi_\infty$  (in particular,  $|\Delta_n^{n-k} \xi_n| \simeq \lambda^{\frac{n}{2}} \xi_\infty$  when  $k$  is close to  $m$ ), hence the contribution of the second term overcomes that of the first term in  $(\Delta_n^k \xi_n - \lambda^k \xi_\infty, \Delta_n^{n-k} \xi_n)$ . For  $k \simeq m$ , we have  $\|\xi_n(\Delta_n^k, \Delta_n^{n-k})\| = O(\lambda^m)$  too, thus in order to estimate the leading term in  $\Sigma_n^1$  we may consider partial derivatives  $\partial_{(0,0)}^\beta$  instead of  $\partial_{\xi_n(\Delta_n^k, \Delta_n^{n-k})}^\beta$ ; indeed,

$$\partial_{22} S_{\xi_n(\Delta_n^k, \Delta_n^{n-k})} (\Delta_n^{n-k} \xi_n)^2 = \partial_{22} S_{(0,0)} \lambda^{2(n-k)} \xi_\infty^2 + O(\lambda^{\frac{3n}{2}}).$$

Therefore, it holds

$$\Sigma_n^1 = -\xi_\infty^2 \partial_{22} S_{(0,0)} \sum_{k=0}^{m-1} \lambda^{2(n-k)} + o(\lambda^n) = -\xi_\infty^2 \partial_{22} S_{(0,0)} \frac{1}{\lambda^{-1} - \lambda} \lambda^n + o(\lambda^n).$$

As we observed,  $S \circ \mathcal{I}_0 = S$ , and  $\xi, \eta$  play symmetric roles in the computations, hence  $\partial_{22} S_{(0,0)} = \partial_{11} S_{(0,0)} = \frac{1}{2} \text{tr}(d^2 S_{(0,0)})$ , and thus,

$$(5.6) \quad \Sigma_n^1 = -\frac{\xi_\infty^2}{2} \text{tr}(d^2 S_{(0,0)}) \frac{1}{\lambda^{-1} - \lambda} \lambda^n + o(\lambda^n).$$

We argue similarly for  $\Sigma_n^2$ . Indeed,

$$\Sigma_n^2 := 2 \sum_{k=n+1}^{+\infty} \left( h(s(k), s(k+1)) - h(s_\infty(k), s_\infty(k+1)) \right),$$

and  $dh(s(k), s(k+1)) = 0$  for all  $k \in \mathbb{Z}$ , since  $x(k), x(k+1) \in \{r=0\}$  are associated to perpendicular bounces, hence we only have to care about second order terms in the expansion. Let us now express the sum  $\Sigma_n^2$  in  $(\xi, \eta)$ -coordinates:

$$(5.7) \quad \begin{aligned} \Sigma_n^2 &= 2 \sum_{k=m}^{+\infty} (S(0, 0) - S(\xi_\infty(2k+1), 0)) = -2 \sum_{k=m}^{+\infty} (S(\lambda^k \xi_\infty, 0) - S(0, 0)) \\ &= -\xi_\infty^2 \sum_{k=m}^{+\infty} \partial_{11} S_{(0,0)} \lambda^{2k} + o(\lambda^n) = -\frac{\xi_\infty^2}{2} \text{tr}(d^2 S_{(0,0)}) \frac{1}{\lambda^{-1} - \lambda} \lambda^n + o(\lambda^n). \end{aligned}$$

It follows from (5.6)-(5.7) that

$$\Sigma_n^1 + \Sigma_n^2 = -\xi_\infty^2 \text{tr}(d^2 S_{(0,0)}) \frac{1}{\lambda^{-1} - \lambda} \lambda^n + o(\lambda^n),$$

which concludes the proof.  $\square$

**Remark 5.2.** *The parameter  $\xi_\infty$  in the present paper is different from – although related to – the quantity identified by the same symbol in [BDKL]. Indeed, in [BDKL], we chose  $R$  to be a linearization of the dynamics in a neighborhood of the 2-periodic point with a suitable normalization (see the sentence above [BDKL, (8)]). On the other hand, in this paper, the map  $R$  is the conjugacy with the Birkhoff Normal Form determined in Corollary 2.7. Of course at first order the two conjugacies coincide up to a normalization, but since we have chosen two different normalizations, the corresponding formulae involving  $\xi_\infty$  (see e.g. [BDKL, (31)]) therefore differ from the ones obtained in this paper (see e.g. the estimates in Proposition 5.1).*

**5.3.  $\mathcal{MLS}$ -determination of the Birkhoff data.** Let us now assume that the billiard table  $\mathcal{D}$  presents additional symmetries in the sense of Definition 1.3, i.e.,  $\mathcal{D} \in \mathbf{B}_{\text{sym}}$ . As a consequence of the above estimates on the Marked Length Spectrum, we can conclude the following result.

**Corollary 5.3.** *Let  $\mathcal{D} \in \mathbf{B}_{\text{sym}}$ . With the same notations as above, the value of the parameter  $\xi_\infty = \xi_\infty(\mathcal{D}, 1, 2)$  associated to the orbit  $h_\infty = h_\infty(\mathcal{D}, 1, 2)$ <sup>14</sup> homoclinic to (12) is a  $\mathcal{MLS}$ -invariant.*

*Proof.* By the estimates obtained in Proposition 5.1, the stable eigenvalue  $\lambda$  is a  $\mathcal{MLS}$ -invariant; indeed, since  $\text{tr}(d^2S_{(0,0)}) > 0$  (see Lemma A.2 and (5.9) below), it holds

$$(5.8) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \log |\mathcal{L}(h_n) - (n+1)\mathcal{L}(\sigma) - \mathcal{L}^\infty| = \log \lambda,$$

the left hand side being spectrally determined<sup>15</sup>. Still by Proposition 5.1, we deduce that  $\xi_\infty^2 \text{tr}(d^2S_{(0,0)})$  is also  $\mathcal{MLS}$ -invariant. We claim that in fact, each of the two quantities  $\text{tr}(d^2S_{(0,0)})$  and  $\xi_\infty$  is  $\mathcal{MLS}$ -invariant.

Indeed, the trace  $\text{tr}(d^2S_{(0,0)})$  of the Hessian of  $S$  at  $(0, 0)$  can also be computed in  $(s, s'')$ -coordinates: letting  $h^{(2)}(s, s'') = h(s, s') + h(s', s'')$  be the generating function of  $T = \mathcal{F}^2$  at a point  $(s, s'')$ , for  $s' = s'(s, s'')$  chosen as in Subsection 5.1, and noting that the point  $(0, 0)$  is critical for  $h^{(2)}$ , it holds

$$(5.9) \quad \text{tr}(d^2S_{(0,0)}) = \text{tr}(P^T A P),$$

where  $A$  is the matrix of the Hessian  $d^2h_{(0,0)}^{(2)}$  at  $(0, 0)$ , and  $P = D\psi_{(0,0)}$  is the differential of the change of coordinates  $\psi: (\xi, \eta) \mapsto (s, s'')$ . By (5.2), we have

$$(5.10) \quad \begin{aligned} \partial_1 h^{(2)}(s, s'') &= \partial_1 h(s, s'(s, s'')) = -\sin \varphi, \\ \partial_2 h^{(2)}(s, s'') &= \partial_2 h(s'(s, s''), s'') = \sin \varphi'', \end{aligned}$$

letting  $\varphi = \varphi(s, s'')$ , resp.  $-\varphi'' = -\varphi''(s, s'')$  be the angle between the normal at  $s$ , resp.  $s''$ , with the segment connecting  $s$  to  $s'$ , resp.  $s''$  to  $s'$ .

Recall the formula of the differential  $(\delta s, \delta \varphi) \mapsto (\delta s', \delta \varphi')$  of the billiard map in  $(s, \varphi)$ -coordinates (see Chernov-Markarian [CM, p. 35]): at a point  $(s, \varphi)$ , it holds

$$\begin{pmatrix} \delta s' \\ \delta \varphi' \end{pmatrix} = -\frac{1}{\cos \varphi'} \begin{pmatrix} \mathcal{L}\mathcal{K} + \cos \varphi & \mathcal{L} \\ \mathcal{L}\mathcal{K}\mathcal{K}' + \mathcal{K} \cos \varphi' + \mathcal{K}' \cos \varphi & \mathcal{L}\mathcal{K}' + \cos \varphi' \end{pmatrix} \begin{pmatrix} \delta s \\ \delta \varphi \end{pmatrix},$$

<sup>14</sup> See the definitions of  $h_\infty$  and  $\xi_\infty$  in Section 3.

<sup>15</sup> Of course this is a particular case of Theorem A.1.

letting  $(s', \varphi')$  be the image of  $(s, \varphi)$  by the billiard map,  $\mathcal{L} := h(s, s')$ , and  $\mathcal{K}, \mathcal{K}'$  be the respective curvatures at the points  $s, s'$ . In particular, the differential of the square of the billiard map is equal to

$$(5.11) \quad \begin{pmatrix} \delta s \\ \delta \varphi \end{pmatrix} \mapsto \begin{pmatrix} \delta s'' \\ \delta \varphi'' \end{pmatrix} = \frac{1}{\cos \varphi' \cos \varphi''} \begin{pmatrix} a & b \\ * & * \end{pmatrix} \begin{pmatrix} \delta s \\ \delta \varphi \end{pmatrix},$$

with

$$\begin{aligned} a &:= 2\mathcal{L}\mathcal{L}'\mathcal{K}\mathcal{K}' + (\mathcal{L} + \mathcal{L}')\mathcal{K} \cos \varphi' + 2\mathcal{L}'\mathcal{K}' \cos \varphi + \cos \varphi \cos \varphi', \\ b &:= 2\mathcal{L}\mathcal{L}'\mathcal{K}' + (\mathcal{L} + \mathcal{L}') \cos \varphi', \end{aligned}$$

letting  $(s'', \varphi'')$  be the second iterate of  $(s, \varphi)$  under the billiard map,  $\mathcal{L}'' := h(s', s'')$ , and  $\mathcal{K}''$  be the curvature at the point  $s''$ .

In (5.10), the angle  $\varphi = \varphi(s, s'')$  is seen as a function of  $(s, s'')$ . In order to compute  $\partial_1 \varphi(s, s'')$ , we set  $\delta s'' = 0$  in (5.11), which gives  $a\delta s + b\delta \varphi = 0$ , with  $\delta \varphi = \partial_1 \varphi(s, s'')\delta s$ , so that

$$(5.12) \quad \partial_1 \varphi(s, s'') = -\frac{a}{b} = -\mathcal{K} - \frac{2\mathcal{L}'\mathcal{K}' \cos \varphi + \cos \varphi \cos \varphi'}{2\mathcal{L}\mathcal{L}'\mathcal{K}' + (\mathcal{L} + \mathcal{L}') \cos \varphi'}.$$

Similarly, to compute  $\partial_2 \varphi(s, s'')$ , we set  $\delta s = 0$  in (5.11), which gives  $b\delta \varphi = \cos \varphi' \cos \varphi'' \delta s''$ , with  $\delta \varphi = \partial_2 \varphi(s, s'')\delta s''$ , so that

$$(5.13) \quad \partial_2 \varphi(s, s'') = \frac{\cos \varphi' \cos \varphi''}{b} = \frac{\cos \varphi' \cos \varphi''}{2\mathcal{L}\mathcal{L}'\mathcal{K}' + (\mathcal{L} + \mathcal{L}') \cos \varphi'}.$$

Differentiating (5.10) with respect to  $s$ , and thanks to (5.12), we deduce that

$$\partial_{11} h^{(2)}(s, s'') = -\cos \varphi \cdot \partial_1 \varphi(s, s'') = \mathcal{K} \cos \varphi + \frac{2\mathcal{L}'\mathcal{K}' \cos^2 \varphi + \cos^2 \varphi \cos \varphi'}{2\mathcal{L}\mathcal{L}'\mathcal{K}' + (\mathcal{L} + \mathcal{L}') \cos \varphi'}.$$

Similarly, differentiating (5.10) with respect to  $s''$ , and by (5.13), we get

$$\partial_{21} h^{(2)}(s, s'') = -\cos \varphi \cdot \partial_2 \varphi(s, s'') = -\frac{\cos \varphi \cos \varphi' \cos \varphi''}{2\mathcal{L}\mathcal{L}'\mathcal{K}' + (\mathcal{L} + \mathcal{L}') \cos \varphi'}.$$

Moreover, by the time reversal symmetry  $(s, \varphi) \mapsto (s, -\varphi)$ , we have  $\partial_{22} h^{(2)}(s, s'') = \partial_{11} h^{(2)}(s'', s)$ , with  $s'(s'', s) = s'(s, s'')$ , and the orbit segment  $(s, \varphi) \mapsto (s', \varphi') \mapsto (s'', \varphi'')$  corresponds to the orbit segment  $(s'', -\varphi'') \mapsto (s', -\varphi') \mapsto (s, -\varphi)$ , so that

$$\partial_{22} h^{(2)}(s, s'') = \mathcal{K}'' \cos \varphi'' + \frac{2\mathcal{L}\mathcal{K}' \cos^2 \varphi'' + \cos^2 \varphi'' \cos \varphi'}{2\mathcal{L}\mathcal{L}'\mathcal{K}' + (\mathcal{L} + \mathcal{L}') \cos \varphi'}.$$

At the point  $(s, s'') = (0, 0)$ , we have  $\varphi = \varphi' = \varphi'' = 0$ ,  $\mathcal{L} = \mathcal{L}' = \frac{\mathcal{L}(12)}{2}$  is the length of the two-periodic orbit (12), and  $\mathcal{K} = \mathcal{K}' = \mathcal{K}''$  is the common curvature at the bouncing points, hence the matrix  $A$  of the Hessian  $d^2 h_{(0,0)}^{(2)}$  is equal to

$$(5.14) \quad A = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} := \begin{pmatrix} \frac{2(\mathcal{L}\mathcal{K}+1)^2-1}{2\mathcal{L}(\mathcal{L}\mathcal{K}+1)} & -\frac{1}{2\mathcal{L}(\mathcal{L}\mathcal{K}+1)} \\ -\frac{1}{2\mathcal{L}(\mathcal{L}\mathcal{K}+1)} & \frac{2(\mathcal{L}\mathcal{K}+1)^2-1}{2\mathcal{L}(\mathcal{L}\mathcal{K}+1)} \end{pmatrix}.$$

Besides, the change of coordinates  $\psi: (\xi, \eta) \mapsto (s, s'')$  is the composition  $\psi = Q^{-1} \circ R^{-1}$ , where  $R: (s, r) = (s, \sin \varphi) \mapsto (\xi, \eta)$  is the map to go to Birkhoff coordinates,

and  $Q: (s, s'') \mapsto (s, \sin \varphi(s, s''))$ . By (5.12)-(5.13), the matrix of  $DQ_{(0,0)}^{-1}$  is

$$(5.15) \quad DQ_{(0,0)}^{-1} = -\frac{1}{\beta} \begin{pmatrix} -\beta & 0 \\ \alpha & 1 \end{pmatrix}.$$

Moreover,  $DR_{(0,0)}^{-1}$  maps  $(1, 0)$  to a vector  $v_s = (v_1, -v_2)$  in the stable space of  $D\mathcal{F}_{(0,0)}^2$  at  $(0, 0)$ , and maps  $(0, 1)$  to a vector  $v_u$  in the unstable space of  $D\mathcal{F}_{(0,0)}^2$ ; since  $R$  maps  $\{\varphi = 0\}$  to  $\{\xi = \eta\}$ , we also have  $v_u = (v_1, v_2)$ , and  $2v_1v_2 = \det(v_s, v_u) = 1$ , as the change of coordinates  $R$  is symplectic. Moreover, due to the symmetry at the two-periodic orbit (12), the matrix of  $D\mathcal{F}_{(0,0)}^2$  is equal to  $B^2$ , with

$$B := \begin{pmatrix} \mathcal{L}\mathcal{K} + 1 & \mathcal{L} \\ \mathcal{L}\mathcal{K}^2 + 2\mathcal{K} & \mathcal{L}\mathcal{K} + 1 \end{pmatrix}.$$

After computations, the matrix of  $DR_{(0,0)}^{-1}$  is

$$(5.16) \quad DR_{(0,0)}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \theta^{-1} & \theta^{-1} \\ -\theta & \theta \end{pmatrix}, \quad \theta^2 := \frac{\sqrt{(\mathcal{L}\mathcal{K} + 1)^2 - 1}}{\mathcal{L}}.$$

By (5.9), and as  $P = D\psi_{(0,0)} = DQ_{(0,0)}^{-1} \circ DR_{(0,0)}^{-1}$ , with the expressions of  $A$ ,  $DQ_{(0,0)}^{-1}$ ,  $DR_{(0,0)}^{-1}$  obtained in (5.14)-(5.15)-(5.16), we deduce that

$$\begin{aligned} \text{tr}(d^2S_{(0,0)}) &= \text{tr}(P^T AP) = \frac{1}{\beta^2} \text{tr} \left( \begin{pmatrix} \alpha(\alpha^2 - \beta^2) & \alpha^2 - \beta^2 \\ \alpha^2 - \beta^2 & \alpha \end{pmatrix} \begin{pmatrix} \theta^{-2} & 0 \\ 0 & \theta^2 \end{pmatrix} \right) \\ &= \frac{\alpha(\alpha^2 - \beta^2)\theta^{-2} + \alpha\theta^2}{\beta^2} = 4\tau(2\tau^2 - 1)\sqrt{\tau^2 - 1}, \end{aligned}$$

with  $\tau := \frac{1}{2}\text{tr}(B) = \mathcal{L}\mathcal{K} + 1$ . By definition,  $\mathcal{L} = \frac{\mathcal{L}(12)}{2}$  is  $\mathcal{MLS}$ -invariant; moreover, by Theorem A.5, the curvature  $\mathcal{K}$  at the bouncing points of the orbit (12) is also  $\mathcal{MLS}$ -invariant, hence, by the above calculation,  $\text{tr}(d^2S_{(0,0)})$  is  $\mathcal{MLS}$ -invariant. Since we observed at the beginning of the proof that the quantity  $\xi_\infty^2 \text{tr}(d^2S_{(0,0)})$  is also  $\mathcal{MLS}$ -invariant, we conclude that  $\xi_\infty$  is  $\mathcal{MLS}$ -invariant.  $\square$

**Remark 5.4.** Note that the parameter  $\xi_\infty$  can be interpreted in terms of some area in parameter space. Indeed, let us consider the area  $\mathcal{A}_n$  of the quadrilateral of the  $(s, r)$ -plane bounded by the stable/unstable manifolds of the fixed point  $(s(2, 1), 0)$  and the stable/unstable manifolds of the point  $(s_n(n+1), 0)$ . Since the change of coordinates  $R$  is symplectic, this area is equal to

$$(5.17) \quad \mathcal{A}_n = \frac{1}{2} \tan\left(\frac{\theta}{2}\right) s_n(n+1)^2 + o(\lambda^n) = \lambda^n \xi_\infty^2 + o(\lambda^n),$$

where  $\theta \in (0, \pi)$  is the angle between the stable/unstable subspaces at  $(s(2, 1), 0)$ .

Actually, this remark gives another way to see that  $\xi_\infty$  is a  $\mathcal{MLS}$ -invariant; indeed, as in the work of Otal (see e.g. the proof of [O, Théorème 2]), it can be shown that the area  $\mathcal{A}_n$  is a  $\mathcal{MLS}$ -invariant, and since  $\lambda$  is also a  $\mathcal{MLS}$ -invariant, so is  $\xi_\infty$ , by (5.17). Moreover (although we will not need it for the present paper, where the analysis is done for the periodic orbit (12)), the same construction can be done to show that generically, the Birkhoff Normal Form of any periodic point is a  $\mathcal{MLS}$ -invariant: as for the orbit (12), we consider a sequence of periodic orbits  $(\hat{h}_n)_{n \geq 0}$  in

the horseshoe generated by some homoclinic intersection between the invariant manifolds of the periodic point (see Subsection 4.4 for more details). Then, the analogue of Corollary 4.22 shows that if the first Birkhoff coefficient does not vanish, then in fact, all the Birkhoff coefficients are  $\mathcal{MLS}$ -invariants, up to a homoclinic parameter  $\hat{\xi}_\infty$ ; moreover, by the previous remark,  $\hat{\xi}_\infty$  can be expressed as an area which can also be computed from  $\mathcal{MLS}$ .

The above result allows us to conclude:

**Corollary 5.5.** *Let  $\mathcal{D} \in \mathbf{B}_{\text{sym}}$ , and let  $\mathcal{F} = \mathcal{F}(\mathcal{D})$  be the associated billiard map. We consider the 2-periodic orbit (12). Let  $N = N(\mathcal{D}, 1, 2): (\xi, \eta) \mapsto (\Delta(\xi\eta)\xi, \Delta(\xi\eta)^{-1}\eta)$  be the Birkhoff Normal Form of  $\mathcal{F}^2$  associated to the orbit (12), with  $\Delta = \Delta(\mathcal{D}, 1, 2): z \mapsto \lambda + \sum_{\ell=1}^{+\infty} a_\ell z^\ell$ . If  $a_1 \neq 0$ , then*

- the Birkhoff Normal Form  $N$  is a  $\mathcal{MLS}$ -invariant;
- the differential of the gluing map  $\mathcal{G}$  at any point  $(\xi, \eta) \in \Gamma_\infty$  is also a  $\mathcal{MLS}$ -invariant, where  $\mathcal{G} = \mathcal{G}(\mathcal{D}, 1, 2)$  and  $\Gamma_\infty = \Gamma_\infty(\mathcal{D}, 1, 2)$  are taken as in Subsection 4.1.

*Proof.* Recall that by Corollary 4.22, the parameters  $\{\bar{a}_\ell, \bar{\gamma}_\ell, \bar{g}_\ell\}_{\ell \geq 0}$  are  $\mathcal{MLS}$ -invariants, provided that  $\bar{a}_1 \neq 0$ ; by the above corollary (recall (4.9)) we thus conclude that  $\{a_\ell, \gamma_\ell, g_\ell\}_{\ell \geq 0}$  are  $\mathcal{MLS}$ -invariants, as well as the expressions of  $\Delta: z \mapsto \lambda + \sum_{\ell=1}^{+\infty} a_\ell z^\ell$  and of the Birkhoff Normal Form  $N = R_0 \mathcal{F}^2 R_0^{-1}$  in a neighborhood of  $(s(2, 1), 0)$ , but also of  $\gamma$  and  $g = \partial_2 G^-(\cdot, \gamma(\cdot))$ . By (4.4), we deduce that the differential  $D\mathcal{G}_{(\xi, \eta)}$  of the gluing map  $\mathcal{G} = R \circ \mathcal{F}^2 \circ R^{-1}|_{\Omega_\infty}$  at any point  $(\xi, \eta) \in \Gamma_\infty$  is also a  $\mathcal{MLS}$ -invariant, where  $\mathcal{G}, \Omega_\infty$  and  $\Gamma_\infty$  are taken as in Subsection 4.1.  $\square$

## 6. RECONSTRUCTING THE GEOMETRY FROM THE MARKED LENGTH SPECTRUM

In this section, we assume that the billiard table  $\mathcal{D}$  has additional symmetries, i.e.,  $\mathcal{D} \in \mathbf{B}_{\text{sym}}$ . It follows from the previous part that if  $T := \mathcal{F}^2: (s, r) \mapsto (s'', r'')$  denotes the square of the billiard map  $\mathcal{F} = \mathcal{F}(\mathcal{D}): (s, r) \mapsto (s', r')$ , then under some twist condition, the Birkhoff Normal Form  $N = N(\mathcal{D}, 1, 2) = RTR^{-1}$  of  $T$  in a neighborhood of  $(s(2, 1), 0)$  is completely determined by the Marked Length Spectrum  $\mathcal{MLS}(\mathcal{D})$ , assuming that  $s(2, 1)$  is the arc-length parameter of the point of  $\mathcal{M}_2$  in the 2-periodic orbit (12).

In this part, our goal is to see which information on the geometry of the billiard table  $\mathcal{D}$  can be reconstructed, and conclude the proof of our Main Theorem:

**Theorem 6.1.** *For an open and dense set of billiard tables  $\mathcal{D} \in \mathbf{B}_{\text{sym}}$ , the Marked Length Spectrum  $\mathcal{MLS}(\mathcal{D})$  determines completely the geometry of  $\mathcal{D}$ .*

Fix a billiard table  $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^3 \mathcal{O}_i \in \mathbf{B}_{\text{sym}}$ , and let  $\mathcal{F} := \mathcal{F}(\mathcal{D})$  be the associated billiard map. After possibly applying some isometry, we assume that in the plane with  $(\bar{x}, \bar{y})$ -coordinates, the trace of the point in the 2-periodic orbit (12) which is on the first obstacle, resp. second obstacle, has coordinates  $(-\frac{1}{2}\ell, 0)$ , resp.  $(\frac{1}{2}\ell, 0)$ , where  $\ell = \ell(\mathcal{D}) := \frac{1}{2}\mathcal{L}(12)$  is the half-length of the orbit (12). In particular, the axis of symmetry is the vertical axis  $\{\bar{x} = 0\}$ .

**6.1. A construction for symmetric billiard tables.** We consider the following construction. If  $1, 2, 3$  are the labels of the three obstacles of  $\mathcal{D}$ , we define a new billiard table  $\mathcal{D}^* = \mathcal{D}^*(\mathcal{D})$  formed by three obstacles  $1^*, 2^*, 3^*$  (see Figure 6)

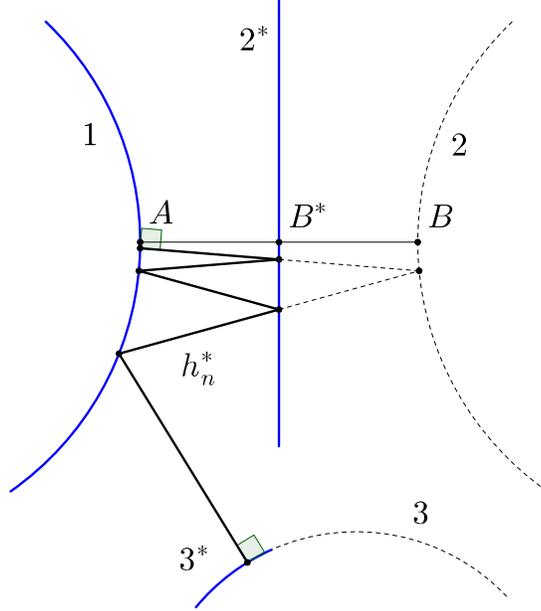


FIGURE 6. Defining a new table using the  $\mathbb{Z}_2$ -symmetry of the pair  $\{\mathcal{O}_1, \mathcal{O}_2\}$ .

- we consider a half-plane  $\{\bar{x} \leq 0\}$  or  $\{\bar{x} \geq 0\}$  such that it contains the trace of at least one of the two points  $x_\infty^{(j)}(0) \in \{r = 0\}$ ,  $j \in \{1, 2\}$ , in the respective homoclinic orbits  $h_\infty^{(1)} = (\dots 212131212 \dots)$  and  $h_\infty^{(2)} = (\dots 121232121 \dots)$ ; in the following we assume that this point is  $x_\infty(0) = x_\infty^{(1)}(0)$  and that its trace is in the half-plane  $\{\bar{x} \leq 0\}$ ;
- in this case, we let the obstacle with label  $1^*$  in  $\mathcal{D}^*$  be the same as the one of  $\mathcal{D}$  with label 1;
- we define the obstacle  $2^*$  as the vertical line segment  $\{0\} \times [-\ell^*, \ell^*]$  for some  $\ell^* > 0$  such that  $\{0\} \times [-\ell^*, \ell^*]$  does not cross the third obstacle, and the intersection of  $\{0\} \times (-\ell^*, \ell^*)$  and of the line segment between the points of parameters  $x_\infty(1) = \mathcal{F}(x_\infty(0))$  and  $x_\infty(2) = \mathcal{F}^2(x_\infty(0))$  is non-empty; we parametrize this line segment in arc-length in such a way that the image of the point in the 2-periodic orbit  $(1^*2^*)$  is associated to the parameter 0;
- we let the obstacle  $3^*$  be some small arc in the obstacle 3 such that it is in the half-plane  $\{\bar{x} \leq 0\}$  and contains a neighborhood of the point with coordinates  $x_\infty(0)$ .

By construction, the billiard table  $\mathcal{D}^*$  satisfies the non-eclipse condition and is of the same type as  $\mathcal{D}$ , except that the obstacle  $2^*$  is now flat. Moreover, by Proposition 2.1, for any sufficiently large integer  $n$ , each palindromic orbit

$h_n = h_n^{(1)} = (31 \underbrace{2121 \dots 21}_{2n})$  as above in  $\mathcal{D}$  shadows either  $h_\infty = h_\infty^{(1)}$  or its image under  $\mathcal{I}: (s, r) \mapsto (s, -r)$ , and can thus be associated to a periodic orbit in  $\mathcal{D}^*$ , denoted by  $h_n^*$ , which is defined as follows:

- we start at the image of the point  $x_n(0) \in \{r = 0\}$  of  $h_n$ ; it is close to the image of  $x_\infty(0)$  so it is indeed on the obstacle  $3^*$ ;
- the trace of the first orbit's segment is the same as for  $h_n$ ;
- the trace of the second orbit's segment is the first part of the trace of the second segment of  $h_n$  which is contained in the half-plane  $\{\bar{x} \leq 0\}$ ;
- the trace of the third orbit's segment is the second part of the second segment of  $h_n$ , which is contained in  $\{\bar{x} \geq 0\}$ ; it is also the continuation of the second segment of  $h_n^*$  under the billiard flow of  $\mathcal{D}^*$ , after it gets reflected on  $2^*$  according to the usual law of reflection of angles;
- we repeat this folding procedure along the trajectory each time we hit the axis  $\{\bar{x} = 0\}$  until we reach the image of the point  $x_n(2n + 2)$ , so that the trace of the orbit of  $h_n^*$  is contained in  $\{\bar{x} \geq 0\}$ .

The points of  $h_n^*$  which are on the boundary  $\partial\mathcal{D}^*$  of the new table still define an orbit under the dynamics of the associated billiard map  $\mathcal{F}^* = \mathcal{F}(\mathcal{D}^*): (s, r) \mapsto (s', r')$ , with the same length  $\mathcal{L}(h_n^*) = \mathcal{L}(h_n)$  as the original orbit  $h_n$ . The symbolic coding of  $h_n^*$  is

$$h_n^* = (3^*1^* \underbrace{(2^*1^*)(2^*1^*)(2^*1^*)(2^*1^*) \dots (2^*1^*)(2^*1^*)}_{2 \times 2n = 4n}),$$

where each word  $(2^*1^*)$  with even index replaces a 1 and each word  $(2^*1^*)$  with odd index replaces a 2 in the previous coding. Formally, we obtain

**Lemma 6.2.** *The maps  $\mathbf{B}_{\text{sym}} \ni \mathcal{D} \mapsto \mathcal{D}^*(\mathcal{D})$  and  $h_n \mapsto h_n^*$  satisfy the following properties:*

- there exists an integer  $m_0 \geq 0$  such that the subset of palindromic orbits  $(h_{2m-1})_{m \geq m_0}$  of  $\mathcal{F}$  embeds into the set of palindromic orbits of  $\mathcal{F}^*$  by the map  $h_n \mapsto h_n^*$  defined above;
- for each  $n = 2m - 1$ ,  $m \geq m_0$ , we have  $\mathcal{L}(h_n) = \mathcal{L}(h_n^*)$ ;
- for each  $n = 2m - 1$ ,  $m \geq m_0$ , we have  $\text{LE}(h_n) = \text{LE}(h_n^*)$ .

*Proof.* The fact that each  $h_n^*$  is palindromic follows from the preservation of angles under reflections. It remains to show the third point about Lyapunov exponents. Indeed, let  $x = (s, r) \in \mathcal{M}$  be a point of the orbit  $h_n^*$  associated to a bounce on the obstacle  $1^*$ , and set  $(s^*, r^*) := \mathcal{F}^*(s, r)$ ,  $(s', r') := (\mathcal{F}^*)^2(s, r)$ . We also denote by  $\mathcal{L}_1^* := h(s, s^*)$ ,  $\mathcal{L}_2^* := h(s^*, s')$  the respective distances between the points of collision, and let  $\mathcal{K} := \mathcal{K}(s)$ ,  $\mathcal{K}' := \mathcal{K}(s')$  be the respective curvatures at  $s, s'$ .<sup>16</sup> Set  $\nu := \sqrt{1 - r^2}$ ,  $\nu^* := \sqrt{1 - (r^*)^2}$ , and  $\nu' := \sqrt{1 - (r')^2}$ . It follows from (1.2) that

$$D(\mathcal{F}^*)_x^2 = \begin{pmatrix} \frac{1}{\nu}((\mathcal{L}_1^* + \mathcal{L}_2^*)\mathcal{K} + \nu) & \frac{\mathcal{L}_1^* + \mathcal{L}_2^*}{\nu\nu'} \\ (\mathcal{L}_1^* + \mathcal{L}_2^*)\mathcal{K}\mathcal{K}' + \mathcal{K}\nu' + \mathcal{K}'\nu & \frac{1}{\nu}((\mathcal{L}_1^* + \mathcal{L}_2^*)\mathcal{K}' + \nu') \end{pmatrix}.$$

Note that  $\mathcal{L} := \mathcal{L}_1^* + \mathcal{L}_2^*$  is equal to the distance between the associated bounces on the initial billiard table  $\mathcal{D}$ , and then, the matrix of  $D(\mathcal{F}^*)_x^2$  is equal to the matrix of  $-D\mathcal{F}_x$ . Therefore, each new collision created by the introduction of the auxiliary

<sup>16</sup> Note that the curvature at  $s^*$  vanishes, as this bounce is on the flat piece  $2^*$ .

obstacle  $2^*$  at a shorter distance does not affect the differential, nor the Lyapunov exponent, as  $D_x(\mathcal{F}^*)^{4n+2} = D_x\mathcal{F}^{2n+2}$ .  $\square$

As a consequence of the previous observations, we obtain:

**Proposition 6.3.** *We consider a billiard table  $\mathcal{D} \in \mathbf{B}_{\text{sym}}$ , with Marked Length Spectrum  $\mathcal{MLS}(\mathcal{D})$ . We let  $\mathcal{D}^* = \mathcal{D}^*(\mathcal{D})$  be the billiard table defined above and denote by  $\mathcal{F}^* = \mathcal{F}^*(\mathcal{D}^*)$  the associated billiard map. Let  $N^* = N^*(\mathcal{D}^*, 1, 2): (\xi, \eta) \mapsto (\Delta^*(\xi\eta)\xi, \Delta^*(\xi\eta)^{-1}\eta)$  be the Birkhoff Normal Form of  $T^* := (\mathcal{F}^*)^2: (s, r) \mapsto (s'', r'')$  in a neighborhood of the point  $(0_{1^*}, 0)$  in the period two orbit  $(1^*2^*)$  which is on the first obstacle  $1^*$ , with  $\Delta^*: z \mapsto \sum_{j=0}^{+\infty} a_j z^j$ . If  $a_1 \neq 0$ , then  $\mathcal{MLS}(\mathcal{D})$  determines the Birkhoff Normal Form  $N^*$ .*

*Proof.* There is a new 2-periodic orbit  $(1^*2^*)$  for the map  $\mathcal{F}^*$  which bounces perpendicularly at the points with  $(\bar{x}, \bar{y})$ -coordinates  $(-\frac{1}{2}\ell, 0)$  and  $(0, 0)$ . Moreover, the image  $h_\infty^*$  in  $\mathcal{D}^*$  of the homoclinic trajectory  $h_\infty$  in  $\mathcal{D}$  can be defined following the same "folding" procedure as above. It is also homoclinic to  $(1^*2^*)$ , and similarly, it is accumulated by the orbits  $(h_n^*)_n$  defined above. The point  $(0_{1^*}, 0)$  with trace  $(-\frac{1}{2}\ell, 0)$  is a saddle fixed point for the dynamics of  $T^* = (\mathcal{F}^*)^2$ , hence we may consider the Birkhoff Normal Form  $N^*: (\xi, \eta) \mapsto (\Delta^*(\xi\eta)\xi, \Delta^*(\xi\eta)^{-1}\eta)$  of  $T^*$  in a neighborhood of this point. The orbits  $(h_n^*)_n$  are still palindromic, hence the analogue of Lemma 3.2 and Lemma 4.5 remains true in this case. We also note that the homoclinic parameter  $\xi_\infty$  is preserved by the unfolding construction. By Lemma 6.2, the Lyapunov exponent of each orbit  $h_n^*$  is  $\mathcal{MLS}(\mathcal{D})$ -invariant. Therefore, if  $a_1 \neq 0$ , then by the same method as in Lemmata 4.8, 4.10, 4.11, 4.16, 4.21, Corollary 4.22 and Corollary 5.3, we can recover the Birkhoff invariants of  $N^*$  by considering the series expansion of  $\text{LE}(h_n^*)$  with respect to  $n$ . As a result, the Birkhoff Normal Form  $N^*$  is entirely determined by the Marked Length Spectrum  $\mathcal{MLS}(\mathcal{D})$  of the initial table.  $\square$

Let  $(0_{2^*}, 0)$  be the  $(s, r)$ -coordinates of the point in the orbit  $(1^*2^*)$  whose trace is on the second obstacle  $2^*$ . The Birkhoff Normal Forms of  $(\mathcal{F}^*)^2$  at the two points  $(0_{1^*}, 0)$  and  $(0_{2^*}, 0)$  in the orbit  $(1^*2^*)$  coincide:

**Lemma 6.4.** *Let  $\mathcal{D} \in \mathbf{B}_{\text{sym}}$  and let  $\mathcal{F}^* = \mathcal{F}^*(\mathcal{D}^*)$ . The Birkhoff Normal Form of  $T^* = (\mathcal{F}^*)^2$  in a neighborhood of the point  $\mathcal{F}^*(0_{1^*}, 0) = (0_{2^*}, 0)$  coincides with the map  $N^* = N^*(\mathcal{D}^*, 1, 2)$  defined in Proposition 6.3.*

*Proof.* Let  $\mathcal{U}^* \subset \mathbb{R}^2$  be an open neighborhood of  $(0_{1^*}, 0)$ , and let  $R^*: \mathcal{U}^* \rightarrow \mathbb{R}^2$  be a conjugacy map such that  $R^*T^*|_{\mathcal{U}^*} = N^*R^*|_{\mathcal{U}^*}$ . Then  $\mathcal{F}^*(\mathcal{U}^*)$  is an open neighborhood of  $(0_{2^*}, 0)$ , since  $\mathcal{F}^*(0_{1^*}, 0) = (0_{2^*}, 0)$ . The map  $\tilde{R}^* := R^* \circ (\mathcal{F}^*)^{-1}$  is symplectic, and for any  $y = \mathcal{F}^*(x)$  with  $x \in \mathcal{U}^*$ , it holds

$\tilde{R}^* \circ T^*(y) = R^* \circ (\mathcal{F}^*)^{-1} \circ (\mathcal{F}^*)^2(\mathcal{F}^*(x)) = R^* \circ T^*(x) = N^* \circ R^*(x) = N^* \circ \tilde{R}^*(y)$ , which concludes, by uniqueness of the Birkhoff Normal Form.  $\square$

**6.2. Recovering the geometry of a symmetric billiard table.** In the following, we use the same notation as in the last part. Given  $\mathcal{D} \in \mathbf{B}_{\text{sym}}$ , then by definition, after rotation by an angle of  $-\frac{\pi}{2}$ , near the point  $(0, \frac{\ell}{2})$ , the first obstacle 1 (which is the same as the obstacle  $1^*$ ) can be represented as a graph

$$\mathcal{C} = \left\{ \left( t, \frac{\ell}{2} + \beta_2 t^2 + \beta_4 t^4 + \dots \right) : t \in I \right\},$$



*Proof.* Fix  $r > 0$ . For each integer  $j \geq 0$ , we define a map  $a_j: \mathbf{B}_{\text{sym}}(3, r) \rightarrow \mathbb{R}$  in such a way that for all  $\mathcal{D} \in \mathbf{B}_{\text{sym}}(3, r)$ , the  $j^{\text{th}}$  Birkhoff invariant of  $N^* = N^*(\mathcal{D}^*, 1, 2)$  is equal to  $a_j(\mathcal{D})$ , i.e.,  $N^*: (\xi, \eta) \mapsto (\Delta^*(\xi\eta)\xi, \Delta^*(\xi\eta)^{-1}\eta)$ , with  $\Delta^*: z \mapsto \sum_{j=0}^{+\infty} a_j(\mathcal{D})z^j$ . In particular, with the topology introduced in Definition 1.4, and by Cauchy's integral formula, the map  $a_j$  is continuous.

As for the Birkhoff Normal Form  $N(\mathcal{D}, 1, 2)$ , the coefficient  $\lambda := a_0(\mathcal{D})$  is related to the Lyapunov exponent  $\text{LE}(1^*2^*)$  and only depends on  $\mathcal{L}(12)$  and on the curvature  $\mathcal{K}$  at the bouncing points of the 2-periodic orbit (12).

Besides, by the construction of the Birkhoff Normal Form (see e.g. [Bi, Ste, Mos]), the first coefficient  $a_1 := a_1(\mathcal{D})$  of  $N^*$  is determined by the jet of order three of  $T^*$ . Together with Lemma 6.5, we thus have

$$a_1 = c_0^* \beta_4 + f_0^*(\ell, \beta_2),$$

for some constant  $c_0^* \neq 0$  and some continuous function  $f_0^*: \mathbb{R}^2 \rightarrow \mathbb{R}$ . Equivalently,  $\beta_2, \beta_4$  can be interpreted in terms of the curvature  $\mathcal{K}$  and its second derivative  $\mathcal{K}''$  at the bouncing points of the 2-periodic orbit (12),<sup>17</sup> as

$$\mathcal{K} = 2\beta_2, \quad \mathcal{K}'' = 24(\beta_4 - \beta_2^3),$$

which gives (6.1).

According to (6.1), it is therefore possible to make the first Birkhoff invariant  $a_1$  non-zero by modifying the shape of the obstacles  $\mathcal{O}_1, \mathcal{O}_2$  so as to change the value of  $\mathcal{K}''$ , but keeping  $\ell, \mathcal{K}$  fixed. Let us now reformulate it in terms of the topology on  $\mathbf{B}_{\text{sym}}(3, r)$  introduced in Definition 1.4. After possibly applying some isometry, we have  $\mathcal{O}_1 = \mathcal{O}(f)$ , with  $f \in C_r^\omega(\mathbb{T}, \mathbb{R}^2)$ ,  $\theta \mapsto \varrho(\theta)(\cos(\theta), \sin(\theta))$ , for some even<sup>18</sup> function  $\varrho \in C_r^\omega(\mathbb{T}, \mathbb{R})$ ,  $\theta \mapsto \sum_{j=0}^{+\infty} \hat{\varrho}_j e^{ij\theta}$ . The curvature  $\mathcal{K} = \mathcal{K}(0)$  and its second derivative  $\mathcal{K}'' = \mathcal{K}''(0)$  satisfy

$$\begin{aligned} \mathcal{K} &= \frac{1}{\varrho(0)} - \frac{\varrho''(0)}{\varrho^2(0)}, \\ \mathcal{K}'' &= -\frac{\varrho''(0)}{\varrho^2(0)} + \frac{3(\varrho''(0))^2}{\varrho^3(0)} + \frac{3(\varrho''(0))^3}{\varrho^4(0)} - \frac{\varrho''''(0)}{\varrho^2(0)}, \end{aligned}$$

with  $\varrho(0) = \sum_{j=0}^{+\infty} \hat{\varrho}_j$ ,  $\varrho''(0) = -\sum_{j=0}^{+\infty} j^2 \hat{\varrho}_j$ , and  $\varrho''''(0) = \sum_{j=0}^{+\infty} j^4 \hat{\varrho}_j$ . For any table  $\mathcal{D}$  such that  $a_1(\mathcal{D})$  vanishes, it is then sufficient to perturb the first Fourier coefficients of  $\varrho$  to get a new function  $\tilde{\varrho} \in C_r^\omega(\mathbb{T}, \mathbb{R})$  such that the associated table  $\tilde{\mathcal{D}}$  satisfies  $a_1(\tilde{\mathcal{D}}) \neq 0$ . In this way, we see that for the topology introduced in Definition 1.4, the condition  $a_1 \neq 0$  holds for a dense subset of  $\mathbf{B}_{\text{sym}}(3, r)$ , and clearly, this condition is also open, as the map  $a_1: \mathbf{B}_{\text{sym}}(3, r) \rightarrow \mathbb{R}$  is continuous.  $\square$

By Lemma 6.6, in order to prove Theorem 6.1, it is sufficient to show that the Marked Length Spectrum determines the geometry for the set of billiard tables in  $\mathbf{B}_{\text{sym}}$  such that the first invariant  $a_1$  is non-zero. In the following, we fix a table  $\mathcal{D} \in \mathbf{B}_{\text{sym}}$  satisfying the non-degeneracy condition  $(\star)$  and show that the geometry of  $\mathcal{D}$  is determined by the Marked Length Spectrum  $\mathcal{MLS} := \mathcal{MLS}(\mathcal{D})$ .

**Corollary 6.7.** *The coefficients  $(\beta_{2k})_{k \geq 1}$  of the graph  $\mathcal{C}$  are  $\mathcal{MLS}$ -invariants. Therefore, by analyticity, the geometry of  $\mathcal{O}_1, \mathcal{O}_2$  can be reconstructed from  $\mathcal{MLS}$ .*

<sup>17</sup> The first derivative of the curvature vanishes due to the symmetries of the table.

<sup>18</sup> Due to the  $\mathbb{Z}_2$ -symmetry of  $\mathcal{O}_1$ .

*Proof.* By  $(\star)$ , Proposition 6.3 and Lemma 6.4, the Birkhoff Normal Form  $N^*$  of the map  $T^*$  in a neighborhood of the point  $(0_{2^*}, 0)$  is determined by the Marked Length Spectrum  $\mathcal{MLS}$ . By the construction of the Normal Form given by Moser [Mos] (see in particular the equations (3.2), (3.3) and (3.4) on pp. 680–681), the Birkhoff invariants are determined inductively by the jet of  $T^*$ . More precisely, for each  $k_1 \geq 1$ , the coefficients  $(a_k(\mathcal{D}))_{1 \leq k \leq k_1}$  of  $N^*$  are related to the  $(2k_1 + 1)^{\text{th}}$  jet of  $T^*$  by some invertible triangular system, and thus, there is a one-to-one correspondence between the jets of  $T^*$  and  $N^*$  at the point  $(0, 0)$ . By the previous discussion, we deduce that the jet of  $T^*$  at the point  $(0_{2^*}, 0)$  is determined by  $\mathcal{MLS}$ . Now, by Lemma 6.5, the jet of  $T^*$  at the point  $(0_{2^*}, 0)$  is also in one-to-one correspondence with the coefficients  $(\beta_{2k})_{k \geq 1}$ . Therefore, the coefficients  $(\beta_{2k})_{k \geq 1}$  can be recovered from the Marked Length Spectrum, which concludes the proof.  $\square$

To conclude the proof of Theorem 6.1, it remains to show that the geometry of the third scatterer can also be recovered. While the auxiliary table  $\mathcal{D}^*$  and the associated Birkhoff Normal Form  $N^*$  were useful to determine the geometry of  $\mathcal{O}_1, \mathcal{O}_2$ , now, we focus again on the initial billiard table  $\mathcal{D}$ . We denote by  $\mathcal{F} = \mathcal{F}(\mathcal{D})$  and  $T := \mathcal{F}^2$  the billiard map of  $\mathcal{D}$  and its square, let  $N = N(\mathcal{D}, 1, 2)$  be the Birkhoff Normal Form of  $\mathcal{F}^2$  associated to the 2-periodic orbit (12), and assume that the first Birkhoff invariant of  $N$  is non-zero.

**Corollary 6.8.** *The geometry of  $\mathcal{O}_3$  can be reconstructed from  $\mathcal{MLS}$ .*

*Proof.* In the following, we use the notation introduced in Subsections 4.1–4.2. By Corollary 5.5, the Marked Length Spectrum  $\mathcal{MLS}$  determines the function  $\gamma$  and the differential  $D\mathcal{G}_{(\xi, \eta)}$  of the gluing map  $\mathcal{G} = R_- \circ T \circ R_+^{-1}|_{\Omega_\infty}$ , at any point  $(\xi, \eta) \in \Gamma_\infty$ . Restricted to  $\mathcal{O}_\infty := R_+^{-1}(\Omega_\infty)$ , resp.  $T(\mathcal{O}_\infty)$ , we have  $R_+ = R_{m_0}$ , resp.  $R_- = R_{-m_0}$ , for some integer  $m_0 \geq 1$ , where  $R_{\pm m_0} := N^{\pm m_0} R_0 T^{\mp m_0}$ , and  $R_0$  is the canonical conjugacy map given by Lemma 2.7. By Corollary 5.5, the map  $N$  is a  $\mathcal{MLS}$ -invariant. Similarly, the map  $R_0$  is also a  $\mathcal{MLS}$ -invariant, as it only depends on the obstacles  $\mathcal{O}_1, \mathcal{O}_2$  whose geometry is also known, by Corollary 6.7. Besides, in the definition of  $R_{\pm m_0}$ , we take iterates of  $T$  between the first two obstacles, so their expression is also known. In other words, restricted to  $\mathcal{O}_\infty$ , resp.  $T(\mathcal{O}_\infty)$ , the map  $R_+ = R_{m_0}$ , resp.  $R_- = R_{-m_0}$  is determined by  $\mathcal{MLS}$ , as it only depends on the obstacles  $\mathcal{O}_1, \mathcal{O}_2$  whose geometry is a  $\mathcal{MLS}$ -invariant. Since  $\gamma$  and  $R_+$  are  $\mathcal{MLS}$ -invariants, then the arc  $\mathcal{A}_\infty := R_+^{-1}(\Gamma_\infty) = \{R_+^{-1}(\eta, \xi_\infty + \gamma(\eta)) : |\eta| \text{ small}\}$  can be recovered. Moreover, we know the differential  $D\mathcal{G}_{(\xi, \eta)}$  at any point  $(\xi, \eta) \in \Gamma_\infty$ , with  $\mathcal{G} = R_- \circ T \circ R_+^{-1}|_{\Omega_\infty}$ , hence we can determine  $D\mathcal{F}_x^2$ , for any  $x = (s, r) = R^{-1}(\eta, \xi_\infty + \gamma(\eta)) \in \mathcal{A}_\infty$ , with  $|\eta|$  small. By definition, given any such point  $x$ , we have  $\mathcal{F}(x) = x' = (s', 0)$  for some parameter  $s' \in \mathbb{R}$ , and  $\mathcal{F}^2(x) = (s, -r)$ , by Lemma 4.1. Let us denote by  $\mathcal{K}(s), \mathcal{K}(s')$  the respective curvatures at the points  $x = (s, r), x' = (s', 0)$ , by  $\mathcal{L} := h(s, s')$  the length of the line segment between their traces on the table, and set  $\nu := \sqrt{1 - r^2}$ . By (1.2), we have

$$D\mathcal{F}_x^2 = \begin{pmatrix} \frac{2aa'}{\nu} - 1 & \frac{2a'\mathcal{L}}{\nu^2} \\ \frac{2a}{\mathcal{L}}(aa' - \nu) & \frac{2aa'}{\nu} - 1 \end{pmatrix},$$

where

$$a := \mathcal{L}\mathcal{K}(s) + \nu, \quad a' := \mathcal{L}\mathcal{K}(s') + 1.$$

The values of  $\mathcal{K}(s)$  and  $\nu$  are already known, thus by considering the first line in the expression of this differential, we deduce the value of  $aa'$  and  $a'\mathcal{L}$ , and then, of  $\mathcal{L}$  and  $\mathcal{K}(s')$ . In particular, the geometry of  $\mathcal{O}_3$  is completely determined: for any such  $x \in \mathcal{A}_\infty$ , we draw a line segment of length  $\mathcal{L}$  starting from the associated point in the table, such that the angle of this segment with the normal to  $\partial\mathcal{O}_2$  at this point is equal to  $\varphi = \arcsin(r)$ . Then, the endpoint belongs to  $\partial\mathcal{O}_3$ , and the curvature at this point is equal to  $\mathcal{K}(s')$ . In particular, we recover the geometry of an arc in  $\mathcal{O}_3$ , hence the third obstacle  $\mathcal{O}_3$  is entirely determined by  $\mathcal{MLS}$ , by analyticity.

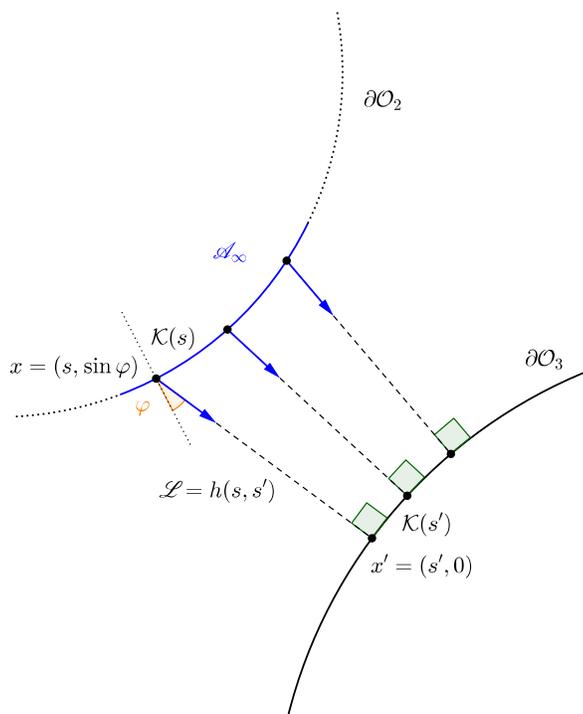


FIGURE 8. Reconstructing the geometry of the third obstacle  $\mathcal{O}_3$ .

□

**Remark 6.9.** We can also provide an alternative proof of the above result. Since  $N$  and  $\gamma$  are  $\mathcal{MLS}$ -invariants, then for any sufficiently large integer  $n = 2m - 1 \geq 0$ , we can compute the coordinates  $(\xi_n, \eta_n) = R_-(x_n(1))$  of the associated point in the orbit  $h_n$ . The conjugacy map  $R_-$  is entirely determined by the obstacles  $\mathcal{O}_1, \mathcal{O}_2$ , whose geometry is known, thus we can recover the coordinates of  $x_n(1) = (s_n(1), r_n(1))$ , and also of the other points in this orbit which are on  $\partial\mathcal{O}_1, \partial\mathcal{O}_2$ , i.e.,  $x_n(k)$ , for  $k = 2, \dots, 2n + 1$ . Let  $\mathcal{L}_n$  be the length of the line segment connecting the points of parameters  $x_n(0)$  and  $x_n(1)$ . Then  $\mathcal{L}_n$  is determined by  $\mathcal{MLS}$ : indeed, we know the total length  $\mathcal{L}(h_n)$ , as well as the length of the other orbit segments in  $h_n$ , as they are associated to the points  $x_n(k)$ , for  $k = 1, \dots, 2n + 1$ . Therefore, starting from the trace of the point  $x_n(1)$ , then the endpoint of the outward line segment of length  $\mathcal{L}_n$  based at this point and making an angle  $-\varphi_n(1) = -\arcsin(r_n(1))$  with the normal to  $\partial\mathcal{O}_2$  gives a point on  $\partial\mathcal{O}_3$ , associated to the parameter  $x_n(0)$ .

For different values of  $n$ , the corresponding points are pairwise distinct (they have different periods and all start perpendicularly to  $\partial\mathcal{O}_3$ ), and since they accumulate to the trace of the homoclinic point  $x_\infty(0)$  as  $n \rightarrow +\infty$ , this completely determines the geometry of  $\mathcal{O}_3$ , again by analyticity.

## 7. CONCLUSIONS AND FURTHER QUESTIONS

In this article we showed that for a generic class of chaotic billiards obtained by removing from the plane  $m \geq 3$  convex analytic scatterers with some symmetries, the Marked Length Spectrum determines the geometry of the billiard.

Our result leads to a number of natural questions:

**Question 1.** *Is it possible to remove the assumption about the mirror symmetry between  $\mathcal{O}_1$  and  $\mathcal{O}_2$ ?*

**Question 2.** *Is it possible to remove the assumption about each of the scatterers  $\mathcal{O}_1$  and  $\mathcal{O}_2$  being symmetric around the period-two collision?*

This would be a major step (similar to the one leading from [Z1] to [Z2, Z3]), and the techniques involving classical Birkhoff normal forms seem inadequate to this task.

**Question 3.** *Is it possible to obtain similar results for the unmarked Length Spectrum?*

As in [Z2, Z3], one could ask if simply marking the length of the 2-periodic orbit, of the associated Lyapunov exponent and possibly requiring some non-degeneracy of the Spectrum for those values, would suffice to recover the coefficients that describe the dynamics in the Birkhoff coordinates. It is unclear to us if such an approach could be carried out successfully, since it is not easy to distinguish between the many homoclinic orbits that accumulate on the periodic orbit, and our strategy hinges on a very fine asymptotic analysis of a specific family of approximating homoclinic orbits. We ultimately believe that such an approach should be possible, but for sure our strategy would have to be modified to deal with all such homoclinic orbits at the same time.

Let us also note that several quantities are length-spectral invariants. For instance, any periodic orbit of period at least three bounces on the three scatterers; as 2-periodic orbits correspond to minimizers of the distance between two scatterers, the two smallest elements in the Length Spectrum are the lengths of two 2-periodic orbits.

Recall that the topological entropy of the subshift of finite type described in the first part is equal to  $\log(m-1)$ , where  $m \geq 3$  is the number of obstacles. By [Mor2], this number is also a length-spectral invariant.

## APPENDIX A. THE MARKED LYAPUNOV SPECTRUM

In this appendix we collect some rather technical results that are needed for this paper. Stronger versions of such results have been stated in the paper [BDKL], but their proofs appear to be incorrect. In this work we only need results in the (much simpler) situation in which the scatterers are symmetric: we therefore state and prove such results here, in order for the arguments in this paper to be complete.

**Theorem A.1.** *The Lyapunov exponent of any palindromic periodic orbit in any billiard table in  $\mathbf{B}(m)$  is a  $\mathcal{MLS}$ -invariant.*

**Lemma A.2.** *Let  $(\bar{s}_0, \dots, \bar{s}_{p-1})$  denote the collision points of a palindromic periodic orbit of (least) period  $p = 2q$ , so that  $\bar{s}_i = \bar{s}_{p-i}$  for any  $0 < i < p$ . For  $n \geq 1$ , define  $L_n(s_0, s_1, \dots, s_n) = \sum_{j=0}^{n-1} h(s_j, s_{j+1})$ . Then*

$$L_q(s_0, \dots, s_q) - L_q(\bar{s}_0, \dots, \bar{s}_q) = Q_q(s_0 - \bar{s}_0, \dots, s_q - \bar{s}_q) + R_q,$$

where  $Q_q$  is a positive definite quadratic form and  $R_q$  is a remainder term that satisfies the estimate:

$$R_q = O(\|(s_0 - \bar{s}_0, \dots, s_q - \bar{s}_q)\|^3).$$

*Proof.* Since  $(\bar{s}_0, \dots, \bar{s}_{p-1})$  is palindromic,  $L_q(s_0, \dots, s_q)$  has a critical point (in fact, a minimum) at  $(\bar{s}_0, \dots, \bar{s}_q)$ ; recall in fact that the periodic orbit has necessarily orthogonal collisions at the points  $s_0$  and  $s_q$ . This amounts to say that  $\partial_j L_q(\bar{s}_0, \dots, \bar{s}_q) = 0$  for any  $j = 0, \dots, q$ . Hence the lemma follows from Taylor's formula, provided that we show that the Hessian of  $L_q$  at  $(\bar{s}_0, \dots, \bar{s}_q)$  is positive definite.

It is immediate from the definition of  $L_q$  that  $\partial_{ij} L_q = 0$  if  $|i-j| > 1$ ; in other terms, the Hessian of  $L_q$  is a tridiagonal (symmetric) matrix. For notational convenience, let  $h^{ij} = h(\bar{s}_i, \bar{s}_j)$ ; let  $\bar{\varphi}_j \in [-\frac{\pi}{2}, \frac{\pi}{2}]$  denote the angle formed by the outgoing trajectory at the  $j$ -th collision point and the unit normal vector to the domain at  $\bar{s}_j$ ; finally, let  $\mathcal{K}_j$  denote reciprocal of the radius of curvature at the point  $\bar{s}_j$ . Recall that by convention  $\mathcal{K}_j < 0$  for any  $j$ .

A direct computation shows that the diagonal terms are given by:

$$\begin{aligned} \partial_{00} L_q &= \frac{1}{h^{01}} \cos^2 \bar{\varphi}_0 - \mathcal{K}_0 \cos \bar{\varphi}_0 \\ \partial_{jj} L_q &= \left[ \frac{1}{h^{j-1j}} + \frac{1}{h^{jj+1}} \right] \cos^2 \bar{\varphi}_j - 2\mathcal{K}_j \cos \bar{\varphi}_j \text{ for } 0 < j < q \\ \partial_{qq} L_q &= \frac{1}{h^{q-1q}} \cos^2 \bar{\varphi}_q - \mathcal{K}_q \cos \bar{\varphi}_q, \end{aligned}$$

while the off-diagonal terms are given by:

$$\partial_{jj+1} L_q = \frac{1}{h^{jj+1}} \cos \bar{\varphi}_j \cos \bar{\varphi}_{j+1}.$$

Using the above expressions it is simple to prove the following lemma.

**Lemma A.3.** *For  $0 \leq n \leq q$ , let  $f_n$  denote the determinant of the  $(n+1) \times (n+1)$  top-left minor of  $\partial_{ij} L_q$ . Then  $f_n > 0$  for all  $0 \leq n \leq q$ .*

The above lemma implies that all eigenvalues of  $\partial_{ij} L_q$  are positive, which completes the proof of our result.  $\square$

*Proof of Lemma A.3.* We will in fact prove a slightly stronger statement which holds for  $0 \leq n < q$ , namely:

$$(A.1) \quad f_n > f_{n-1} \frac{\cos^2 \bar{\varphi}_n}{h^{nn+1}};$$

The proof will follow by induction. Since the matrix is tridiagonal, it is known that the determinants  $f_n$  can be expressed by the following recursive relation:

$$f_n = \partial_{nn}L_q f_{n-1} - (\partial_{n-1n}L_q)^2 f_{n-2},$$

with the convention  $f_{-1} = 1$  and  $f_{-2} = 0$ .

The base case is  $f_0 = \partial_{00}L_q$ , for which (A.1) immediately holds. Assuming by inductive hypothesis that (A.1) holds for  $n-1$ , let us prove it for  $n$ . The recursive relation yields, for  $n < q$ :

$$\begin{aligned} f_n - f_{n-1} \frac{\cos^2 \bar{\varphi}_n}{h^{nn+1}} &= f_{n-1} \left( \partial_{nn}L_q - \frac{\cos^2 \bar{\varphi}_n}{h^{nn+1}} \right) - (\partial_{n-1n}L_q)^2 f_{n-2} = \\ &= f_{n-1} \left( \frac{\cos^2 \bar{\varphi}_n}{h^{n-1n}} - 2\mathcal{K}_n \cos \bar{\varphi}_n \right) - \left( \frac{\cos \bar{\varphi}_{n-1} \cos \bar{\varphi}_n}{h^{n-1n}} \right)^2 f_{n-2} > \\ &> f_{n-2} \frac{\cos^2 \bar{\varphi}_{n-1} \cos^2 \bar{\varphi}_n}{h^{n-1n} h^{n-1n}} - 2f_{n-1} \mathcal{K}_n \cos \bar{\varphi}_n - \left( \frac{\cos \bar{\varphi}_{n-1} \cos \bar{\varphi}_n}{h^{n-1n}} \right)^2 f_{n-2} > \\ &> -2f_{n-1} \mathcal{K}_n \cos \bar{\varphi}_n > 0. \end{aligned}$$

This shows the statement for up to  $n = q-1$ ; an analogous computation then shows that also  $f_q > 0$ , which concludes the proof of our lemma.  $\square$

**Lemma A.4.** *Let  $\sigma = (\sigma_0 \cdots \sigma_{q-1})$  with  $\sigma_i = \sigma_{q-i}$  encode a palindromic periodic orbit of period  $q$  and Lyapunov exponent  $\text{LE}(\sigma) = -\frac{1}{q} \log \lambda$ ,  $\lambda = \lambda(\sigma)$  being the contracting eigenvalue of  $D\mathcal{F}^q$  at  $\sigma$ ; let  $\tau$  (of length  $p$ ) be so that both  $\sigma\tau$  and  $\tau\sigma$  are admissible. Let  $h_n(\sigma, \tau)$  denote the periodic orbit encoded by  $(\tau\sigma^n)$  of period  $p+nq$ . There exists  $C_0(\sigma, \tau)$  and  $C_1(\sigma, \tau)$  so that:*

$$(A.2) \quad \mathcal{L}(h_n(\sigma, \tau)) - n\mathcal{L}(\sigma) = C_0(\sigma, \tau) + C_1(\sigma, \tau)\lambda^n + o(\lambda^n).$$

*Proof.* The existence of  $C_0(\sigma, \tau)$  follows by the same arguments used in the definition of  $\mathcal{L}^\infty$  (see also Proposition 2.1). Obtaining the remaining terms follows step-by-step by the proof of Proposition 5.1, where it is proved in the special case  $\sigma = (12)$  and  $\tau = (32)$ , and obtains an explicit expression for the coefficient  $C_1((12), (32))$ , which we do not need in this lemma. It is omitted in the interest of keeping the length of this paper under control.  $\square$

*Proof of Proposition A.1.* Using Lemma A.4, and taking the limit of (A.2) for  $n \rightarrow \infty$ , we conclude that  $C_0(\sigma, \tau)$  is a  $\mathcal{MLS}$ -invariant, since the left hand side is spectrally determined. Hence, again by (A.2), we gather:

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \log |\mathcal{L}(h_n(\sigma, \tau)) - n\mathcal{L}(\sigma) - C_0(\sigma, \tau)|.$$

Since the right hand side is spectrally determined, we conclude that  $\lambda$  is  $\mathcal{MLS}$ -invariant. Similar considerations show that  $C_1(\sigma, \tau)$  is also  $\mathcal{MLS}$ -invariant.  $\square$

Proposition A.1 has the following immediate corollary:

**Theorem A.5.** *Let  $\mathcal{D} \in \mathbf{B}_{\text{sym}}(m)$ ; consider the 2-periodic orbit encoded by the word  $\sigma = (12)$  and let  $R = \mathcal{K}^{-1}$  be the (common) curvature radius at the collision points of the orbit. Then  $R$  is a  $\mathcal{MLS}$ -invariant.*

*Proof.* By Theorem A.1, the Lyapunov exponent  $\text{LE}(\sigma)$  of the 2-periodic orbit is a  $\mathcal{MLS}$ -invariant. Hence,  $\mathcal{MLS}(D)$  determines the eigenvalues of the linearization of

the square of the billiard map  $D\mathcal{F}^2$  at the collision points of the 2-periodic orbit; by (1.2) we have

$$D\mathcal{F}_{(0,0)}^2 = \begin{pmatrix} \mathcal{L}\mathcal{K} + 1 & \mathcal{L} \\ \mathcal{L}\mathcal{K}^2 + 2\mathcal{K} & \mathcal{L}\mathcal{K} + 1 \end{pmatrix}^2,$$

where  $\mathcal{L} := \frac{1}{2}\mathcal{L}((12))$ . In particular, we have that  $\lambda^{1/2} + \lambda^{-1/2} = 2(\mathcal{L}\mathcal{K} + 1)$ ; since it is of course spectrally determined, we conclude that  $R = \mathcal{K}^{-1}$  is  $\mathcal{MLS}$ -invariant.  $\square$

**Remark A.6.** *As a matter of fact, Theorem A.1 holds also for non-palindromic orbits. It must be noted that in the general case there is no analog of Lemma A.3, since there is no sub-orbit contained in a periodic orbit that is a length minimizer (otherwise we would have an orthogonal collision, which would force the orbit to be palindromic). Because of this, one needs to show an additional cancellation for  $\Sigma_n^1$  and  $\Sigma_n^2$ ; the proofs are marginally more involved but, in the interest of keeping this paper short but self-contained, we have only stated the result in the palindromic case.*

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