

ON THE RIGIDITY OF THE STABLE NORM AND MATHER'S β -FUNCTION FOR GEODESIC FLOWS

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ABSTRACT. We investigate rigidity phenomena associated to the stable norm and Mather's β -function for Riemannian geodesic flows on closed manifolds. Given two metrics g_1 and g_2 , we compare pointwise the corresponding β -functions. Our main result establishes that they satisfy an inequality, which becomes an equality at an homology class h if and only if the two metrics are homothetic on the Mather set of homology h . In the case of conformally equivalent metrics, this yields a pointwise criterion for homothety on the projected Mather set. Some consequences are discussed, including a pointwise rigidity result on the 2-torus implying that if a metric has the same Mather's β -function at some non-zero homology class as a normalized flat metric in the same conformal class, then the metric must be flat. This result can be considered a pointwise version of a similar global result by Bangert. Finally, an extension of these results to Mañé's perturbations of general Tonelli Lagrangians is discussed.

1. INTRODUCTION

The study of Riemannian geodesic flows reveals several fundamental objects that capture the asymptotic behavior of minimal geodesics and play a crucial role in understanding their structural and dynamical properties. Two key objects central to this work—the *stable norm* and the *Mather's β -function*—can be interpreted as manifestations of a homogenization effect at large scales.

Let (M, g) be a closed Riemannian manifold. We denote by $\|\cdot\|_x$ the associated norm in each fiber $T_x M$. Let us begin by recalling the definitions of stable norm and Mather's β -function and their relationship.

The *stable norm* provides a geometric way to measure real homology classes by capturing the large-scale, or homogenized, geometry of (M, g) : on a macroscopic level, the Riemannian metric “behaves” like a norm on the homology group of the manifold. It was introduced in the context of geometric measure theory by Federer [Fed75] and later became a staple of asymptotic Riemannian geometry through the work of Gromov [Gro81, Gro99] and others [BBI01]. More precisely, let $H_1(M; \mathbb{R})$ denote the first real homology group of M . The stable norm $\|\cdot\|_{g, \text{stable}} : H_1(M; \mathbb{R}) \rightarrow \mathbb{R}$ is a function

$$\|h\|_{g, \text{stable}} := \inf \left\{ \sum |r_i| \mathcal{L}_g(\gamma_i) \right\},$$

where the infimum is taken over all cycles $\sum r_i \gamma_i$ representing the homology class h , with γ_i being 1-simplices, and where $\mathcal{L}_g(\gamma_i)$ is the g -length of γ_i . Although the infimum may

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not be achieved in general, when the dimension of M is two, it is indeed attained for every rational homology class (see [CMP04, Proposition 2.1] or [BM08, Proposition 5.6]). This function indeed defines a norm on $H_1(M; \mathbb{R})$; we refer to [Gro81, Gro99, BB101] for a comprehensive treatment.

From a dynamical perspective, one can consider the geodesic flow on (M, g) . It is the Euler-Lagrange flow of the Lagrangian $L_g: TM \rightarrow \mathbb{R}$ given by

$$L_g(x, v) = \frac{1}{2}g_x(v, v). \quad (1)$$

Let $\mathcal{M}(TM)$ denote the set of Borel probability measures on TM that are *closed* and have finite first momentum. We recall that a Borel probability measure μ on TM is said to be closed if $\int_{TM} df_x(v) d\mu = 0$ for every $f \in C^1(M)$, while it has finite first momentum if $\int_{TM} \|v\|_x d\mu < \infty$. We endow this space with the topology for which $\lim_{n \rightarrow +\infty} \mu_n = \mu$ if and only if $\lim_{n \rightarrow \infty} \int_{TM} f d\mu_n = \int_{TM} f d\mu$ for any continuous function $f: TM \rightarrow \mathbb{R}$ with linear growth, i.e.,

$$\sup_{(x,v) \in TM} \frac{|f(x, v)|}{1 + \|v\|_x} < +\infty.$$

To any $\mu \in \mathcal{M}(TM)$, one can associate its *rotation vector* (or *Schwartzman asymptotic cycle*) $\rho(\mu) \in H_1(M; \mathbb{R})$. The resulting map $\rho: \mathcal{M}(TM) \rightarrow H_1(M; \mathbb{R})$ is surjective and continuous (see [Sor15, Proposition 3.2.2]). The *Mather's β -function* is then defined as:

$$\beta_g: H_1(M; \mathbb{R}) \longrightarrow \mathbb{R}$$

$$h \longmapsto \beta_g(h) := \min_{\mu \in \rho^{-1}(h)} \int_{TM} L_g(x, v) d\mu.$$

The probability measures achieving this minimum are invariant (see [Mn96, Theorem B]) and are called *Mather measures* (or *minimizing measures*) of rotation vector h . Their collection is denoted by $\mathfrak{M}_g^h$. The union of their supports defines an invariant set, called the *Mather set of homology class h* :

$$\widetilde{\mathcal{M}}_g^h = \bigcup_{\mu \in \mathfrak{M}_g^h} \text{supp}(\mu) \subset TM. \quad (2)$$

This set is compact and it is supported on a Lipschitz graph, namely the projection $\pi: TM \rightarrow M$ restricts to a bi-Lipschitz homeomorphism from $\widetilde{\mathcal{M}}_g^h$ onto its image $\pi(\widetilde{\mathcal{M}}_g^h) \subset M$ (see [Mat91, Theorem 2] or [Sor15, Theorem 3.2.7]). We call $\mathcal{M}_g^h := \pi(\widetilde{\mathcal{M}}_g^h)$ the *projected Mather set*.

A fundamental result, which bridges the above geometric and dynamical viewpoints, is the following relation [Mas96, Proposition 1.4.2] (see also [Mas97]):

$$\beta_g(h) = \frac{1}{2} \|h\|_{g, \text{stable}}^2 \quad \forall h \in H_1(M; \mathbb{R}). \quad (3)$$

Thus, the stable norm and Mather's β -function are dual manifestations of the same underlying asymptotic geometry.

Remark 1. Observe that if μ is a Mather measure for L_g with $\rho(\mu) = h \neq 0$, then

$$\begin{aligned}\beta_g(h) &= \frac{1}{2} \|h\|_{g,\text{stable}}^2 = \frac{1}{2} \int_{TM} g_x(v, v) d\mu \\ &\geq \frac{1}{2} \left(\int_{TM} \sqrt{g_x(v, v)} d\mu \right)^2 \geq \frac{1}{2} \|h\|_{g,\text{stable}}^2,\end{aligned}\tag{4}$$

where the last inequality follows from the stable norm seen as minimal total mass of closed measures in the class. Therefore, all inequalities in (4) must be equality, forcing the equality in the Cauchy-Schwarz inequality, yielding to $g_x(v, v)$ being constant and equal to $\|h\|_{g,\text{stable}}^2$ for μ -a.e. (x, v) . In particular, this implies that the support of μ does not intersect the zero section.

A natural question arises: *to what extent do the stable norm or Mather's β -function allow one to recover information on the underlying Riemannian metric g ?*

This question can be approached from different directions:

- (Q1) How do the local or global regularity properties of $\|\cdot\|_{g,\text{stable}}$ or β_g reflect the fine structure of the geodesic flow?
- (Q2) Can a metric or a class of metrics be identified from knowledge of their stable norms or Mather's β -functions?

The differentiability properties of Mather's β -function are subtle and, in some cases, intimately tied to the underlying dynamics of the geodesic flow and the structure of Mather sets. Fine differentiability properties of β_g have been thoroughly investigated in the case of closed surfaces, whereas the situation in higher dimensions presents greater complexity and defies simple characterization; we refer to [BM08, Mas96, Mas97, Mas03, Mas09] for several interesting results in this direction.

A very interesting answer to (Q1) was provided in [Ban94, Theorem 5.3] (see also [Ban88, Mat90, MS11] for related results). Recall that a homology class h is k -irrational if k is the dimension of the smallest subspace of $H_1(M, \mathbb{R})$ generated by integer classes and containing h .

Theorem 2 (Bangert). *When $M = \mathbb{T}^2$, β_g is differentiable at a 1-irrational homology class h if and only if the Mather set $\widetilde{\mathcal{M}}_g^h$ consists of an invariant torus foliated by periodic orbits.*

On the other hand, question (Q2) is more delicate, since the stable norm or Mather's β -function are built from *minimizing* geodesics and measures, hence they typically only capture information on the subset of the phase space traversed by such minimizing geodesics.

A positive result in this direction can be found in [Ban94, Theorem 6.1]:

Theorem 3 (Bangert). *Suppose a Riemannian metric g on the 2-torus $\mathbb{T}^2$ has the same stable norm or Mather's β -function as a flat metric g_0 on $\mathbb{T}^2$. Then, $(\mathbb{T}^2, g)$ and $(\mathbb{T}^2, g_0)$ are isometric by an isometry homotopic to the identity.*

The proof relies on Hopf's theorem, which states that a 2-torus without conjugate points is flat. In particular, the absence of conjugate points follows from the global differentiability of Mather's β -functions and the above mentioned consequence of the differentiability at 1-irrational homology classes.

A similar rigidity result was proved by [Osu10] on the n -dimensional torus, $n \geq 3$, but requires that not only the stable norm of the metric coincides with that of a flat one, but also its $(n - 1)$ -dimensional counterpart (i.e., the one obtained by a similar construction on $H_{n-1}(M; \mathbb{R})$).

Remark 4. We stress that without the assumption on the absence of conjugate points, the stable norm and Mather's β -function are very weak invariants. For example, on a Riemannian 2-torus “with a big bump” (see [Ban88, p. 46]), there exists an open set on which one can arbitrarily increase the metric without changing the stable norm. This construction can be generalized starting from any (M, g) satisfying

$$\text{cl}(\Sigma_g) \subsetneq M, \quad (5)$$

where

$$\Sigma_g := \cup_{h \in H_1(M; \mathbb{R})} \mathcal{M}_g^h. \quad (6)$$

One considers any smooth function $\lambda \geq 0$ on M , not identically zero, that has support in the complement of the closure of Σ_g . Then, one checks that the metrics $g_1 := e^\lambda g$ and g have the same Mather's β -functions.

Therefore, a more reasonable question is the following:

- (Q3)** For a closed Riemannian manifold (M, g) , does the stable norm or Mather's β -function *locally* determine the metric g on the Mather sets? For example, can specific features of the stable norm or Mather's β -function reveal whether two metrics are related on their Mather sets?

In the present paper, we explore question (Q3). More specifically, we conduct a pointwise comparison at individual homology classes. Analogous results for billiard systems are obtained in [BBS25]: there, the authors provide an isoperimetric-like inequality for Mather's β -function of the billiard map in a strongly convex domain, and deduce that when the equality is reached at certain rotation numbers (*i.e.*, *homology classes*), the domain must be disk (see [BBS25][Theorem 4.1]). Extensions to other billiard-like models are also discussed. See also [Fie26] for a recent follow-up, in which the author investigates Mather's β function associated to non-circular elliptic billiards, and proves that a similar result does not hold locally.

Inspired by these ideas, we investigate similar phenomena in the context of Riemannian geodesic flows and Tonelli Lagrangian systems in any dimension. This requires an adaptation of the methods, which involve properties of Mather's measures and Mather sets. Here, we obtain, among other things, a pointwise version of the above-mentioned Bangert's Theorem 3 (see Corollary 12). Moreover, in Section 3, an extension of these results to Mañé's perturbations of general Tonelli Lagrangians is discussed (see Theorem 14).

Remark 5. On the 2-torus, the β -function is closely related to the so-called *marked length spectrum*. In [GLP25], for Riemannian surfaces with Anosov geodesic flow, the authors show that the marked length spectrum determines the conformal class of the underlying complex structure in Teichmüller space (see [GLP25, Proposition 3.7]). By a result of Katok [Kat88], this in turn implies that the marked length spectrum actually determines the metric up to isometry.

By the uniformization theorem, every Riemannian metric on $\mathbb{T}^2$ is conformally equivalent to a flat metric. We may then ask whether the stable norm or Mather's β -function determines the class of the underlying complex structure in the Teichmüller space of $\mathbb{T}^2$. As noted in Remark 4, Mather's β -function is a weak invariant, e.g. in the presence of a “big bump”, so we generally expect the answer to be no. We can then ask a weaker question:

- (Q4) Given two metrics g_1 and g_2 on $\mathbb{T}^2$ with the same Mather's β -function, does there exist a diffeomorphism $\psi: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ isotopic to the identity such that $\psi^* g_2 = e^f g_1$ on $\text{cl}(\Sigma_{g_1})$ (see (6)), for some $f \in C^\infty(\mathbb{T}^2)$?

2. MAIN RESULTS FOR RIEMANNIAN GEODESIC FLOWS

Let M be a closed manifold and let g_1 and g_2 be two Riemannian metrics on M . First of all, we need to introduce some *normalization factor* in order to compare two different metrics and deal with the fact that we can multiply each of them by a constant factor, without changing the dynamical properties. We define

$$\mathcal{C}_{g_1}(g_2) := \max_{x \in M} \max_{\substack{v \in T_x M \\ g_{1,x}(v,v)=1}} g_{2,x}(v,v) = \max_{x \in M} \sup_{\substack{v \in T_x M \\ v \neq 0}} \left(\frac{g_{2,x}(v,v)}{g_{1,x}(v,v)} \right),$$

which can be interpreted as the maximal distortion of the g_2 -length of tangent vectors at x , with respect to the metric g_1 . If g_1 and g_2 are conformally equivalent metrics on M , i.e., $g_2 = \varphi \cdot g_1$ for some positive function $\varphi \in C^\infty(M)$, then it is easy to check that $\mathcal{C}_{g_1}(g_2) = \max_M \varphi$.

We can now state our Main result, which will imply, among other consequences, a pointwise version of Bangert's result. We present our result in terms of Mather's β -functions, but it can be equivalently rephrased in terms of the corresponding stable norms by means of (3).

Main Theorem. *Let M be a closed manifold and let g_1 and g_2 be two Riemannian metrics on M . Let $\beta_{g_i}: H_1(M; \mathbb{R}) \rightarrow \mathbb{R}$ be the Mather's β -function associated to g_i , $i = 1, 2$. Then:*

(i) *We have*

$$\beta_{g_2}(h) \leq \mathcal{C}_{g_1}(g_2) \beta_{g_1}(h), \quad \forall h \in H_1(M; \mathbb{R}). \quad (7)$$

(ii) *Let $h \in H_1(M; \mathbb{R})$. Then the following are equivalent:*

- $\beta_{g_2}(h) = \mathcal{C}_{g_1}(g_2) \beta_{g_1}(h)$;
- $\mathfrak{M}_{g_1}^h \subseteq \mathfrak{M}_{g_2}^h$ and $g_{2,x}(v,v) = \mathcal{C}_{g_1}(g_2) g_{1,x}(v,v)$, $\forall (x,v) \in \widetilde{\mathcal{M}}_{g_1}^h \subseteq \widetilde{\mathcal{M}}_{g_2}^h$.

(iii) Let us assume that g_1 and g_2 are conformally equivalent, i.e., $g_2 = \varphi \cdot g_1$ for some positive function $\varphi \in C^\infty(M)$ with $m := \max_M \varphi$. Then:

$$\beta_{g_2}(h) \leq m \beta_{g_1}(h), \quad \forall h \in H_1(M; \mathbb{R}).$$

Moreover, for $h \in H_1(M; \mathbb{R}) \setminus \{0\}$ we have

$$\beta_{g_2}(h) = m \beta_{g_1}(h) \iff \mathfrak{M}_{g_1}^h \subseteq \mathfrak{M}_{g_2}^h \text{ and } \varphi \equiv m \text{ on } \mathcal{M}_{g_1}^h \subseteq \mathcal{M}_{g_2}^h.$$

Remark 6. The Main Theorem implies, under the assumption $\beta_{g_2}(h) = C_{g_1}(g_2)\beta_{g_1}(h)$, that g_1 and g_2 are homothetic (more specifically, $g_2 = C_{g_1}(g_2)g_1$) on the subset of TM (hence, on its closure):

$$\bigcup_{\substack{h \in H_1(M; \mathbb{R}) \\ \beta_{g_2}(h) = C_{g_1}(g_2)\beta_{g_1}(h)}} \mathbb{R}_+ \cdot \widetilde{\mathcal{M}}_{g_1}^h,$$

where we denote $\mathbb{R}_+ \cdot \widetilde{\mathcal{M}}_{g_1}^h := \{(x, tv) : (x, v) \in \widetilde{\mathcal{M}}_{g_1}^h, t \in \mathbb{R}_+\}$.

Remark 7. For a rational class $h \in H_1(M; \mathbb{R}) \setminus \{0\}$, we observe that

$$\sqrt{2\beta_g(h)} = \inf\{\mathcal{L}_g(\gamma) : \gamma \text{ closed curve in } [h], \text{ up to multiplicity}\},$$

where $\mathcal{L}_g(\gamma)$ denotes the g -length of γ . The infimum is attained on surfaces (see [CMP04, Proposition 2.1] and [BM08, Proposition 5.6]). Therefore, on surfaces, equality at a rational $h \neq 0$ implies that the g_1 -shortest closed geodesics in $[h]$ are also the g_2 -shortest ones, with the corresponding shortest length rescaled by a factor $\sqrt{C_{g_1}(g_2)}$.

Remark 8. The Main Theorem continues to hold for Finsler metrics. In fact, given a Finsler metric $F: TM \rightarrow \mathbb{R}^+$, the corresponding Lagrangian is defined as $L_F(x, v) := \frac{1}{2}F_x(v)$ and the β -Mather's function is defined as in (1). The Main Theorem then holds when considering Finsler metrics F_1, F_2 (instead of Riemannian metrics g_1, g_2), and setting the normalization factor as

$$C_{F_1}(F_2) := \sup_{x \in M} \sup_{\substack{v \in T_x M \\ F_{1,x}(v)=1}} F_{2,x}(v).$$

We can deduce from point (iii) of the Main Theorem the following corollary.

Corollary 9. Let M be a closed manifold and let g_1 and g_2 be two conformally equivalent Riemannian metrics on M , i.e., $g_2 = \varphi \cdot g_1$ for some positive function $\varphi \in C^\infty(M)$ with $m := \max_M \varphi$. Let us denote

$$\mathcal{H}_{g_1} := \{h \in H_1(M; \mathbb{R}) \setminus \{0\} : \mathcal{M}_{g_1}^h = M\}.$$

Then, $\beta_{g_2}(h) = m \beta_{g_1}(h)$ for some $h \in \mathcal{H}_{g_1}$ if and only if g_1 and g_2 are homothetic, i.e., $g_2 = m g_1$.

Remark 10. In some cases one can describe the set of rotation vectors $\mathcal{H}_g$ appearing in Corollary 9 or show that it is not empty.

- (i) In the case of a flat metric g_0 on $\mathbb{T}^n$, we have $\mathcal{H}_{g_0} := H_1(\mathbb{T}^n; \mathbb{R}) \setminus \{0\}$. In fact, the system is completely integrable and it is easy to check that all Mather sets correspond to invariant Lagrangian graphs (see for example [Sor16, Section 4]).
- (ii) In the case of Liouville metrics on $\mathbb{T}^n$, *i.e.*, metrics of the form $g = (f_1(x_1) + \dots + f_n(x_n))(dx_1^2 + \dots + dx_n^2)$, we have (we identify $H_1(\mathbb{T}^n; \mathbb{R}) \simeq \mathbb{R}^n$):

$$\mathcal{H}_g \supseteq \{h \in \mathbb{R}^n : h_1 \cdot \dots \cdot h_n \neq 0\}.$$

In fact, for h with $h_i = 0$ and f_i nonconstant, the projected Mather set $\mathcal{M}_g^h$ is a proper subset of M (see [Hen25, Section 1.1] for a description of the invariant foliation determined by the corresponding integrals of motion).

- (iii) The above cases are examples of *integrable* metrics (in the sense of Arnold-Liouville). It follows from KAM theory that for sufficiently small perturbations $\tilde{g}$ of such metrics g the set $\mathcal{H}_{\tilde{g}}$ is not empty (actually, this set has almost full Lebesgue measure).
- (iv) In [BS12], the authors considered Riemannian manifolds (M, g) with a weaker notion of integrability – which does not require the manifold to be diffeomorphic to a torus – under which the set $\mathcal{H}_g$ is non-empty (actually, it is everything). Consider the following example (see [BS12, Theorem 1.3]). Let G be a simply-connected *amenable* Lie group; recall that a topological group is amenable if it admits a left-invariant, finitely additive, Borel probability measure. Let $\Gamma \leq G$ be a lattice subgroup, and g be a metric on $\Gamma \backslash G$ induced by a left-invariant metric on G . Then, for all $h \in H_1(\Gamma \backslash G; \mathbb{R})$, the Mather set $\widetilde{\mathcal{M}}_g^h$ projects over the whole $\Gamma \backslash G$. Therefore, in this case $\mathcal{H}_g = H_1(\Gamma \backslash G; \mathbb{R}) \setminus \{0\}$.

On $\mathbb{T}^2$ every Riemannian metric g is conformally equivalent to a flat metric. Let us choose a flat metric g_0 in the conformal class of g such that $\mathcal{C}_{g_0}(g) = 1$; this is always possible because a flat metric remains flat under constant rescaling. We call this flat metric g_0 the *normalized conformally equivalent flat metric* associated to g . In particular, the normalized conformally equivalent flat metric g_0 is uniquely determined by g .

Remark 11. Let us observe that the set

$$\mathcal{H}_g := \{h \in H_1(M; \mathbb{R}) \setminus \{0\} : \mathcal{M}_g^h = M\}$$

is invariant if we multiply the metric by a constant factor $c > 0$. Indeed, if μ minimizes $\int_{TM} g_x(v, v) d\mu(x, v)$ among closed probability measures with rotation vector h , then it also minimizes $\int_{TM} c g_x(v, v) d\mu(x, v)$, and viceversa.

Main Theorem (iii), Corollary 9, Remark 10 (i) and Remark 11 imply the following result that, in some sense, can be considered a pointwise version of Bangert's result for metrics in the same conformal class (yet, it does not answer to (Q4), see Section 1).

Corollary 12. *Let g be a Riemannian metric on $\mathbb{T}^2$ and let g_0 its normalized conformally equivalent flat metric. Then, $\beta_g(h) \leq \beta_{g_0}(h)$ for every $h \in H_1(\mathbb{T}^2; \mathbb{R})$. Moreover, equality holds at some $h \in H_1(\mathbb{T}^2; \mathbb{R}) \setminus \{0\}$ if and only if $g = g_0$, *i.e.*, g is flat.*

We now prove the Main Theorem.

Proof of Main Theorem. (i) If $h = 0$, the inequality holds trivially since $\beta_{g_1}(0) = \beta_{g_2}(0) = 0$. Let h be an element in $H_1(M; \mathbb{R}) \setminus \{0\}$ and let μ_h be a Mather measure for the metric g_1 with rotation vector h , i.e., such that $\beta_{g_1}(h) = \frac{1}{2} \int_{TM} g_{1,x}(v, v) d\mu_h(x, v)$. Then:

$$\begin{aligned} \beta_{g_2}(h) &\leq \frac{1}{2} \int_{TM} g_{2,x}(v, v) d\mu_h(x, v) \\ &\leq \frac{\mathcal{C}_{g_1}(g_2)}{2} \int_{TM} g_{1,x}(v, v) d\mu_h(x, v) = \mathcal{C}_{g_1}(g_2) \beta_{g_1}(h), \end{aligned} \quad (8)$$

where the first inequality follows from the definition of $\beta_{g_2}(h)$ as the minimum over all closed Borel probability measures of rotation vector h , while the second using that $g_{2,x}(v, v) \leq \mathcal{C}_{g_1}(g_2) g_{1,x}(v, v)$ for all $(x, v) \in TM$.

(ii) For $h = 0$, the result is trivially true. In fact, $\beta_g(0) = 0$ and $\widetilde{\mathcal{M}}_g^0 = M \times \{0\}$ for every metric g .

In the following, let us assume that $h \in H_1(M; \mathbb{R}) \setminus \{0\}$.

($\implies$) Assume that $\beta_{g_2}(h) = \mathcal{C}_{g_1}(g_2) \beta_{g_1}(h)$. Since $h \neq 0$, we have that $\beta_{g_i}(h) \neq 0$ for $i = 1, 2$ (see (3)), hence all inequalities in (8) are indeed equalities. Moreover, from Remark 1, one has that the support of any Mather measure of rotation vector $h \neq 0$ for g_1 does not intersect the zero section. Consequently, on such a support, the ratio $\frac{g_{2,x}(v, v)}{g_{1,x}(v, v)}$ is well-defined and is constantly equal to $\mathcal{C}_{g_1}(g_2)$. From the definition of the Mather set of homology class h (recall (2)) we conclude that

$$\frac{g_{2,x}(v, v)}{g_{1,x}(v, v)} \equiv \mathcal{C}_{g_1}(g_2) \quad \forall (x, v) \in \widetilde{\mathcal{M}}_{g_1}^h.$$

Observe that the fact that all inequalities in (8) are equalities, also implies that any Mather measure of rotation number h for g_1 is also minimizing for g_2 , i.e.,

$$\mathfrak{M}_{g_1}^h \subseteq \mathfrak{M}_{g_2}^h,$$

which by (2) yields

$$\widetilde{\mathcal{M}}_{g_1}^h \subseteq \widetilde{\mathcal{M}}_{g_2}^h.$$

($\impliedby$) Let $\mu_h \in \mathfrak{M}_{g_1}^h$. Since $\mathfrak{M}_{g_1}^h \subseteq \mathfrak{M}_{g_2}^h$, we have that $\mu_h \in \mathfrak{M}_{g_2}^h$ and therefore:

$$\begin{aligned} \beta_{g_2}(h) &= \frac{1}{2} \int_{TM} g_{2,x}(v, v) d\mu_h(x, v) \\ &= \frac{1}{2} \int_{TM} \frac{g_{2,x}(v, v)}{g_{1,x}(v, v)} \cdot g_{1,x}(v, v) d\mu_h(x, v) \\ &= \frac{\mathcal{C}_{g_1}(g_2)}{2} \int_{TM} g_{1,x}(v, v) d\mu_h(x, v) = \mathcal{C}_{g_1}(g_2) \beta_{g_1}(h), \end{aligned} \quad (9)$$

where we have used that $\frac{g_{2,x}(v, v)}{g_{1,x}(v, v)} \equiv \mathcal{C}_{g_1}(g_2)$ on $\widetilde{\mathcal{M}}_{g_1}^h \subseteq \widetilde{\mathcal{M}}_{g_2}^h$.

(iii) Observe that if g_1 and g_2 are conformally equivalent on M , i.e., $g_2 = \varphi \cdot g_1$ for some positive function $\varphi \in C^\infty(M)$, then for every $x \in M$ and $v \in T_x M \setminus \{0\}$, we have

$$\frac{g_{2,x}(v, v)}{g_{1,x}(v, v)} = \varphi(x)$$

(it is independent of v) and in particular $\mathcal{C}_{g_1}(g_2) = \max_M \varphi = m$. Since the function φ is independent of v , we deduce that

$$g_2 = \mathcal{C}_{g_1}(g_2)g_1 \text{ on } \widetilde{\mathcal{M}}_{g_1}^h \iff \varphi \equiv m \text{ on } \mathcal{M}_{g_1}^h.$$

Thus, (iii) is a direct consequence of (i)-(ii). $\square$

3. MAÑÉ'S PERTURBATIONS OF TONELLI LAGRANGIANS

In the above discussion, we focused on Riemannian geodesic flows, but the same ideas can be implemented for general Tonelli Lagrangians. Let M be a closed manifold. A Lagrangian $L: TM \rightarrow \mathbb{R}$ is said to be *Tonelli* if

- (i) $L \in C^2(TM)$;
- (ii) L is strictly convex in the fibers, i.e., if the second partial derivative $\partial^2 L / \partial v^2(x, v)$ is positive definite for all $(x, v) \in TM$;
- (iii) L is superlinear in each fiber, i.e., $\lim_{\|v\| \rightarrow +\infty} \frac{L(x, v)}{\|v\|} = +\infty$, uniformly in $x \in M$.

The same construction of Mather's β -function and Mather sets described above for L_g associated to a Riemannian metric g – which is clearly Tonelli – holds for a general Tonelli Lagrangian L . Let us introduce some notation:

- we denote by $\beta_L: H_1(M; \mathbb{R}) \rightarrow \mathbb{R}$ the associated β -function:

$$\beta_L(h) := \min_{\mu \in \rho^{-1}(h)} \int_{TM} L(x, v) d\mu. \quad (10)$$

- we denote by $\mathfrak{M}_L^h$ the set of Mather measures of rotation vector h , i.e., the set of measures $\mu \in \mathcal{M}(TM)$ realizing the minimum in (10).
- we denote by $\widetilde{\mathcal{M}}_L^h$ the associated Mather set of homology h , i.e., the union of the supports of all Mather measures of rotation vector h , and $\mathcal{M}_L^h := \pi(\widetilde{\mathcal{M}}_L^h)$ its projection on M .

We refer to [Sor15] for a more detailed description.

Given a Tonelli Lagrangian L , we can perturb it in the following way. Let $V \in C^2(M)$ and consider the Lagrangian $L_V := L + V$, which is still of Tonelli type. This kind of perturbation is called *in the sense of Mañé* (see [Mn96]). Since we can modify V by a constant without changing the dynamics, we normalize V by assuming that $\max_M V = 0$.

Remark 13. The study of the Mather sets for a generic perturbation in the sense of Mañé was done in [BC08]: they show that, for a residual set of potentials, there is abundance of minimizing ergodic measures in every homology class. The analogous result is obtained in [CM20] for perturbations of geodesic flows on a closed manifold given by a conformal perturbation of the metric.

Similarly to what we did in the previous section, one can prove the following results.

Theorem 14. *Let M be a closed manifold. Let $L: TM \rightarrow \mathbb{R}$ be a Tonelli Lagrangian and let $V \in C^2(M)$ be such that $\max_M V = 0$. Then:*

- (i) *We have $\beta_{L_V}(h) \leq \beta_L(h)$ for every $h \in H_1(M; \mathbb{R})$.*
- (ii) *Let $h \in H_1(M; \mathbb{R})$. Then, the following conditions are equivalent:*
 - (a) $\beta_{L_V}(h) = \beta_L(h)$;

- (b) $\mathfrak{M}_L^h \subseteq \mathfrak{M}_{L_V}^h$, $\widetilde{\mathcal{M}}_L^h \subseteq \widetilde{\mathcal{M}}_{L_V}^h$ and $V \equiv 0$ on $\mathcal{M}_L^h \subseteq \mathcal{M}_{L_V}^h$.
 (iii) Let $M = \mathbb{T}^n$ and let $L(x, v) = \ell(v)$ be a completely integrable Lagrangian. Denote $\ell_V := \ell + V$. Then, for every $h \in H_1(\mathbb{T}^n; \mathbb{R})$, one has

$$\beta_{\ell_V}(h) = \beta_\ell(h) \iff V \equiv 0 \text{ on } \mathbb{T}^n.$$

Proof. (i) Let $h \in H_1(M; \mathbb{R})$ and let μ_h be a Mather measure for L with rotation vector h , i.e., $\mu_h \in \mathfrak{M}_L^h$. Then:

$$\begin{aligned} \beta_{L_V}(h) &\leq \int_{TM} (L(x, v) + V(x)) d\mu_h(x, v) \\ &= \beta_L(h) + \int_{TM} V(x) d\mu_h(x, v) \leq \beta_L(h), \end{aligned} \quad (11)$$

where the last inequality follows from the fact that $V \leq 0$. This concludes the proof of point (i).

(ii) Let $h \in H_1(M; \mathbb{R})$.

(a) $\implies$ (b). Assume that $\beta_{L_V}(h) = \beta_L(h)$. For every $\mu_h \in \mathfrak{M}_L^h$, we deduce from (11) that $\int_{TM} V(x) d\mu_h(x, v) = 0$. Being $V \leq 0$, this implies that $V \equiv 0$ on the support of μ_h . Since this holds for every $\mu_h \in \mathfrak{M}_L^h$, we conclude that

$$V \equiv 0 \quad \text{on } \mathcal{M}_L^h.$$

Moreover, this proves that any $\mu_h \in \mathfrak{M}_L^h$ is also minimizing for L_V , showing that

$$\mathfrak{M}_L^h \subseteq \mathfrak{M}_{L_V}^h \quad \text{and} \quad \widetilde{\mathcal{M}}_L^h \subseteq \widetilde{\mathcal{M}}_{L_V}^h.$$

(b) $\implies$ (a). Let $\mu_h \in \mathfrak{M}_L^h$. Since $\mathfrak{M}_L^h \subseteq \mathfrak{M}_{L_V}^h$ and $V \equiv 0$ on $\mathcal{M}_L^h \subset \mathcal{M}_{L_V}^h$, we conclude:

$$\beta_{L_V}(h) = \int_{TM} (L(x, v) + V(x)) d\mu_h(x, v) = \int_{TM} L(x, v) d\mu_h(x, v) = \beta_L(h).$$

(iii) Identify $H_1(\mathbb{T}^n; \mathbb{R})$ with $\mathbb{R}^n$. One can easily show (see for instance [Sor16, Section 4]) that $\beta_\ell(h) = \ell(h)$ for every $h \in H_1(\mathbb{T}^n; \mathbb{R})$ and that $\widetilde{\mathcal{M}}_\ell^h = \mathbb{T}^n \times \{h\}$. Therefore, the claim follows from the result in (ii). $\square$

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REFERENCES

- [Ban88] V. Bangert. Mather sets for twist maps and geodesics on tori. In Dynamics reported, Vol. 1, volume 1 of Dynam. Report. Ser. Dynam. Systems Appl., pages 1–56. Wiley, Chichester, 1988.
- [Ban94] V. Bangert. Geodesic rays, Busemann functions and monotone twist maps. Calc. Var. Partial Differential Equations, 2(1):49–63, 1994.
- [BBI01] D. Burago, Y. Burago, and S. Ivanov. A course in metric geometry, volume 33 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001.
- [BBS25] S. Baranzini, M. Bialy, and A. Sorrentino. Isoperimetric-type inequalities for Mather's β -function of convex billiards. [arXiv:2509.06915](https://arxiv.org/abs/2509.06915), 2025.
- [BC08] P. Bernard and G. Contreras. A generic property of families of Lagrangian systems. Ann. of Math. (2), 167(3):1099–1108, 2008.
- [BM08] F. Balacheff and D. Massart. Stable norms of non-orientable surfaces. Ann. Inst. Fourier (Grenoble), 58(4):1337–1369, 2008.
- [BS12] L. T. Butler and A. Sorrentino. Weak Liouville-Arnol'd theorems and their implications. Comm. Math. Phys., 315(1):109–133, 2012.
- [CM20] G. Contreras and J. A. G. Miranda. On finite quotient Aubry set for generic geodesic flows. Math. Phys. Anal. Geom., 23(2):Paper No. 14, 11, 2020.
- [CMP04] G. Contreras, L. Macarini, and G. P. Paternain. Periodic orbits for exact magnetic flows on surfaces. Int. Math. Res. Not., (8):361–387, 2004.
- [Fed75] H. Federer. Real flat chains, cochains and variational problems. Indiana Univ. Math. J., 24:351–407, 1974/75.
- [Fie26] C. Fierobe. Spectral rigidity among ellipses, Bialy's conjecture and local extrema of Mather's beta function. [arXiv:2603.09439](https://arxiv.org/abs/2603.09439), 2026.
- [GLP25] C. Guillarmou, T. Lefeuvre, and G. P. Paternain. Marked length spectrum rigidity for Anosov surfaces. Duke Math. J., 174(1):131–157, 2025.
- [Gro81] Mikhael Gromov. Structures métriques pour les variétés riemanniennes, volume 1 of Textes Mathématiques [Mathematical Texts]. CEDIC, Paris, 1981.
- [Gro99] M. Gromov. Metric structures for Riemannian and non-Riemannian spaces, volume 152 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1999. Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.
- [Hen25] J. Henheik. Deformational rigidity of integrable metrics on the torus. Ergodic Theory Dynam. Systems, 45(2):467–503, 2025.
- [Kat88] A. Katok. Four applications of conformal equivalence to geometry and dynamics. Ergodic Theory Dynam. Systems, 8*:139–152, 1988.
- [Mas96] D. Massart. Normes stables des surfaces. PhD thesis, École Normale Supérieure de Lyon, 1996.
- [Mas97] D. Massart. Stable norms of surfaces: local structure of the unit ball of rational directions. Geom. Funct. Anal., 7(6):996–1010, 1997.
- [Mas03] D. Massart. On Aubry sets and Mather's action functional. Israel J. Math., 134:157–171, 2003.
- [Mas09] D. Massart. Vertices of Mather's beta function. II. Ergodic Theory Dynam. Systems, 29(4):1289–1307, 2009.
- [Mat90] J. N. Mather. Differentiability of the minimal average action as a function of the rotation number. Bol. Soc. Brasil. Mat. (N.S.), 21(1):59–70, 1990.
- [Mat91] J. N. Mather. Action minimizing invariant measures for positive definite Lagrangian systems. Math. Z., 207(2):169–207, 1991.
- [Mn96] R. Mañé. Generic properties and problems of minimizing measures of Lagrangian systems. Nonlinearity, 9(2):273–310, 1996.

- [MS11] D. Massart and A. Sorrentino. Differentiability of Mather’s average action and integrability on closed surfaces. Nonlinearity, 24(6):1777–1793, 2011.
- [Osu10] O. Osuna. Rigidity of the stable norm on tori. Rev. Colombiana Mat., 44(1):15–21, 2010.
- [Sor15] A. Sorrentino. Action-minimizing methods in Hamiltonian dynamics, volume 50 of Mathematical Notes. Princeton University Press, Princeton, NJ, 2015. An introduction to Aubry-Mather theory.
- [Sor16] A. Sorrentino. Lecture notes on Mather’s theory for Lagrangian systems. Publ. Mat. Urug., 16:169–192, 2016.

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