SMOOTH CONJUGACY CLASSES OF 3D AXIOM A FLOWS

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Abstract. We show a rigidity result for 3-dimensional contact *Axiom A* flows: given two 3D contact Axiom A flows Φ_1, Φ_2 whose restrictions $\Phi_1|_{\Lambda_1}, \Phi_2|_{\Lambda_2}$ to basic sets Λ_1, Λ_2 are orbit equivalent, we prove that if periodic orbits in correspondence have the same length, then the conjugacy is as regular as the flows and respects the contact structure, extending a previous result due to Feldman-Ornstein [23]. Some of the ideas are reminiscent of the work of Otal [54]. As an application, we show that the billiard maps of two open dispersing billiards without eclipse and with the same marked length spectrum are *smoothly* conjugated.

Contents

1. Introduction, statement of the results

The concept of *rigidity* arises in several ways in dynamics; one of them is the problem of knowing when two smooth systems which are topologically conjugated are actually *smoothly* conjugated. It appears for instance in the framework of diffeomorphisms of the circle. In [2] Arnold proved the first $\mathcal{C}^\omega$-linearization result. More precisely, he showed that an analytic diffeomorphism with Diophantine rotation number α and sufficiently close to the rotation R_α is analytically conjugated to R_α. A global result in the $\mathcal{C}^\infty$ category is due to Herman, in [31], where he also proved the optimality of the Diophantine condition in the smooth case; see also [65], [40] for related works.

For low dimensional *Anosov* systems, the question of rigidity has been investigated in many works, see for instance the series of papers by de la Llave, Marco and Moriyón [44, 17, 45, 20], [19], and [18]. While renormalization is one of the main tools behind the study of rigidity for circle diffeomorphisms, the approach for *hyperbolic* systems is quite different. Indeed, for such systems, *periodic orbits* are abundant, and each of them carries with itself an obstruction to smooth conjugacy, namely the associated eigenvalues of the differential. In the aforementioned works of de la Llave-Marco-Moriyón, it is shown that those obstructions are actually complete invariants for smooth conjugacy classes. The Anosov assumption can be relaxed, namely, we may consider systems where hyperbolicity is only observed on a subset of the phase space. In particular, when the *non-wandering set* is hyperbolic, this leads to the notion of *Axiom A* systems. In [60], Pinto-Rand showed that Lipschitz conjugacy classes of hyperbolic basic sets on surfaces, which possess an invariant measure absolutely continuous with respect to Hausdorff measure, can be characterised in many ways, in particular, in terms of eigenvalues at periodic points. Let us also mention the works [59] and [5], where other rigidity results for hyperbolic sets have been obtained. In the context of expanding maps in any dimension, Gogolev and Rodriguez-Hertz [26] have shown that, open and densely, smooth conjugacy classes are determined by the value of the Jacobian of the return maps at periodic points.

Let us now say a few words on rigidity questions in *geometric* frameworks. A natural setting is that of hyperbolic *geodesic flows*. In this case, the general hope is that periodic data, in particular, the *length spectrum*, may be sufficient to characterize not only smooth conjugacy classes, but also to recover some *geometric* information. The question of *spectral rigidity* asks whether the (marked) length spectrum is sufficient to determine the metric up to isometry. There exist various instances of this problem, both local and non-local. Guillemin-Kazhdan [30] have shown that compact negatively curved surfaces are spectrally rigid in the *deformative* sense: a family $(g_s)_{s\in(0,1)}$ of isospectral negatively curved metrics is *isometric*, that is, for each $s \in (0,1)$, there exists a diffeomorphism ϕ_s such that $g_s = \phi_s^* g_0$. Later, Paternain-Salo-Uhlmann [55] proved that any Anosov surface is spectrally rigid in the deformative sense. Let us recall that for hyperbolic surfaces, periodic trajectories can be naturally marked by free homotopy classes. The question of spectral rigidity for hyperbolic surfaces was addressed by Otal [54] and independently by Croke [15], who obtained the following *global* result: two negatively curved metrics g_0 and g_1 on a closed surface with the same *marked length spectrum* are isometric (see also [16] for the multidimensional case). Recently, Guillarmou-Lefeuvre [29] proved that in all dimensions, the marked length spectrum of a Riemannian manifold with Anosov geodesic flow and non-positive curvature locally determines the metric. See also the recent work [27] where a sharpened version of Otal and Croke's result was obtained. Other works have also investigated the case where the hyperbolic set is not the whole manifold. For instance, in [28], Guillarmou considers a smooth one-parameter family $(g_s)_{s\in(0,1)}$ of metrics on a smooth connected compact manifold with strictly convex boundary. When the metrics have no conjugate points, and the trapped set is a hyperbolic set for the geodesic flow, he proved that if all the metrics in the family are lens equivalent, then they are isometric. Following this work, Lefeuvre [42] studied the X-ray transform on a smooth compact connected Riemannian manifold with hyperbolic trapped set. Other results in this direction have been recently obtained also by Chen, Erchenko and Gogolev in [9].

Another setting where rigidity questions for the length spectrum have been investigated is the case of *planar billiards*, in particular, for convex domains. One of the first results was obtained by Colin de Verdière [14], who established dynamical spectral rigidity for convex domains with analytic boundary and a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry. De Simoi-Kaloshin-Wei [22] have proved dynamical spectral rigidity for $\mathbb{Z}_2$-symmetric strictly convex domains close to a circle; see also the recent work [3] by Ayub-De Simoi for ellipses of eccentricity smaller than 0.30. Moreover, recently, for smoothly conjugate billiard maps of Birkhoff billiards, Kaloshin-Koudjinan [38] study rigidity in the form of Marvizi-Melrose invariants. Let us also recall that the question of spectral rigidity for convex billiards can be considered for other kinds of spectra as well: one of the most famous examples concerns the *Dirichlet (or Neumann) spectrum*, which has been investigated at length, in particular, in a series of works by Zelditch [66] and Hezari-Zelditch [32, 33, 34, 35]. . . , but also in many others.

Yet, even more than convex billiards, hyperbolic billiards, in particular, *dispersing billiards* are the most natural analogue of hyperbolic geodesic flows; indeed, although convex billiards may exhibit some hyperbolicity, for dispersing billiards, hyperbolicity is present on the whole phase space. The case of *Sinai billiards* is very interesting, due to the abundance of periodic orbits; yet, the complicated structure of the set of periodic orbits as well as the presence of singularities make them hard to deal with. Many works have been dedicated to the study of *open dispersing billiards* (see [25, 43, 47, 48, 49, 62] for instance). Recall that the dynamics of open dispersing billiards is of *Axiom A* type, and that their non-wandering set can be described symbolically (see [47] for instance), which allows to define a marked length spectrum. The rigidity of scattering lengths and travelling times has been investigated in a series of works by Noakes, Stoyanov, and Petkov (see [51, 52, 53, 63] and also the book [57]). In particular, *lens rigidity* was established by Noakes-Stoyanov [51, 53]: open dispersing billiards in $\mathbb{R}^d$, $d \geq 2$, are uniquely determined by the travelling times of billiard orbits and also by their scattering length spectra. It is interesting to observe that as in the present work, the conjugacy between the billiard flows of two billiards with the same spectral data plays an important role: yet, Noakes-Stoyanov consider the conjugacy between the billiard flows *outside the trapped set*, while here, we study the conjugacy precisely *on the trapped set*; this is due to the fact that these works deal with different spectra.

Let us conclude this introduction by mentioning recent rigidity results for hyperbolic billiards in terms of the *length spectrum*. In [10], Chen-Kaloshin-Zhang established the dynamical spectral rigidity of piecewise analytic Bunimovich stadia and squash type stadia. In [21], De Simoi-Kaloshin and the second author solved the question of marked length spectral determination for non-eclipsing open dispersing billiards with analytic boundary and two partial symmetries, under some mild non-degeneracy condition. Observe that in the $\mathcal{C}^k$ category, $k \in \mathbb{N}_{\geq 3} \cup \{+\infty\}$, the marked length spectrum is insufficient to fully determine the geometry of such tables; indeed, periodic orbits are not dense in the whole phase space, so it is possible to deform the geometry of the arcs of the table which are not "seen" by the trapped set, i.e., which come from "gaps" of the projection on the table of the Cantor set on which we have information through periodic orbits.

In the present work, we generalize the result of Feldman-Ornstein [23] from contact Anosov flows on 3-manifolds to contact Axiom A flows on 3-manifolds. More precisely, equality of the length data allows us to upgrade an orbit equivalence to a flow conjugacy as regular as the flows, see Theorem A. We apply this result to billiards exhibiting some hyperbolicity, and obtain a *dynamical* rigidity result: for $k \geq 3$, we show that two $\mathcal{C}^k$ open dispersing billiards[1] whose billiard maps are topologically conjugated on some horseshoe and have the same length data are actually *smoothly* conjugated, in a *canonical* way (see Theorem C and Corollary D). In a previous version of this work, we were discussing geometric implications of the smoothness of the conjugacy, in connection with the question of spectral rigidity; yet, an error was found in this part by Jacopo De Simoi.

1.1. **Preliminaries.** Let $\Phi = (\Phi^t)_{t\in\mathbb{R}}$ be a continuous flow defined on a manifold M. For each point $x \in M$, we denote by $\mathcal{O}_\Phi(x) := \{\Phi^t(x)\}_{t\in\mathbb{R}}$ the Φ-orbit of x. We denote by $\mathrm{Fix}(\Phi) := \{x \in M : \Phi^t(x) = x \text{ for all } t \in \mathbb{R}\}$ the set of *fixed points* of Φ, and we denote by $\mathrm{Per}(\Phi) := \{y \in M : \Phi^T(y) = y \text{ for some } T > 0\}$ the set of *periodic points* of Φ; for any $x \in \mathrm{Per}(\Phi)$, we let $T_\Phi(x) = T_\Phi(\mathcal{O}_\Phi(x)) > 0$ be the prime period of x. Recall that the *non-wandering set* $\Omega(\Phi) \subset M$ is the set of points $x \in M$ such that for any open set $U \ni x$, any $T_0 > 0$, there exists $T > T_0$ such that $\Phi^T(U) \cap U \neq \emptyset$. When Φ is a differentiable flow on some smooth manifold M, we denote by $X_\Phi(\cdot) := \frac{d}{dt}|_{t=0}\Phi(\cdot, t)$ its *flow vector field*.

In the following, given an integer $n \geq 1$, and $\beta \in (0,1)$, we say that a function f is of class $\mathcal{C}^{n,\beta}$ if f is $\mathcal{C}^n$, and its n^{th} derivative is β-Hölder continuous.

Definition 1.1 (Orbit equivalence)**.** For $i = 1, 2$, let $\Phi_i = (\Phi_i^t)_{t\in\mathbb{R}}$ be a flow defined on a manifold M_i, and let $\Lambda_i \subset M_i$ be a Φ_i-invariant subset. We say that the flows Φ_1, Φ_2 are *orbit equivalent* on Λ_1, Λ_2 if there exists a homeomorphism $\Psi \colon \Lambda_1 \to \Lambda_2$ such that for some continuous function $\theta \colon \Lambda_1 \times \mathbb{R} \to \mathbb{R}$, we have for each $x \in \Lambda_1$:

- $\theta(x, 0) = 0$, and $\theta(x, \cdot)$ is an increasing $\mathcal{C}^{1,\beta}$ homeomorphism of $\mathbb{R}$, for some $\beta \in (0,1)$;
- $\Psi \circ \Phi_1^t(x) = \Phi_2^{\theta(x,t)} \circ \Psi(x)$, for all $t \in \mathbb{R}$.

In other words, Ψ sends Φ_1-orbits to Φ_2-orbits:

$$\Psi(\mathcal{O}_{\Phi_1}(x)) = \mathcal{O}_{\Phi_2}(\Psi(x)), \quad \text{for all } x \in \Lambda_1.$$

Recall that Ψ is automatically $\mathcal{C}^\delta$ for some $\delta \in (0,1)$, if Λ_1, Λ_2 are compact hyperbolic sets (see e.g. Katok-Hasselblatt [39, Theorem 19.1.2]).

Moreover, we say that Ψ is *iso-length-spectral* if

$$T_{\Phi_1}(x) = T_{\Phi_2}(\Psi(x)), \quad \forall\, x \in \mathrm{Per}(\Phi_1) \cap \Lambda_1,$$

i.e., the flows Φ_1, Φ_2 have the same periodic length data.

If M_1, M_2 are smooth, and Φ_1, Φ_2 are differentiable flows, we abbreviate as $X_i := X_{\Phi_i}$ the flow vector field, for $i = 1, 2$, and we say that Ψ is *differentiable along* Φ_1*-orbits* (in Λ_1) if the Lie derivative

$$\Lambda_1 \ni x \mapsto L_{X_1}\Psi(x) := \lim_{t\to 0} \frac{1}{t}\big(\Psi \circ \Phi_1^t(x) - \Psi(x)\big) \in \mathbb{R}X_2 \circ \Psi(x)$$

is a well-defined continuous function.

[1] Actually, the same result also holds for more general billiards, see Theorem C.

Definition 1.2 (Adapted contact form)**.** Given a smooth (connected) 3-manifold M, recall that a *contact form* is a smooth differential one-form that satisfies the *non-integrability condition* $\alpha \wedge d\alpha \neq 0$; without loss of generality, we may assume that $\alpha \wedge d\alpha > 0$.

Let $k \geq 2$, and let $\Phi = (\Phi^t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^k$ Axiom A flow defined on a smooth 3-manifold M. Given a basic set $\Lambda \subset M$ for Φ, we say that a contact form α is *adapted to* Λ if it satisfies the following Reeb conditions:

(a) $\imath_X\alpha|_\Lambda \equiv 1$;
(b) $X|_{\mathcal{W}^{cs}_\Phi(\Lambda)} \in \ker d\alpha|_{\mathcal{W}^{cs}_\Phi(\Lambda)}$ and $X|_{\mathcal{W}^{cu}_\Phi(\Lambda)} \in \ker d\alpha|_{\mathcal{W}^{cu}_\Phi(\Lambda)}$.

In the following, we fix a $\mathcal{C}^\infty$ smooth Riemannian manifold M, and we consider a $\mathcal{C}^2$ flow $\Phi = (\Phi^t)_{t\in\mathbb{R}}$ on M.

Definition 1.3 (Hyperbolic set)**.** A Φ-invariant compact subset $\Lambda \subset M \setminus \mathrm{Fix}(\Phi)$ is called a *(uniformly) hyperbolic* set (for Φ) if there exists a $D\Phi$-invariant splitting

$$T_xM = E^s(x) \oplus \mathbb{R}X(x) \oplus E^u(x), \qquad \forall\, x \in \Lambda,$$

where the *(strong) stable bundle* E^s_Φ, resp. the *(strong) unstable bundle* E^u_Φ is uniformly contracted, resp. expanded, i.e., there exist $C > 0$, $\lambda \in (0,1)$ such that

$$\begin{aligned} \|D\Phi^t(x)\cdot v\| &\leq C\lambda^t\|v\|, \qquad \forall\, x \in \Lambda,\, \forall\, v \in E^s_\Phi(x),\, \forall\, t \geq 0,\\ \|D\Phi^{-t}(x)\cdot v\| &\leq C\lambda^t\|v\|, \qquad \forall\, x \in \Lambda,\, \forall\, v \in E^u_\Phi(x),\, \forall\, t \geq 0. \end{aligned}$$

We also denote by E^{cs}_Φ, resp. E^{cu}_Φ, the *weak stable bundle* $E^{cs}_\Phi := E^s_\Phi \oplus \mathbb{R}X$, resp. the *weak unstable bundle* $E^{cu}_\Phi := \mathbb{R}X \oplus E^u_\Phi$.

Let us recall the definition of an *Axiom A* flow:

Definition 1.4 (Axiom A flow)**.** A flow $\Phi\colon M \times \mathbb{R} \to M$ is called *Axiom A* if the non-wandering set $\Omega(\Phi) \subset M$ can be written as a disjoint union $\Omega(\Phi) = \Lambda \cup F$, where Λ is a closed hyperbolic set such that periodic orbits are dense in Λ, and $F \subset \mathrm{Fix}(\Phi)$ is a finite union of hyperbolic fixed points.

Definition 1.5 (Lamination)**.** Let $n \geq 1$, $\beta \in (0,1)$. A $\mathcal{C}^{n,\beta}$-lamination of a set $\Lambda \subset M$ is a disjoint collection of $\mathcal{C}^{n,\beta}$ submanifolds of a given same dimension, which vary continuously in the $\mathcal{C}^{n,\beta}$-topology, and whose union contains the set Λ.

Let $\Phi\colon M \times \mathbb{R} \to M$ be an Axiom A flow with a decomposition $\Omega(\Phi) = \Lambda \cup F$ as in Definition 1.4. The stable bundle E^s_Φ, resp. the unstable bundle E^u_Φ, over Λ integrates to a continuous lamination $\mathcal{W}^s_\Phi$, resp. $\mathcal{W}^u_\Phi$, called the *(strong) stable lamination*, resp. the *(strong) unstable lamination*. Similarly, E^{cs}_Φ, resp. E^{cu}_Φ integrates to a continuous lamination $\mathcal{W}^{cs}_\Phi$, resp. $\mathcal{W}^{cu}_\Phi$, called the *weak stable lamination*, resp. the *weak unstable lamination*. For each point $x \in \Lambda$, a local orbit segment in $\mathcal{O}_\Phi(x)$ containing x will also be denoted as $\mathcal{W}^c_{\Phi,\mathrm{loc}}(x) = \mathcal{W}^{cs}_{\Phi,\mathrm{loc}}(x) \cap \mathcal{W}^{cu}_{\Phi,\mathrm{loc}}(x)$. Each of these laminations is invariant under the dynamics, i.e., $\Phi^t(\mathcal{W}^*_\Phi(x)) = \mathcal{W}^*_\Phi(\Phi^t(x))$, for all $x \in M$ and $* = s, u, c, cs, cu$. For each subset $S \subset \Lambda$, we also denote $\mathcal{W}^*_\Phi(S) := \cup_{x\in S}\mathcal{W}^*_\Phi(x)$, for $* = s, u, c, cs, cu$.

Besides, we have $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_m$ for some integer $m \geq 1$, where for each $i \in \{1, \ldots, m\}$, Λ_i is a hyperbolic set such that $\Phi|_{\Lambda_i}$ is transitive, and $\Lambda_i = \cap_{t\in\mathbb{R}}\Phi^t(U_i)$ for some open set $U_i \supset \Lambda_i$. The set Λ_i is called a *basic set* of Φ.

Remark 1.6. In general, the stable/unstable distributions $E^{s/u}_{\mathcal{F}}$ at a hyperbolic invariant set Λ of some diffeomorphism $\mathcal{F}$ are only Hölder continuous, but according to Pinto-Rand [58], when the stable, resp. unstable leaves are one-dimensional, and Λ has local product structure, then the stable holonomies, resp. unstable holonomies are of class $\mathcal{C}^{1,\beta}$, $\beta \in (0,1)$. In our case, both distributions are one-dimensional, so the holonomies will be $\mathcal{C}^{1,\beta}$, for some $\beta \in (0,1)$.

Let us recall the following version of the extension theorem due to Whitney [64]. It legitimates the notion of differentiability in Whitney sense.

Theorem 1.7. *Fix an integer $k \geq 1$. Let $A \subset \mathbb{R}^n$ be a closed subset, $n \geq 1$, and let $f_0, \ldots, f_k \colon A \to \mathbb{R}$ be continuous functions such that for some $\beta \in (0,1)$, it holds*

$$f_0(y) - f_0(x) = \sum_{j=1}^{k} \frac{f_j(x)}{j!}(y-x)^j + O(|y-x|^{k+\beta}), \quad \forall\, x, y \in A. \tag{1.1}$$

Then, there exists a $\mathcal{C}^{k,\beta}$ function $f \colon \mathbb{R}^n \to \mathbb{R}$ such that $f|_A = f_0|_A$, $f^{(j)}|_A = f_j|_A$ for $j = 1, \ldots, k$, and $f|_{\mathbb{R}^n \setminus A}$ is $\mathcal{C}^\omega$. A function $f_0 \colon A \to \mathbb{R}$ which satisfies (1.1) for some functions $f_1, \ldots, f_k \colon A \to \mathbb{R}$ is said to be $\mathcal{C}^{k,\beta}$ in Whitney sense.

1.2. **Dynamical spectral rigidity of contact Axiom A flows.** Our main dynamical result is the following.

Theorem A (Length spectral rigidity on basic sets)**.** *Fix $k \geq 2$. For $i = 1, 2$, let $\Phi_i = (\Phi_i^t)_{t \in \mathbb{R}}$ be a $\mathcal{C}^k$ Axiom A flow defined on a 3-manifold M_i. Let Λ_i be a basic set for Φ_i, and assume that there exists a smooth contact form α_i on M_i that is adapted to Λ_i. If there exists an orbit equivalence $\Psi_0 \colon \Lambda_1 \to \Lambda_2$ between $\Phi_1|_{\Lambda_1}$ and $\Phi_2|_{\Lambda_2}$ that is differentiable along Φ_1-orbits and iso-length-spectral, then*

(1) $\Phi_1|_{\Lambda_1}$, $\Phi_2|_{\Lambda_2}$ are $\mathcal{C}^k$-conjugate; more precisely, there exists a Hölder continuous homeomorphism $\Psi \colon \Lambda_1 \to \Lambda_2$ that is $\mathcal{C}^k$ in Whitney sense, such that

$$\Psi \circ \Phi_1^t(x) = \Phi_2^t \circ \Psi(x), \quad \textit{for all } (x,t) \in \Lambda_1 \times \mathbb{R};$$

(2) Ψ preserves the contact form, i.e., $\Psi^* \alpha_2|_{\Lambda_1} = \alpha_1|_{\Lambda_1}$.

In other terms, iso-length-spectral orbit equivalence classes between basic sets of $\mathcal{C}^k$ Axiom A flows with an adapted contact form are in one-to-one correspondence with $\mathcal{C}^k$ flow conjugacy classes between these basic sets, where the conjugacy preserves the contact form. Besides, it will be clear from the proof that the $\mathcal{C}^k$-regularity is actually needed on Λ_i (in Whitney sense).

Remark 1.8. Let Φ_1, Φ_2, and let Λ_1, Λ_2 be as in Theorem A. The flow conjugacy $\Psi \colon \Lambda_1 \to \Lambda_2$ between $\Phi_1|_{\Lambda_1}$ and $\Phi_2|_{\Lambda_2}$ given by Theorem A is essentially unique. Indeed, for any other flow conjugacy $\widetilde{\Psi} \colon \Lambda_1 \to \Lambda_2$, it holds

$$(\Psi^{-1} \circ \widetilde{\Psi}) \circ \Phi_1^t = \Phi_1^t \circ (\Psi^{-1} \circ \widetilde{\Psi}) \quad \text{on } \Lambda_1,$$

that is, $\Psi^{-1} \circ \widetilde{\Psi}$ is in the diffeomorphism centralizer of $\Phi_1|_{\Lambda_1}$. By [4, Theorem 1.4], the centralizer is trivial, hence $\widetilde{\Psi} = \Psi \circ \Phi_1^T$, for some $T \in \mathbb{R}$. In Subsection 3.1, we explain that in some cases (when the system has a time-reversal symmetry) there is a natural way to choose T so as to make the conjugacy *canonical*.

Since the Hausdorff dimension is preserved by Lipschitz continuous homeomorphisms, and since the stable/unstable Hausdorff dimensions are constant on Λ (see for instance [56]), we deduce from Theorem A the following result:

Corollary B. *Let $\Phi = (\Phi^t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^k$ Axiom A flow defined on a smooth 3-manifold M, $k \geq 2$. Let Λ be a basic set for Φ with an adapted smooth contact form α. Then, the Hausdorff dimensions $\dim_H(\Lambda)$, $\delta^{(s)}(\Lambda)$, $\delta^{(u)}(\Lambda)$ are invariant under iso-length-spectral orbit equivalences, where for $* = s, u$, we let $\delta^{(*)}(\Lambda) := \dim_H(\Lambda \cap \mathcal{W}^*_\Phi(x))$, for any $x \in \Lambda$.*

1.3. **Open dispersing billiards.** We consider a billiard table $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^{\ell} \mathcal{O}_i$ obtained by removing from the plane $\ell \geq 3$ *obstacles* $\mathcal{O}_1, \dots, \mathcal{O}_\ell$, each $\mathcal{O}_i$ being a convex domain with $\mathcal{C}^k$ boundary $\partial\mathcal{O}_i$, for some $k \geq 3$, such that $\overline{\mathcal{O}_1}, \dots, \overline{\mathcal{O}_\ell}$ are pairwise disjoint. For each $i \in \{1, \dots, \ell\}$, we let $|\partial\mathcal{O}_i|$ be the corresponding perimeter, and parametrize each $\partial\mathcal{O}_i$ counterclockwisely in arc-length by some map $\Upsilon_i \in \mathcal{C}^k(\mathbb{T}_i, \mathbb{R}^2)$, $s \mapsto \Upsilon_i(s)$, where $\mathbb{T}_i := \mathbb{R}/(|\partial\mathcal{O}_i|\mathbb{Z})$. The set of all such billiard tables will be denoted by $\mathbf{B}$, and for each $\ell \geq 3$, we let $\mathbf{B}(\ell) \subset \mathbf{B}$ be the subset of tables with ℓ obstacles.

Let $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^{\ell} \mathcal{O}_i \in \mathbf{B}$, for some $\ell \geq 3$. We denote the collision space by

$$\mathcal{M} := \bigcup_i \mathcal{M}_i, \qquad \mathcal{M}_i := \{(q, v),\ q \in \partial\mathcal{O}_i,\ v \in \mathbb{R}^2,\ \|v\| = 1,\ \langle v, n\rangle \geq 0\},$$

where n is the unit normal vector to $\partial\mathcal{O}_i$ pointing outside $\mathcal{O}_i$. For each $x = (q, v) \in \mathcal{M}$, we have $q = \Upsilon_i(s)$, for some $i \in \{1, \dots, \ell\}$ and some arc-length parameter $s \in \mathbb{T}_i$; we let $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ be the oriented angle between n and v, and set $r := \sin\varphi$. Therefore, each $\mathcal{M}_i$ can be seen as a cylinder $\mathbb{T}_i \times [-1, 1]$ endowed with coordinates (s, r). In the following, given a point $x = (s, r) \in \mathcal{M}$, we let $\Upsilon(s) := q$ be the associated point of $\partial\mathcal{D}$.

For each pair $(s_1, r_1), (s_2, r_2) \in \mathcal{M}$, we denote by

$$h(s_1, s_2) := \|\Upsilon(s_1) - \Upsilon(s_2)\| \tag{1.2}$$

the Euclidean length of the segment connecting the associated points of the table.

Let $\mathfrak{M} := \{(q, v) \in \mathcal{D} \times \mathbb{S}^1\}/\sim$ be the quotient of $\mathcal{D} \times \mathbb{S}^1$ by the relation $\sim$:

$$(q_1, v_1) \sim (q_2, v_2) \quad\Longleftrightarrow\quad q_1 = q_2 \in \partial\mathcal{D} \text{ and } v_2 = \mathcal{R}_{q_1}(v_1),$$

where $\mathcal{R}_{q_1}$ is the reflection in $\mathbb{R}^2$ with respect to the tangent line $T_{q_1}\partial\mathcal{D}$. An element of $\mathfrak{M}$ will be denoted as $[(q, v)]$. In the following, we identify a point $[(q, v)] \in \mathfrak{M}$, $q \in \partial\mathcal{D}$, with the corresponding element $(q, v) \in \mathcal{M}$. Let $\Phi = (\Phi^t)_{t\in\mathbb{R}}$ be the associated billiard flow on $\mathfrak{M}$. We can describe this flow with coordinates (x, y, ω), where $(x, y) \in \mathbb{R}^2$ are the Cartesian coordinates of some point $q \in \mathcal{D}$ on the table and $\omega \in [0, 2\pi)$ denotes the couterclockwise angle between the positive x axis and the velocity vector v. In the recent paper [41, Section 3] by Küster-Schütte-Weich, the authors give a detailed description of *smooth models* for the billiard flow, which remain of *contact* type; besides, such smooth models are unique in a strong sense, hence can be considered as intrinsic to the billiard system.

For each $x \in \mathcal{M}$, we let $\tau(x) \in \mathbb{R}_+ \cup \{+\infty\}$ be the first return time of the Φ-orbit of x to $\mathcal{M}$, and denote by

$$\mathcal{F} = \mathcal{F}(\mathcal{D})\colon \mathcal{M} \cap \{\tau \neq +\infty\} \to \mathcal{M}, \quad x \mapsto \Phi^{\tau(x)}(x)$$

the associated billiard map, which we see as a map $\mathcal{F}\colon (s,r)\mapsto(s',r')$, with $s'=s'(s,r)$ and $r'=r'(s,r)$.

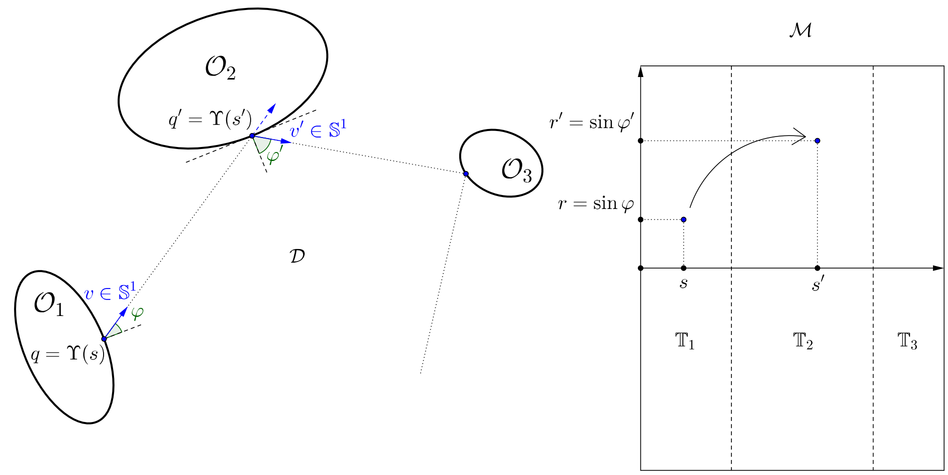


FIGURE 1. An open dispersing billiard and its phase space.

For any point $x=(s,r)\in\mathcal{M}$ with a well-defined image $(s',r')=\mathcal{F}(s,r)$, recall that $h=h(s,s')$ is the distance between the two points of collision. Note that $h(s,s')=h(s,s'(s,r))=\tau(s,r)$ is the first return time of $(s,r)\in\mathcal{M}$ to $\mathcal{M}$. Let $\mathcal{K}:=\mathcal{K}(s)$, $\mathcal{K}':=\mathcal{K}(s')$ be the respective curvatures, and set $\nu=\nu(r):=\sqrt{1-r^2}$, $\nu':=\nu(r')=\sqrt{1-(r')^2}$. By the formulas in Chernov-Markarian [12], the differential of the billiard map is

$$D\mathcal{F}(s,r)=-\begin{bmatrix}\frac{1}{\nu'}(h\mathcal{K}+\nu) & \frac{h}{\nu\nu'}\\ h\mathcal{K}\mathcal{K}'+\mathcal{K}\nu'+\mathcal{K}'\nu & \frac{1}{\nu}(h\mathcal{K}'+\nu')\end{bmatrix}. \tag{1.3}$$

The map $\mathcal{F}$ is exact symplectic for the Liouville form $\lambda=-rds$:

$$\mathcal{F}^*\lambda-\lambda=d\tau. \tag{1.4}$$

Fix a lift $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ to $\mathbb{R}\times[-1,1]$. We let $|\partial\mathcal{D}|:=|\partial\mathcal{O}_1|+\cdots+|\partial\mathcal{O}_m|$ be the total perimeter, and extend the definition of h by letting $h(s+p|\partial\mathcal{D}|,s'+q|\partial\mathcal{D}|)=h(s,s')$, for any $p,q\in\mathbb{Z}$. Then, h is a *generating function*[2] for the dynamics of $\widetilde{\mathcal{F}}$ (or $\mathcal{F}$):

$$\begin{cases} r &= \frac{\partial h(s,s')}{\partial s},\\ r' &= -\frac{\partial h(s,s')}{\partial s'}.\end{cases}$$

Observe that $\mathcal{F}$ is a negative twist map, i.e., $\frac{\partial s'}{\partial r}(s,r)<0$, and that $-\frac{\partial^2 h}{\partial s\partial s'}(s,s')>0$. Let us also recall that the time-reversal involution $\mathcal{I}\colon(s,r)\mapsto(s,-r)$ conjugates the billiard map with its inverse, i.e., $\mathcal{F}\circ\mathcal{I}=\mathcal{I}\circ\mathcal{F}^{-1}$.

Due to the strict convexity of the obstacles, the dynamics is of Axiom A type (see [48, 49] or [62, Subsection 2.1] for more details). In connection with Remark 1.6,

[2] In the following, we will also refer to the function $\tau=\tau(s,r)$ as a generating function.

let us also recall that several works have been dedicated to the smoothness of stable/unstable laminations of open dispersing billiards (see Morita [48] and Stoyanov [62]). Besides, if the non-wandering set

$$\Omega(\mathcal{F}) := \bigcap_{j\in\mathbb{Z}} \mathcal{F}^j(\mathcal{M})$$

has no tangential collisions, then it is a hyperbolic set; moreover, we have $\Omega(\mathcal{F}) = \Lambda \cup F$, $\Lambda \cap F = \emptyset$, where F is a finite union of periodic points, and Λ can be written as a disjoint union $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_m$, $m \geq 1$, each Λ_i being a horseshoe such that $\mathcal{F}|_{\Lambda_i}$ is conjugated to a non-trivial subshift of finite type. In the following, for each point $x \in \Omega(\mathcal{F})$, we denote by $\mathcal{W}^s_{\mathcal{F}}(x)$, resp. $\mathcal{W}^u_{\mathcal{F}}(x)$, its stable, resp. unstable manifold for the map $\mathcal{F}$. The non-wandering set $\Omega(\Phi)$ of the billiard flow Φ is the set of all points in the orbit of some $x \in \Omega(\mathcal{F})$. Similarly, when speaking about a basic set for Φ in the following, we mean the union of orbits of all the points in a set Λ_i as above, for some $i \in \{1, \dots, m\}$. Let us define the quotient set $\Lambda_i^\tau := \{(s,r,t) \in \Lambda_i \times \mathbb{R} : 0 \leq t \leq \tau(s,r)\}/\approx$, where

$$((s,r),\tau(s,r)) \approx (\mathcal{F}(s,r), 0).$$

We can identify Λ_i^τ with the set $\{(s,r,t) \in \Lambda_i \times \mathbb{R} : 0 \leq t < \tau(s,r)\}$, and define the projection $\Pi\colon \Lambda_i^\tau \to \partial\mathcal{D}$ as[3]

$$\Pi(s,r,t) := s \simeq \Upsilon(s). \tag{1.5}$$

The billiard flow Φ restricted to the orbits of points in Λ_i is defined at all times and can be seen as a *special flow* induced by the vertical vector field $X = \frac{\partial}{\partial t} = (0,0,1)$ on Λ_i^τ. Actually, the (x,y,ω)-coordinates introduced above are slightly more convenient, as they also allow to describe points which are not in $\Omega(\Phi)$: for any point $(s,r,t) \in \Lambda_i^\tau$, with $r = \sin\varphi \in (-1,1)$, $t \in [0,\tau(s,r))$, we let $U(s,r,t) := (x,y,\omega) \in \mathfrak{M}$ be the corresponding (x,y,ω)-coordinates, with $x = x(s,r,t)$, $y = y(s,r,t)$, and

$$\omega = \omega(s,r) = \angle\big(R_{-\frac{\pi}{2}+\varphi}\big(\Upsilon'(s)\big),(1,0)\big) = \angle\big(R_{-\frac{\pi}{2}+\arcsin r}\big(\Upsilon'(s)\big),(1,0)\big),$$
$$\big(x(s,r,t),y(s,r,t)\big) = \Upsilon(s) + t(\cos\omega,\sin\omega),$$

where $\Upsilon(s)$ is the associated point of $\partial\mathcal{D}$, and for $\theta \in \mathbb{R}$, R_θ is the rotation of angle θ. The map Υ is $\mathcal{C}^k$, hence the change of coordinates U is of class $\mathcal{C}^{k-1}$.

Claim 1.9. *The contact form $\alpha = \lambda + dt$ is adapted to Λ_i^τ (recall Definition 1.2).*

Proof. Let us verify that $\imath_X\alpha = 1$ and $\imath_X d\alpha = 0$. Indeed, for any $(s,r,t) = ((s,r),t) \in \mathcal{M} \times \mathbb{R}$, we have

$$\alpha(s,r,t)\big(X(s,r,t)\big) = (\lambda(s,r) + dt)\frac{\partial}{\partial t} = 1,$$

and

$$d\alpha(s,r,t)(X(s,r,t)) = d\lambda(s,r)\frac{\partial}{\partial t} = 0.$$

Besides, for $W\colon (s,r,t) \mapsto (\mathcal{F}(s,r), t - \tau(s,r))$, we have

$$\begin{aligned} W^*\alpha(s,r,t) &= \alpha \circ W(s,r,t) = \alpha(\mathcal{F}(s,r), t-\tau(s,r)) \\ &= \lambda(\mathcal{F}(s,r)) + d(t - \tau(s,r)) = \mathcal{F}^*\lambda(s,r) + dt - d\tau(s,r) \\ &= \lambda(s,r) + d\tau(s,r) + dt - d\tau(s,r) = \alpha(s,r,t). \end{aligned}$$

[3] By a slight abuse of notation, we will also denote by $\Pi\colon \Lambda_i \to \partial\mathcal{D}$ the projection $(s,r) \mapsto s$.

Therefore, α descends to an adapted contact form on Λ_i^τ. □

Let us also recall how the contact structure looks like in (x, y, ω)-coordinates. For each point $X = (x, y, \omega) \in \mathfrak{M}$, we let

$$T_X\mathfrak{M} \supset T_X^0\mathfrak{M} := \ker\big(-\sin\omega dx + \cos\omega dy\big) \cap \ker\big(d\omega\big),$$
$$T_X\mathfrak{M} \supset T_X^\perp\mathfrak{M} := \ker\big(\cos\omega dx + \sin\omega dy\big).$$

The one-dimensional subbundle $T^0\mathfrak{M} \subset T\mathfrak{M}$ and the two-dimensional subbundle $T^\perp\mathfrak{M} \subset T\mathfrak{M}$ are $D\Phi$-invariant. More precisely, for any $t \in \mathbb{R}$, the differential $D\Phi^t$ acts on $T^0\mathfrak{M} \oplus T^\perp\mathfrak{M}$ as follows (see Chernov-Markarian [12] for more details):

$$D\Phi^t(X) = \begin{bmatrix} 1 & 0 \\ 0 & D^\perp\Phi^t(X) \end{bmatrix}, \quad \forall\, X \in \mathfrak{M}.$$

In particular, $\cos\omega dx + \sin\omega dy$ is the contact form in (x, y, ω)-coordinates, and $T^\perp\mathfrak{M}$ is the associated contact distribution.

Definition 1.10. Let $\mathcal{D}_1, \mathcal{D}_2$ be two billiards with $\mathcal{C}^k$ boundaries, for some $k \geq 3$, and let Φ_1, Φ_2 be the associated billiard flows. If there exist two basic sets $\Lambda_1^{\tau_1} \subset \Omega(\Phi_1)$, $\Lambda_2^{\tau_2} \subset \Omega(\Phi_2)$, and an iso-length-spectral orbit equivalence between $\Phi_1|_{\Lambda_1^{\tau_1}}$ and $\Phi_2|_{\Lambda_2^{\tau_2}}$, we will simply say that $\mathcal{D}_1, \mathcal{D}_2$ are *iso-length-spectral* on $\Lambda_1^{\tau_1}, \Lambda_2^{\tau_2}$.

Theorem C (Smooth conjugacy of billiard maps of isospectral hyperbolic billiards)**.** *Let $\mathcal{D}_1, \mathcal{D}_2$ be two billiards with $\mathcal{C}^k$ boundaries, for some $k \geq 3$, and let Φ_1, Φ_2 be the associated billiard flows. Let us consider a basic set $\Lambda_i^{\tau_i}$ for Φ_i, $i = 1, 2$, and let Λ_i be the* horseshoe[4] *obtained by projecting $\Lambda_i^{\tau_i}$ onto the first two coordinates (s_i, r_i). If $\mathcal{D}_1, \mathcal{D}_2$ are iso-length-spectral on $\Lambda_1^{\tau_1}$, $\Lambda_2^{\tau_2}$, then there exists a map $\widetilde{\Psi}\colon (s_1, r_1, t_1) \mapsto (s_2, r_2, t_2)$ which conjugates Φ_1, Φ_2 on $\Lambda_1^{\tau_1}, \Lambda_2^{\tau_2}$ respectively. The map $\widetilde{\Psi}$ induces a conjugacy $\Psi\colon \Lambda_1 \to \Lambda_2$ between the respective billiard maps $\mathcal{F}_1|_{\Lambda_1}, \mathcal{F}_2|_{\Lambda_2}$ which is $\mathcal{C}^{k-1}$ in Whitney sense, and such that $\Psi^*(ds_2 \wedge dr_2) = ds_1 \wedge dr_1$ on Λ_1.*

Moreover, the respective generating functions τ_1, τ_2 of $\mathcal{F}_1, \mathcal{F}_2$ satisfy

$$\tau_2 \circ \Psi - \tau_1 = \chi \circ \mathcal{F}_1 - \chi \quad \text{on } \Lambda_1, \tag{1.6}$$

for some function $\chi\colon \Lambda_1 \to \mathbb{R}$ which is $\mathcal{C}^{k-1}$ in Whitney sense, such that

$$\Psi^*\lambda_2 - \lambda_1 = d\chi \quad \text{on } \Lambda_1, \quad \text{where } \lambda_i = -r_i ds_i,\ i = 1, 2. \tag{1.7}$$

Let $\mathcal{I}_i\colon (s_i, r_i) \mapsto (s_i, -r_i)$, $i = 1, 2$, be the respective time-reversal involutions. If $\Psi^{-1} \circ \mathcal{I}_2 \circ \Psi \circ \mathcal{I}_1|_{\Lambda_1}$ fixes $\mathcal{F}_1$-orbits, i.e., $\mathcal{I}_2 \circ \Psi(x_1)$ and $\Psi \circ \mathcal{I}_1(x_1)$ are in the same $\mathcal{F}_2$-orbit, for all $x_1 \in \Lambda_1$, and if there exists a 2-periodic point $x_1 \in \Lambda_1$ or a point $x_1 \in \Lambda_1 \cap \{r_1 = 0\}$ whose orbit is dense in Λ_1 and such that $\mathcal{F}_2^m \circ \Psi(x_1) \in \{r_2 = 0\}$, for some $m \in \mathbb{Z}$, then Ψ, resp. χ, can be chosen in a unique way such that

$$\Psi \circ \mathcal{I}_1|_{\Lambda_1} = \mathcal{I}_2 \circ \Psi|_{\Lambda_1}, \qquad \text{resp. } \chi \circ \mathcal{I}_1|_{\Lambda_1} = -\chi|_{\Lambda_1}. \tag{1.8}$$

The proof of Theorem C is given in Section 3.

Remark 1.11. Theorem C applies naturally to open dispersing billiards, as those exhibit uniformly hyperbolic dynamics. Yet, even in the case of convex billiards, generically, hyperbolic dynamics arises from *Aubry-Mather* periodic orbits with

[4]Let us recall that a *horseshoe* for a diffeomorphism f is a transitive, locally maximal hyperbolic set that is totally disconnected and not finite.

transverse heteroclinic intersections (see for instance [1, 36] for more details). Thus, our result may also be applied to the associated horseshoes.

Remark 1.12. The coboundary χ in Theorem C can be interpreted as the difference between stable (or unstable) actions for the billiard maps $\mathcal{F}_1, \mathcal{F}_2$. Indeed, fix a 2-periodic point $p_1 \in \Lambda_1$, and let $p_2 := \Psi(p_1) \in \Lambda_2$. Let us consider a point $x_1 \in \Lambda_1$ in the stable manifold $\mathcal{W}^s_{\mathcal{F}_1}(p_1)$ of p_1, and let $x_2 := \Psi(x_1) \in \mathcal{W}^s_{\mathcal{F}_2}(p_2) \cap \Lambda_2$. For $i = 1, 2$, we define the *stable action* of x_i as the sum of the following convergent series:

$$\mathcal{A}^s_{p_i,\mathcal{F}_i}(x_i) = \mathcal{A}^s_i(x_i) := \sum_{k=0}^{+\infty} \big(\tau_i \circ \mathcal{F}_i^k(x_i) - \tau_i \circ \mathcal{F}_i^k(p_i)\big).$$

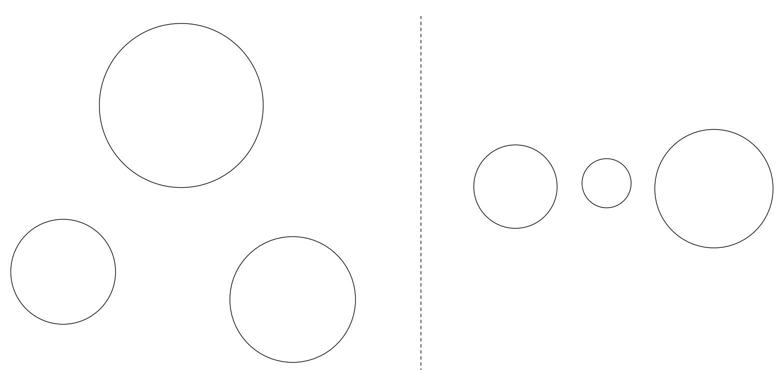
Since the two billiards have the same periodic length data, and since p_1, p_2 are 2-periodic, we have $\tau_1 \circ \mathcal{F}_1^k(p_1) = \tau_2 \circ \mathcal{F}_2^k(p_2)$, for each $k \in \mathbb{Z}$. Observe that $\lim_{k\to+\infty} \chi \circ \mathcal{F}_1^k(x_1) = \chi(p_1) = 0$ (as $\chi(p_1) = \chi \circ \mathcal{I}_1(p_1) = -\chi(p_1)$, by (1.8)). By (1.6), we thus conclude that

$$\mathcal{A}^s_1(x_1) - \mathcal{A}^s_2(x_2) = \sum_{k=0}^{+\infty} \big(\tau_1 \circ \mathcal{F}_1^k(x_1) - \tau_2 \circ \mathcal{F}_2^k(x_2)\big) = \chi(x_1) - \lim_{k\to+\infty} \chi \circ \mathcal{F}_1^k(x_1) = \chi(x_1),$$

i.e., $\chi(x_1)$ is the difference between the stable actions $\mathcal{A}^s_1(x_1)$ and $\mathcal{A}^s_2(\Psi(x_1))$.

1.4. **Open dispersing billiards without eclipse.** We now discuss the following important example (see for instance [47, 21] for more details). Fix an integer $\ell \geq 3$. We let $\mathbf{B}_{ne}(\ell) \subset \mathbf{B}(\ell)$ be the set of all billiards $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^{\ell} \mathcal{O}_i \in \mathbf{B}(\ell)$ which satisfy the following

Non-eclipse condition: The convex hull of any two obstacles is disjoint from any other obstacle.



$\mathcal{D} \in \mathbf{B}_{ne}(3)$ $\mathcal{D} \in \mathbf{B}(3) \setminus \mathbf{B}_{ne}(3)$

Let $\mathcal{F}$, resp. Φ be the associated billiard map, resp. billiard flow. The non-wandering set $\Omega(\mathcal{F})$ is reduced to a single basic set Λ. Moreover, $\mathcal{F}|_\Lambda$ is conjugated by some Hölder homeomorphism to the subshift of finite type associated with the transition matrix $(1 - \delta_{i,j})_{1 \leq i,j \leq \ell}$, where $\delta_{i,j} = 1$, when $i = j$, and $\delta_{i,j} = 0$ otherwise, when $i \neq j$. In other words, any *admissible* word $\varsigma \in \mathrm{Adm}_\infty$, i.e., such that $\varsigma = (\varsigma_j)_j \in \{1, \ldots, \ell\}^{\mathbb{Z}}$ with $\varsigma_{j+1} \neq \varsigma_j$, for all $j \in \mathbb{Z}$, can be realized by an orbit, and by hyperbolicity of the dynamics, this orbit is unique. We denote by $x(\varsigma) \in \Omega(\mathcal{F})$

the point with symbolic coding ς. Let $\mathrm{Adm} \subset \cup_{j\geq 2}\{1,\dots,\ell\}^j$ be the set of all finite words $\sigma = \sigma_1 \dots \sigma_j$, $j \geq 2$, such that $\sigma^\infty := \cdots\sigma\sigma\sigma\cdots \in \mathrm{Adm}_\infty$. It is the set of symbolic codings of periodic orbits. In particular, we may thus define the *marked length spectrum* $\mathcal{MLS}(\mathcal{D})$ as the map

$$\mathcal{MLS}(\mathcal{D})\colon \mathrm{Adm} \to \mathbb{R}, \quad \sigma \mapsto \mathcal{L}(\sigma),$$

where $\mathcal{L}(\sigma) = T_\Phi(x(\sigma^\infty))$ is the perimeter of the periodic orbit encoded by σ.

For any billiards $\mathcal{D}_1, \mathcal{D}_2 \in \mathbf{B}_{ne}(\ell)$ with respective billiard maps $\mathcal{F}_1, \mathcal{F}_2$, the restrictions $\mathcal{F}_1|_{\Omega(\mathcal{F}_1)}$, $\mathcal{F}_2|_{\Omega(\mathcal{F}_2)}$ are topologically conjugated in a canonical way, by sending a point $x_1 \in \Omega(\mathcal{F}_1)$ to the point $x_2 \in \Omega(\mathcal{F}_2)$ with the same coding. The billiard flows Φ_1, Φ_2 are thus orbit equivalent through some Hölder continuous orbit equivalence.

Corollary D. *Fix $\ell \geq 3$, and let $\mathcal{D}_1, \mathcal{D}_2 \in \mathbf{B}_{ne}(\ell)$ with $\mathcal{C}^k$ boundaries, for some $k \geq 3$. If $\mathcal{D}_1, \mathcal{D}_2$ have the same marked length spectrum, then the respective billiard maps $\mathcal{F}_1, \mathcal{F}_2$ are conjugated on $\Omega_1 := \Omega(\mathcal{F}_1)$, $\Omega_2 := \Omega(\mathcal{F}_2)$ by a map $\Psi\colon \Omega_1 \to \Omega_2$ that is $\mathcal{C}^{k-1}$ in Whitney sense, such that $\Psi^*(ds_2 \wedge dr_2) = ds_1 \wedge dr_1$ and $\Psi \circ \mathcal{I}_1 = \mathcal{I}_2 \circ \Psi$ on Ω_1, where $\mathcal{I}_i\colon (s_i, r_i) \mapsto (s_i, -r_i)$, for $i = 1, 2$, is the time-reversal involution. Moreover, (1.6)-(1.7)-(1.8) in Theorem C hold for some coboundary $\chi\colon \Lambda_1 \to \mathbb{R}$ which is $\mathcal{C}^{k-1}$ in Whitney sense.*

Acknowledgements: We thank Péter Bálint, Sylvain Crovisier, Jacques Féjoz, Andrey Gogolev, Colin Guillarmou, Umberto L. Hryniewicz, Thibault Lefeuvre, Jean-Pierre Marco, Federico Rodriguez Hertz, Disheng Xu for their encouragement and several useful discussions. We are especially grateful to Marie-Claude Arnaud, Jacopo De Simoi, Livio Flaminio, Vadim Kaloshin, and Ke Zhang for many conversations about this work, and for having pointed out a mistake in a previous version of the paper.

2. Smooth conjugacy classes for 3D Axiom A flows on basic sets

2.1. **Synchronization of the flows using periodic data.** Let us start by recalling the fact that an orbit equivalence between two hyperbolic flows can be upgraded to a flow conjugacy as long as the lengths of associated periodic orbits coincide.

Proposition 2.1. *Let $k \geq 2$, and let $\Phi_1 = (\Phi_1^t)_{t\in\mathbb{R}}$, resp. $\Phi_2 = (\Phi_2^t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^k$ Axiom A flow defined on a smooth manifold M_1, resp. M_2, and let Λ_1, resp. Λ_2 be a basic set for Φ_1, resp. Φ_2. Assume that there exists an orbit equivalence $\Psi_0\colon \Lambda_1 \to \Lambda_2$ differentiable along Φ_1-orbits, and that*

$$T_{\Phi_1}(x) = T_{\Phi_2}(\Psi(x)), \qquad \textit{for each } x \in \mathrm{Per}(\Phi_1) \cap \Lambda_1. \tag{2.1}$$

Then the flows Φ_1, Φ_2 are topologically conjugate, i.e., there exists a homeomorphism $\Psi\colon \Lambda_1 \to \Lambda_2$ such that

$$\Psi \circ \Phi_1^t(x) = \Phi_2^t \circ \Psi(x), \quad \textit{for all } (x,t) \in \Lambda_1 \times \mathbb{R}.$$

Proof. The proof is classical but we recall it here for completeness.

We fix an orbit equivalence $\Psi_0\colon \Lambda_1 \to \Lambda_2$ that is differentiable along Φ_1-orbits. Let X_1, X_2 be the respective flow vector fields of Φ_1, Φ_2, and let $L_{X_1}\Psi_0$ be the Lie derivative of Ψ_0 along Φ_1. As Ψ_0 sends Φ_1-orbits to Φ_2-orbits, it holds

$$L_{X_1}\Psi_0(x) = v_{\Psi_0}(x) X_2(\Psi_0(x)), \quad \text{for all } x \in \Lambda_1,$$

for some function $v_{\Psi_0}\colon \Lambda_1 \to \mathbb{R}$ which measures the "speed" of Ψ_0 along the flow direction. Observe that $v_{\Psi_0}(x) = \frac{d}{dt}|_{t=0}\theta(x,t)$.

By (2.1), for each $x \in \mathrm{Per}(\Phi_1) \cap \Lambda_1$ we have

$$\int_0^{T_{\Phi_1}(x)} dt = T_{\Phi_1}(x) = T_{\Phi_2}(\Psi_0(x)) = \int_0^{T_{\Phi_1}(x)} \frac{d}{ds}|_{s=0}\theta(\Phi_1^t(x), s)dt = \int_0^{T_{\Phi_1}(x)} v_{\Psi_0}(\Phi_1^t(x))dt,$$

hence

$$\frac{1}{T_{\Phi_1}(x)} \int_0^{T_{\Phi_1}(x)} \big(v_{\Psi_0}(\Phi_1^t(x)) - 1\big)dt = 0, \qquad \text{for each } x \in \mathrm{Per}(\Phi_1) \cap \Lambda_1.$$

We deduce from Livsic's theorem (see [39, Subsection 19.2]) that there exists a continuous function $u\colon \Lambda_1 \to \mathbb{R}$ differentiable along Φ_1-orbits such that $v_{\Psi_0} - 1 = L_{X_1}u$. Let us set $\Psi\colon x \mapsto \Phi_2^{-u(x)} \circ \Psi_0(x)$. Given any $x \in \Lambda_1$, we compute

$$\begin{aligned} v_\Psi(x)X_2(\Psi(x)) &= L_{X_1}\big(\Phi_2^{-u(x)} \circ \Psi_0\big)(x) \\ &= \lim_{t\to 0} \frac{1}{t}\Big(\Phi_2^{\theta(x,t)-u(\Phi_1^t(x))} \circ \Psi_0(x) - \Phi_2^{-u(x)} \circ \Psi_0(x)\Big) \\ &= X_2(\Phi_2^{-u(x)} \circ \Psi_0(x)) \lim_{t\to 0} \frac{1}{t}\Big(\theta(x,t) - u(\Phi_1^t(x)) + u(x)\Big) \\ &= \big(v_{\Psi_0}(x) - L_{X_1}u(x)\big)X_2\big(\Phi_2^{-u(x)} \circ \Psi_0(x)\big) = X_2(\Psi(x)), \end{aligned}$$

i.e., $v_\Psi \equiv 1$ on Λ_1.

As a result, the homeomorphism Ψ is a flow conjugacy between Φ_1 and Φ_2 on Λ_1:

$$\Psi \circ \Phi_1^t(x) = \Phi_2^t \circ \Psi(x), \quad \text{for all } (x,t) \in \Lambda_1 \times \mathbb{R}.$$

□

2.2. **Markov families for Axiom A flows on basic sets.** In this part, we recall some classical facts about Markov families for Axiom A flows on basic sets, following the presentation given in [11].

Let $k \geq 2$, and let $\Phi = (\Phi^t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^k$ Axiom A flow defined on a smooth manifold M.

Definition 2.2 (Rectangle, proper family)**.** A closed subset $R \subset M$ is called a *rectangle* if there is a small closed codimension one smooth disk $D \subset M$ transverse to the flow Φ such that $R \subset D$, and for any $x, y \in R$, the point

$$[x,y]_R := D \cap \mathcal{W}^{cs}_{\Phi,\mathrm{loc}}(x) \cap \mathcal{W}^{cu}_{\Phi,\mathrm{loc}}(y)$$

exists and also belongs to R. A rectangle R is called *proper* if $R = \overline{\mathrm{int}(R)}$ in the topology of D. For any rectangle R and any $x \in R$, we let

$$\mathcal{W}^s_R(x) := R \cap \mathcal{W}^{cs}_{\Phi,\mathrm{loc}}(x), \quad \mathcal{W}^u_R(x) := R \cap \mathcal{W}^{cu}_{\Phi,\mathrm{loc}}(x).$$

A finite collection of proper rectangles $\mathcal{R} = \{R_1, \ldots, R_m\}$, $m \geq 1$, is called a *proper family of size* $\varepsilon > 0$ if:

1. (1) $M = \{\Phi^t(\mathcal{S}) : t \in [-\varepsilon, 0]\}$, where $\mathcal{S} := R_1 \cup \cdots \cup R_m$;
2. (2) $\mathrm{diam}(D_i) < \varepsilon$, for each $i = 1, \ldots, m$, where $D_i \supset R_i$ is a disk as above;
3. (3) for any $i \neq j$, $D_i \cap \{\Phi^t(D_j) : t \in [0,\varepsilon]\} = \emptyset$ or $D_j \cap \{\Phi^t(D_i) : t \in [0,\varepsilon]\} = \emptyset$.

The set $\mathcal{S}$ is called a *cross-section* of the flow Φ.

Notation 2.3. Let $\mathcal{R} = \{R_1, \dots, R_m\}$ be a proper family with $m \geq 1$ elements.

The cross-section $\mathcal{S} := R_1 \cup \cdots \cup R_m$ is associated with a Poincaré map $\mathcal{F} \colon \mathcal{S} \to \mathcal{S}$, where for any $x \in \mathcal{S}$, we let $\mathcal{F}(x) := \Phi^{\tau_{\mathcal{S}}(x)}(x)$, the function $\tau_{\mathcal{S}} \colon \mathcal{S} \to \mathbb{R}_+$ being the first return time on $\mathcal{S}$, i.e., $\tau_{\mathcal{S}}(x) := \inf\{t > 0 : \Phi^t(x) \in \mathcal{S}\} > 0$, for all $x \in \mathcal{S}$.

Besides, for $* = s, u$ and $x \in R_i$, $i \in \{1, \dots, m\}$, we also let $\mathcal{W}^*_{\mathcal{F}}(x) := \mathcal{W}^*_{R_i}(x)$.

Definition 2.4 (Markov family)**.** Given some small $\varepsilon > 0$, and some integer $m \geq 1$, a proper family $\mathcal{R} = \{R_1, \dots, R_m\}$ of size ε, with Poincaré map $\mathcal{F}$, is called a *Markov family* if it satisfies the following Markov property: for any $x \in \mathrm{int}(R_i) \cap \mathcal{F}^{-1}(\mathrm{int}(R_j)) \cap \mathcal{F}(\mathrm{int}(R_k))$, with $i, j, k \in \{1, \dots, m\}$, it holds

$$\mathcal{W}^s_{R_i}(x) \subset \overline{\mathcal{F}^{-1}(\mathcal{W}^s_{R_j}(\mathcal{F}(x)))} \quad \text{and} \quad \mathcal{W}^u_{R_i}(x) \subset \overline{\mathcal{F}(\mathcal{W}^u_{R_k}(\mathcal{F}^{-1}(x)))}.$$

Theorem 2.5 (see Theorem 2.5 in [7], see also Theorem 4.2 in [11])**.** *The restriction of an Axiom A flow to any basic set has a Markov family of arbitrary small size.*

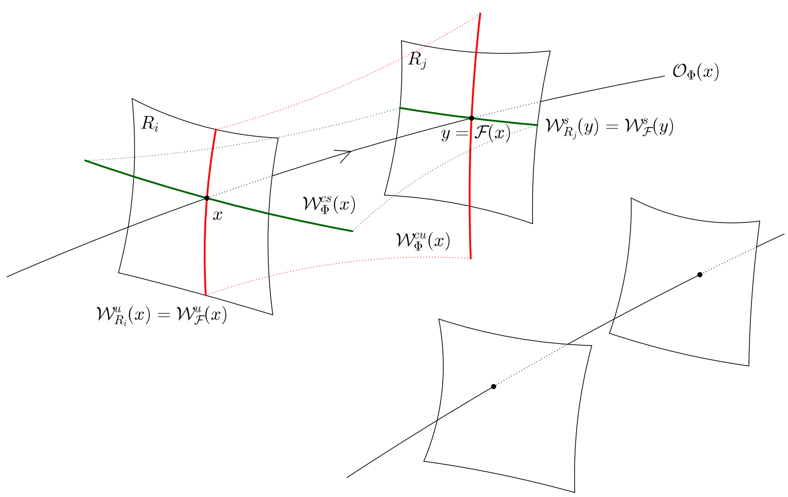


FIGURE 2. Markov family for the flow Φ.

2.3. **Quadrilaterals and temporal displacements.** Let $\Phi = (\Phi^t)_{t \in \mathbb{R}}$ be a $\mathcal{C}^k$ Axiom A flow on a smooth manifold M, with $k \geq 2$, and fix a basic set Λ for Φ.

Definition 2.6 (Quadrilaterals)**.** A *quadrilateral* is a quadruple $\mathscr{Q} = (x_0, x_1, x_2, x_3) \subset \Lambda^4$ such that $x_1 \in \mathcal{W}^s_{\Phi,\mathrm{loc}}(x_0)$, $x_2 \in \mathcal{W}^u_{\Phi,\mathrm{loc}}(x_1)$ and $x_3 \in \mathcal{W}^s_{\Phi,\mathrm{loc}}(x_2) \cap \mathcal{W}^{cu}_{\Phi,\mathrm{loc}}(x_0)$. We let $x_4 = x_4(\mathscr{Q}) := \mathcal{W}^c_{\Phi,\mathrm{loc}}(x_0) \cap \mathcal{W}^u_{\Phi,\mathrm{loc}}(x_3)$. In particular, $x_4 = \Phi^t(x_0)$, for some time $t = t(\mathscr{Q}) \in \mathbb{R}$.

Let us consider a proper Markov family $\mathcal{R} = \{R_1, \dots, R_m\}$ for $\Phi_{|\Lambda}$ of size ε, for some integer $m \geq 1$ and some small $\varepsilon > 0$. Let $\mathcal{F}$ be the associated Poincaré map, and set $\mathcal{S} := R_1 \cup \cdots \cup R_m$. We denote by $\overline{\Lambda} := \Lambda \cap \mathcal{S}$ the trace of Λ on $\mathcal{S}$.

We say that a quadrilateral $\mathscr{Q} = (x_0, x_1, x_2, x_3) \subset \Lambda^4$ is $\mathcal{R}$*-good* if $x_0 \in R_i$ for some $i = i(\mathscr{Q}) \in \{1, \dots, m\}$, and $x_j \in \cup_{t \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})} \Phi^t(R_i)$, for each $j \in \{1, \dots, 4\}$. Note that, up to time translation, there is no loss of generality to assume that $x_0 \in \mathcal{S}$. For any such quadrilateral, and for $j \in \{1, \dots, 4\}$, we denote by $\bar{x}_j$ the projection along the flow line of x_j on R_i, and we let $\overline{\mathscr{Q}} := (\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$. Note that $\bar{x}_0, \dots, \bar{x}_3 \in \overline{\Lambda}$; besides, $\bar{x}_1 \in \mathcal{W}^s_{R_i}(\bar{x}_0)$, $\bar{x}_2 \in \mathcal{W}^u_{R_i}(\bar{x}_1)$, and $\bar{x}_3 \in \mathcal{W}^s_{R_i}(\bar{x}_2) \cap \mathcal{W}^u_{R_i}(\bar{x}_0)$.

Definition 2.7 (s/u-holonomies)**.** Fix $i \in \{1,\dots,m\}$, and let $z_0, z_1 \in R_i \cap \Lambda$ be such that $z_1 \in \mathcal{W}^s_{R_i}(z_0)$. We define the *stable holonomy* $H^s_{\mathcal{S}}(z_0,z_1) \in \mathbb{R}$ as the time $t \in \mathbb{R}$ with smallest absolute value $|t|$ such that $\Phi^t(z_1) \in \mathcal{W}^s_{\Phi,\mathrm{loc}}(z_0)$. Similarly, for any $z_0, z_1 \in R_i \cap \Lambda$, $z_1 \in \mathcal{W}^u_{R_i}(z_0)$, we define the *unstable holonomy* $H^u_{\mathcal{S}}(z_0,z_1) \in \mathbb{R}$ as the time $t \in \mathbb{R}$ with smallest absolute value $|t|$ such that $\Phi^t(z_1) \in \mathcal{W}^u_{\Phi,\mathrm{loc}}(z_0)$.

Lemma 2.8. *For any $i \in \{1,\dots,m\}$, and for any $z_0, z_1 \in \mathcal{W}^s_{R_i}(z_0)$, it holds*

$$H^s_{\mathcal{S}}(z_0,z_1) = \sum_{j=0}^{+\infty} \tau_{\mathcal{S}}(\mathcal{F}^j(z_1)) - \tau_{\mathcal{S}}(\mathcal{F}^j(z_0)).$$

Proof. Fix $i \in \{1,\dots,m\}$, and let $z_0, z_1 \in R_i \cap \Lambda$ be such that $z_1 \in \mathcal{W}^s_{R_i}(z_0)$. We abbreviate $H := H^s_{\mathcal{S}}(z_0,z_1)$ and set $z_2 = \Phi^H(z_1)$. Fix $\varepsilon > 0$ arbitrarily small. As $z_1 \in \mathcal{W}^s_{R_i}(z_0)$ and $z_2 \in \mathcal{W}^s_{\Phi,\mathrm{loc}}(z_0)$, for $n \gg 1$ sufficiently large, it holds

$$\begin{aligned} d(\mathcal{F}^n(z_0), \mathcal{F}^n(z_1)) &< \varepsilon, \\ d(\Phi^{t_n}(z_0), \Phi^{t_n}(z_2)) &< \varepsilon, \end{aligned} \tag{2.2}$$

with $\mathcal{F}^n(z_0) = \Phi^{t_n}(z_0)$ and $t_n := \sum_{j=0}^{n-1} \tau_{\mathcal{S}}(\mathcal{F}^j(z_0))$. Set $u_n := \sum_{j=0}^{n-1} \tau_{\mathcal{S}}(\mathcal{F}^j(z_1))$, so that $\mathcal{F}^n(z_1) = \Phi^{u_n}(z_1)$. The points $\mathcal{F}^n(z_0), \mathcal{F}^n(z_1)$ are exponentially close, and $\tau_{\mathcal{S}}$ is Lipschitz, hence the sequence $(u_n - t_n)_{n \geq 1}$ converges to some limit $\ell \in \mathbb{R}$. Since $z_2 = \Phi^H(z_1)$, and by the triangular inequality, (2.2) yields

$$d(\Phi^{u_n}(z_1), \Phi^{t_n+H}(z_1)) < 2\varepsilon.$$

As we are considering local manifolds, we deduce that $|u_n - t_n - H| < C\varepsilon$, for some uniform constant $C > 0$. Letting $n \to +\infty$, we get $\ell = H$, i.e.,

$$H = \sum_{j=0}^{+\infty} \tau_{\mathcal{S}}(\mathcal{F}^j(z_1)) - \tau_{\mathcal{S}}(\mathcal{F}^j(z_0)).$$

□

Using the same ideas as in Lemma 2.8, we have the following

Lemma 2.9. *For any $i \in \{1,\dots,m\}$, and for any $z_0, z_1 \in \mathcal{W}^u_{R_i}(z_0)$, it holds*

$$H^u_{\mathcal{S}}(z_0,z_1) = \sum_{j=-\infty}^{-1} \tau_{\mathcal{S}}(\mathcal{F}^j(z_0)) - \tau_{\mathcal{S}}(\mathcal{F}^j(z_1)).$$

Let $\mathscr{Q} = (x_0, x_1, x_2, x_3) \subset \Lambda^4$ be a $\mathcal{R}$-good quadrilateral, with $x_0 \in R_i$, $i \in \{1,\dots,m\}$. Let $x_4 = x_4(\mathscr{Q})$, and let $\overline{\mathscr{Q}} := (\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$. As $\bar{x}_1 \in \mathcal{W}^s_{R_i}(\bar{x}_0)$, $\bar{x}_2 \in \mathcal{W}^u_{R_i}(\bar{x}_1)$, and $\bar{x}_3 \in \mathcal{W}^s_{R_i}(\bar{x}_2) \cap \mathcal{W}^u_{R_i}(\bar{x}_0)$, we may define the *temporal displacement* $H(\mathscr{Q}) \in \mathbb{R}$ as

$$H(\mathscr{Q}) := H^s_{\mathcal{S}}(\bar{x}_0,\bar{x}_1) + H^u_{\mathcal{S}}(\bar{x}_1,\bar{x}_2) + H^s_{\mathcal{S}}(\bar{x}_2,\bar{x}_3) + H^u_{\mathcal{S}}(\bar{x}_3,\bar{x}_0). \tag{2.3}$$

By Lemma 2.8 and 2.9, we have:

$$H(\mathscr{Q}) = \lim_{m,n,m',n' \to +\infty} \Big[-\sum_{j=-m'}^{n-1} \tau_{\mathcal{S}}(\mathcal{F}^j(\bar{x}_0)) + \sum_{j=-m}^{n-1} \tau_{\mathcal{S}}(\mathcal{F}^j(\bar{x}_1)) - \sum_{j=-m}^{n'-1} \tau_{\mathcal{S}}(\mathcal{F}^j(\bar{x}_2)) + \sum_{j=-m'}^{n'-1} \tau_{\mathcal{S}}(\mathcal{F}^j(\bar{x}_3)) \Big] \tag{2.4}$$

$$= \lim_{m,n,m',n' \to +\infty} \big[-\tau_{\mathcal{S}}^{m',n}(\bar{x}_0) + \tau_{\mathcal{S}}^{m,n}(\bar{x}_1) - \tau_{\mathcal{S}}^{m,n'}(\bar{x}_2) + \tau_{\mathcal{S}}^{m',n'}(\bar{x}_3) \big], \tag{2.5}$$

where for any point $z \in \mathcal{S}$, and for any integers $m, n \geq 0$, we let

$$\tau_{\mathcal{S}}^{m,n}(z) := \sum_{j=-m}^{n-1} \tau_{\mathcal{S}}(\mathcal{F}^j(z)). \tag{2.6}$$

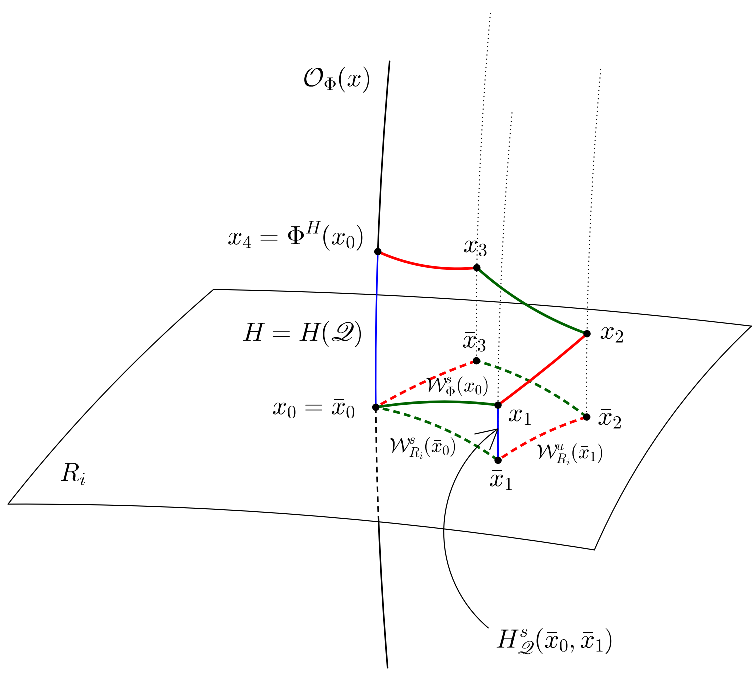


Figure 3. Quadrilaterals and temporal displacements.

2.4. **Periodic approximations of temporal displacements.** Let us recall the following fact.

Lemma 2.10. *For each $i \in \{1, \ldots, m\}$, the stable holonomies $H^s_{\mathcal{S}}(y_0, y_1)$, resp. unstable holonomies $H^u_{\mathcal{S}}(z_0, z_1)$, depend continuously on the points $y_0, y_1 \in R_i \cap \Lambda$, $y_1 \in \mathcal{W}^s_{R_i}(y_0)$, resp. on the points $z_0, z_1 \in R_i \cap \Lambda$, $z_1 \in \mathcal{W}^u_{R_i}(z_0)$.*

Proof. Let us consider the case where $y_0, y_1 \in R_i \cap \Lambda$, $y_1 \in \mathcal{W}^s_{R_i}(y_0)$, the other case is analogous. By definition, the stable holonomy $H^s_{\mathcal{S}}(y_0, y_1)$ satisfies

$$\mathcal{W}^s_{\Phi,\mathrm{loc}}(y_0) \cap \mathcal{W}^c_{\Phi,\mathrm{loc}}(y_1) = \{\Phi^{H^s_{\mathcal{S}}(y_0,y_1)}(y_1)\}.$$

As the invariant manifolds vary continuously, the intersection of the two sets on the left hand side depends continuously on the pair y_0, y_1, with $y_1 \in \mathcal{W}^s_{R_i}(y_0)$. By looking at the right hand side, we conclude that the holonomies are continuous. □

The main goal of this section is to show the following proposition, whose content already appears in the work of Otal [54].

Proposition 2.11. *For any $\mathcal{R}$-good quadrilateral $\mathscr{Q} = (x_0, x_1, x_2, x_3) \in \Lambda^4$, the quantity $H(\mathscr{Q})$ is determined by the lengths of periodic orbits.*

Proposition 2.11 is a direct outcome of Lemma 2.12 and Proposition 2.13 below.

Lemma 2.12. *For any $\mathcal{R}$-good quadrilateral $\mathscr{Q} = (x_0, x_1, x_2, x_3) \in \Lambda^4$, there exists a sequence $(\mathscr{Q}^n)_{n\in\mathbb{N}} \in (\Lambda^4)^{\mathbb{N}}$ of $\mathcal{R}$-good quadrilaterals $\mathscr{Q}^n = (x_0^n, x_1^n, x_2^n, x_3^n)$ with $x_0^n, x_2^n \in \mathrm{Per}(\Phi)$ such that $\lim_{n\to+\infty} \mathscr{Q}^n = \mathscr{Q}$, i.e., $\lim_{n\to\infty} x_j^n = x_j$, for each $j = 0, \dots, 3$. In particular, it holds*

$$H(\mathscr{Q}) = \lim_{n\to+\infty} H(\mathscr{Q}^n).$$

Proof. Fix a $\mathcal{R}$-good quadrilateral $\mathscr{Q} = (x_0, x_1, x_2, x_3) \in \Lambda^4$, with $x_0 \in R_i$, $i \in \{1, \dots, m\}$, and let $\overline{\mathscr{Q}} := (\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$ be the projection of $\mathscr{Q}$ on R_i as before.

As periodic points are dense in Λ, for $j = 0, 2$, there exists a sequence $(\bar{x}_j^n)_{n\in\mathbb{N}} \in \big(\mathrm{Per}(\Phi) \cap R_i\big)^{\mathbb{N}}$ of periodic points such that $\lim_{n\to+\infty} \bar{x}_j^n = \bar{x}_j$. Let $\bar{x}_1^n := [\bar{x}_0^n, \bar{x}_2^n]_{R_i}$ and $\bar{x}_3^n := [\bar{x}_2^n, \bar{x}_0^n]_{R_i}$, so that the lift $\mathscr{Q}^n := (x_0^n, x_1^n, x_2^n, x_3^n)$ of $\overline{\mathscr{Q}}^n := (\bar{x}_0^n, \bar{x}_1^n, \bar{x}_2^n, \bar{x}_3^n)$ is a $\mathcal{R}$-good quadrilateral, where

$$x_0^n := \bar{x}_0^n, \qquad x_1^n := \Phi^{H^s_{\mathcal{S}}(\bar{x}_0^n,\bar{x}_1^n)}(\bar{x}_1^n),$$
$$x_2^n := \Phi^{H^s_{\mathcal{S}}(\bar{x}_0^n,\bar{x}_1^n)+H^u_{\mathcal{S}}(\bar{x}_1^n,\bar{x}_2^n)}(\bar{x}_2^n), \qquad x_3^n := \Phi^{H^s_{\mathcal{S}}(\bar{x}_0^n,\bar{x}_1^n)+H^u_{\mathcal{S}}(\bar{x}_1^n,\bar{x}_2^n)+H^s_{\mathcal{S}}(\bar{x}_2^n,\bar{x}_3^n)}(\bar{x}_3^n),$$

and $x_0^n, x_2^n \in \mathrm{Per}(\Phi)$. Clearly, we have $\lim_{n\to+\infty} \mathscr{Q}^n = \mathscr{Q}$. By the definition (2.3) of temporal displacements in terms of holonomies, and by Lemma 2.10, the function $\widetilde{\mathscr{Q}} \mapsto H(\widetilde{\mathscr{Q}})$ is continuous. Thus, we conclude that $H(\mathscr{Q}) = \lim_{n\to+\infty} H(\mathscr{Q}^n)$. □

In the following, thanks to Theorem 2.5, we can and will assume that the size of the Markov family $\mathcal{R}$ is chosen suitably small (see Definition 2.4 and Theorem 2.5).

Proposition 2.13. *For any $\mathcal{R}$-good quadrilateral $\mathscr{Q} = (x_0, x_1, x_2, x_3) \in \Lambda^4$ such that $x_0, x_2 \in \mathrm{Per}(\Phi)$, the quantity $H(\mathscr{Q})$ is determined by the lengths of periodic orbits. More precisely, there exists a sequence $(x^n)_{n\in\mathbb{N}} \in \mathrm{Per}(\Phi)^{\mathbb{N}}$ of periodic points such that for any $\varepsilon > 0$, there exists an integer $N_0(\varepsilon) \in \mathbb{N}$ such that*

$$\Big| H(\mathscr{Q}) - \big[T_\Phi(x^n) - 2nT_\Phi(x_0) - 2nT_\Phi(x_2)\big] \Big| < \varepsilon, \quad \forall\, n \geq N_0(\varepsilon).$$

Proof. Fix a $\mathcal{R}$-good quadrilateral $\mathscr{Q} = (x_0, x_1, x_2, x_3) \in \Lambda^4$, with $x_0 \in R_i$, $i \in \{1, \dots, m\}$ and $x_0, x_2 \in \mathrm{Per}(\Phi)$, and let $\overline{\mathscr{Q}} := (\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$ be the projection of $\mathscr{Q}$ on R_i. Let $p_0 \geq 1, p_2 \geq 1$ be the prime period of $\bar{x}_0, \bar{x}_2$, respectively, with respect to $\mathcal{F}$. Note that $\bar{x}_1 = [\bar{x}_0, \bar{x}_2]_{R_i}$ and $\bar{x}_3 = [\bar{x}_2, \bar{x}_0]_{R_i}$ are heteroclinic intersections between the invariant manifolds of the periodic points $\bar{x}_0, \bar{x}_2$.

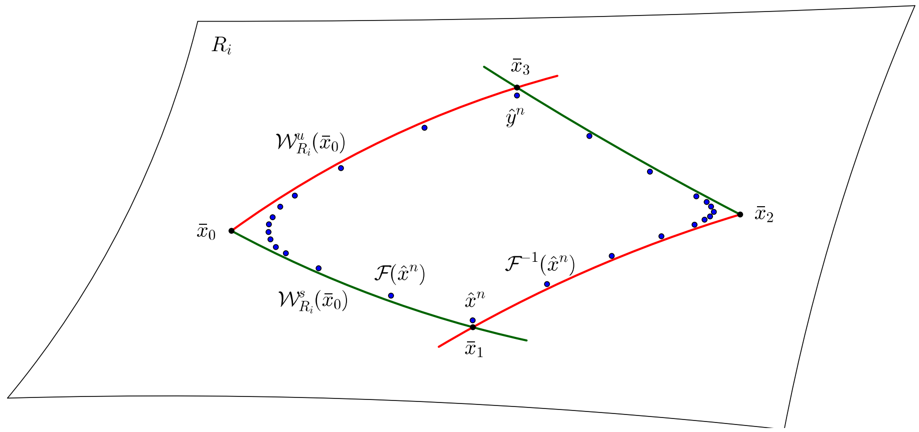


FIGURE 4. Shadowing periodic orbits with a prescribed combinatorics.

As Λ is a basic set, the dynamics can be coded symbolically, using some finite alphabet $\mathscr{A}$. In the following, for each finite word σ in $\mathscr{A}$, we denote by $|\sigma| \in \mathbb{N}$ the length of σ.

The periodic points $\bar{x}_0$, $\bar{x}_2$, correspond to a finite sequence of symbols σ_0, σ_2 respectively, with $|\sigma_0| = p_0$ and $|\sigma_2| = p_2$. In other words, denoting $\sigma_0 = \sigma_0^0 \dots \sigma_0^{p_0-1}$ and $\sigma_2 = \sigma_2^0 \dots \sigma_2^{p_2-1}$, for $j = 0, 2$, it holds

$$\bar{x}_j \longleftrightarrow \dots \sigma_j \sigma_j \underset{\uparrow}{\sigma_j^0} \sigma_j^1 \dots \sigma_j^{p_j-1} \sigma_j \sigma_j \dots$$

The point $\bar{x}_1$ is a heteroclinic intersection between the local manifolds $\mathcal{W}^s_{R_i}(\bar{x}_0)$ and $\mathcal{W}^u_{R_i}(\bar{x}_2)$, hence its symbolic coding is

$$\bar{x}_1 \longleftrightarrow \dots \sigma_2 \sigma_2 \sigma_2^0 \dots \sigma_2^{p_2-1} \underset{\uparrow}{\sigma_0^0} \sigma_0^1 \dots \sigma_0^{p_0-1} \sigma_0 \sigma_0 \dots$$

Similarly, the symbolic coding of $\bar{x}_3$ is

$$\bar{x}_3 \longleftrightarrow \dots \sigma_0 \sigma_0 \sigma_0^0 \dots \sigma_0^{p_0-1} \underset{\uparrow}{\sigma_2^0} \sigma_2^1 \dots \sigma_2^{p_0-1} \sigma_2 \sigma_2 \dots$$

For each integer $n \geq 0$, we define a periodic point $\hat{x}^n$ for $\mathcal{F}$ encoded by the finite word $\hat{\sigma}_n := \underbrace{\sigma_0 \dots \sigma_0}_{2n} \underbrace{\sigma_2 \dots \sigma_2}_{2n}$, i.e.,

$$\hat{x}^n \longleftrightarrow \dots \hat{\sigma}_n \hat{\sigma}_n \underbrace{\sigma_0 \dots \sigma_0}_{2n} \underbrace{\sigma_2 \dots \sigma_2}_{2n} \underset{\uparrow}{\sigma_0^0} \sigma_0^1 \dots \sigma_0^{p_0-1} \underbrace{\sigma_0 \dots \sigma_0}_{2n-1} \underbrace{\sigma_2 \dots \sigma_2}_{2n} \hat{\sigma}_n \hat{\sigma}_n \dots$$

Thus, the point $\hat{x}^n$ is $2(p_0 + p_2)n$-periodic for $\mathcal{F}$. In the following, we will denote by x^n the periodic point for the flow Φ corresponding to the point $\hat{x}^n$.

Lemma 2.14. *For any $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ such that for each integer $n \geq N(\varepsilon)$, the following inequalities hold:*

$$\left|\sum_{k=-np_2}^{np_0-1}\Big[\tau_{\mathcal{S}}(\mathcal{F}^k(\hat{x}^n)) - \tau_{\mathcal{S}}(\mathcal{F}^k(\bar{x}_1))\Big]\right| < \varepsilon, \tag{2.7}$$

$$\left|\sum_{k=-np_0}^{np_2-1}\Big[\tau_{\mathcal{S}}(\mathcal{F}^k(\hat{y}^n)) - \tau_{\mathcal{S}}(\mathcal{F}^k(\bar{x}_3))\Big]\right| < \varepsilon, \tag{2.8}$$

where we have set $\hat{y}^n := \mathcal{F}^{2np_0}(\hat{x}^n) = \mathcal{F}^{-2np_2}(\hat{x}^n)$.

Remark 2.15. Observe that the two sums in (2.7)-(2.8) involving $\hat{x}^n, \hat{y}^n$ add up to

$$\sum_{k=-np_2}^{np_0-1}\tau_{\mathcal{S}}(\mathcal{F}^k(\hat{x}^n)) + \sum_{k=-np_0}^{np_2-1}\tau_{\mathcal{S}}(\mathcal{F}^k(\hat{y}^n)) = T_\Phi(x^n). \tag{2.9}$$

Proof of Lemma 2.14. By looking at the symbolic codings of $\hat{x}^n$ and $\bar{x}_1$, we see that they have the same symbolic past (resp. future) for at least $2np_2$ (resp. $2np_0$) steps of iterations under $\mathcal{F}$. In particular, for each $k \in \{-np_2, \ldots, np_0 - 1\}$, $\mathcal{F}^k(\hat{x}^n)$ and $\mathcal{F}^k(\bar{x}_1)$ have the same symbolic past (resp. future) for at least np_2 (resp. np_0) steps of iterations. By hyperbolicity of $\mathcal{F}$, for some constant $\lambda \in (0,1)$, we thus have

$$d\big(\mathcal{F}^k(\hat{x}^n), \mathcal{F}^k(\bar{x}_1)\big) = O(\lambda^n), \quad \forall\, k \in \{-np_2, \ldots, np_0 - 1\}\,. \tag{2.10}$$

Indeed, without loss of generality (assuming that the size of the Markov family $\mathcal{R}$ is sufficiently small, see Definition 2.4 and Theorem 2.5), we may assume that each of these points belongs to some small neighborhood of the orbit of $\bar{x}_0$ (resp. of $\bar{x}_2$) where the dynamics $\mathcal{F}^{p_0}$ (resp. $\mathcal{F}^{p_2}$) is conjugated to the differential $D\mathcal{F}^{p_0}$ (resp. $D\mathcal{F}^{p_2}$) calculated at some point of the orbit of $\bar{x}_0$ (resp. $\bar{x}_2$). More precisely, let R_i be the rectangle containing the point $\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3$; by Belickii's linearization theorem [6], for $j = 0, 2$, there exist a neighborhood $\mathcal{U}_j$ of $\bar{x}_j$ (which we assume here to contain R_i), a neighborhood $\mathcal{V}_j \subset \mathbb{R}^2$ of $(0,0)$, and a $\mathcal{C}^1$-diffeomorphism $\chi_j : \mathcal{U}_j \to \mathcal{V}_j$, such that

$$\chi_j \circ \mathcal{F}^{p_j} \circ \chi_j^{-1} = D\mathcal{F}^{p_j}(\bar{x}_j).$$

Note that $\hat{x}_n$ tends to $\bar{x}_1$ as $n \to +\infty$: it is a consequence of the fact that the symbolic codings of $\hat{x}_n$ and $\bar{x}_1$ match on larger and larger chunks as $n \to +\infty$. Similarly, $\hat{y}_n = \mathcal{F}^{2np_0}(\hat{x}^n) = \mathcal{F}^{-2np_2}(\hat{x}^n)$ tends to $\bar{x}_3$ as $n \to +\infty$. Thus, considering the chart χ_2, resp. χ_0 near $\bar{x}_1$, and replacing the first n backward iterates under $\mathcal{F}^{p_2}$ with $D\mathcal{F}^{p_2}(\bar{x}_2)$, resp. first n forward iterates under $\mathcal{F}^{p_0}$ with $D\mathcal{F}^{p_0}(\bar{x}_0)$, we obtain the estimate (2.10) with $\lambda := \max(\lambda_0, \lambda_2) \in (0,1)$, denoting by $\lambda_j < 1 < \lambda_j^{-1}$ the eigenvalues of $D\mathcal{F}^{p_j}(\bar{x}_j)$, for $j = 0, 2$.

By (2.10), summing over all the indices $k \in \{-np_2, \ldots, np_0 - 1\}$, as $\tau_{\mathcal{S}}$ is Lipschitz continuous, and as $\mathcal{F}^i$ is Lipschitz continuous for $i = 1, \ldots, p_0$ (resp. $i = 1, \ldots, p_2$) on some fixed neighborhood of the orbit of $\bar{x}_0$ (resp. $\bar{x}_2$), the left hand side in (2.7) is of order at most $O(n\lambda^n)$; therefore, for n sufficiently large, this term is smaller than ε. Inequality (2.8) is proved similarly. □

Let us now conclude the proof of Proposition 2.13. Fix some small $\varepsilon > 0$. By (2.4), there exists $N'(\varepsilon) \geq 1$ such that for $n \geq N'(\varepsilon)$, we have (recall notation (2.6))

$$\left| H(\mathscr{Q}) - \left[-\tau_{\mathcal{S}}^{np_0,np_0}(\bar{x}_0) + \tau_{\mathcal{S}}^{np_2,np_0}(\bar{x}_1) - \tau_{\mathcal{S}}^{np_2,np_2}(\bar{x}_2) + \tau_{\mathcal{S}}^{np_0,np_2}(\bar{x}_3) \right] \right| < \frac{\varepsilon}{2}.$$

By Lemma 2.14, for any $n \geq N(\frac{\varepsilon}{4})$, the periodic point $\hat{x}^n$ satisfies inequalities (2.7)-(2.8) for $\frac{\varepsilon}{4}$ in place of ε. Thanks to (2.9), we also have $\tau_{\mathcal{S}}^{np_2,np_0}(\hat{x}^n) + \tau_{\mathcal{S}}^{np_0,np_2}(\hat{y}^n) = T_\Phi(x^n)$; by (2.7)-(2.8) and the above inequality, for any $n \geq \max(N'(\varepsilon), N(\frac{\varepsilon}{4}))$, we therefore obtain

$$\left| H(\mathscr{Q}) - \left[T_\Phi(x^n) - \tau_{\mathcal{S}}^{np_0,np_0}(\bar{x}_0) - \tau_{\mathcal{S}}^{np_2,np_2}(\bar{x}_2) \right] \right| < \varepsilon,$$

which concludes the proof, observing that for $j = 0, 2$, it holds

$$\tau_{\mathcal{S}}^{np_j,np_j}(\bar{x}_j) = \sum_{j=-np_j}^{np_j-1} \tau_{\mathcal{S}}(\mathcal{F}^j(\bar{x}_j)) = 2nT_\Phi(x_j).$$

□

2.5. **Temporal displacements and areas of quadrilaterals.** Assume that there exists a smooth contact form α on M that is adapted to the basic set Λ in the sense of Definition 1.2. Recall the following fact:

Lemma 2.16. *We have $E_\Phi^s(x) \subset \ker \alpha(x)$, for all $x \in \mathcal{W}_\Phi^s(\Lambda)$, and $E_\Phi^u(x) \subset \ker \alpha(x)$, for all $x \in \mathcal{W}_\Phi^u(\Lambda)$. In particular, it holds*

$$E_\Phi^s(x) \oplus E_\Phi^u(x) = \ker \alpha(x), \quad \forall\, x \in \Lambda. \tag{2.11}$$

Proof. Let $\Gamma = \{\gamma(t) \in t \in [0,1]\} \subset \mathcal{W}_{\Phi,\mathrm{loc}}^s(x)$ be an arc in the local stable manifold of some point $x \in \Lambda$. For each $T > 0$, we have

$$\int_\Gamma \alpha = \int_0^1 \alpha(\gamma(t))(\gamma'(t))dt = \int_0^1 \alpha(\Phi^T \circ \gamma(t))(D\Phi^T(\gamma(t)) \cdot \gamma'(t))dt = \int_{\Phi^T \circ \Gamma} \alpha.$$

As α is uniformly bounded, and $\lim_{T \to +\infty} D\Phi^T(\gamma(t)) \cdot \gamma'(t) \to 0$, for each $t \in [0,1]$, we deduce that $\int_\Gamma \alpha = 0$. Therefore, we have $E_\Phi^s(y) \subset \ker \alpha(y)$, for any $y \in \mathcal{W}_\Phi^s(x)$, $x \in \Lambda$. We argue similarly for the unstable direction.

Let $x \in \Lambda$. The identity (2.11) follows from the inclusions $E_\Phi^s(x) \subset \ker \alpha(x)$, $E_\Phi^u(x) \subset \ker \alpha(x)$, and the equality of the dimensions of the two subspaces. □

Let $\mathscr{Q} = (x_0, x_1, x_2, x_3) \in \Lambda^4$ be a $\mathcal{R}$-good quadrilateral, with $x_0 \in R_i$, for some $i \in \{1, \ldots, m\}$, and let $\overline{\mathscr{Q}} := (\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$ be the projection of $\mathscr{Q}$ on R_i. We define $\widehat{\mathscr{Q}}$ as the set of all points $x \in R_i$ in the closed region bounded by the arcs $\bar{\Gamma}_0$, $\bar{\Gamma}_1$, $\bar{\Gamma}_2$, $\bar{\Gamma}_3$, where for $j = 0, 2$, $\bar{\Gamma}_j \subset \mathcal{W}_{R_i}^s(\bar{x}_j)$ is the stable arc connecting $\bar{x}_j$ to $\bar{x}_{j+1}$, while $\bar{\Gamma}_{j+1} \subset \mathcal{W}_{R_i}^u(\bar{x}_{j+1})$ is the unstable arc connecting $\bar{x}_{j+1}$ to $\bar{x}_{j+2}$, with $\bar{x}_4 := \bar{x}_0$. The set $\widehat{\mathscr{Q}} \subset R_i$ is transverse to the flow direction, i.e.,

$$X(x) \notin T_x\widehat{\mathscr{Q}}, \text{ for each } x \in \widehat{\mathscr{Q}}, \tag{2.12}$$

which ensures that $d\alpha|_{\widehat{\mathscr{Q}}}$ is non-degenerate. Let us define

$$\mathrm{Area}(\mathscr{Q}) := \int_{\widehat{\mathscr{Q}}} d\alpha.$$

Proposition 2.17. *Let $\mathscr{Q}=(x_0,x_1,x_2,x_3)\in\Lambda^4$ be a small quadrilateral, so that $\mathscr{Q}$ is $\mathcal{R}$-good and (2.12) is satisfied. Then*

$$\mathrm{Area}(\mathscr{Q})=-H(\mathscr{Q}).$$

Proof. By Stokes theorem, we have

$$\mathrm{Area}(\mathscr{Q})=\int_{\widehat{\mathscr{Q}}}d\alpha=\sum_{j=0,\dots,3}\int_{\bar\Gamma_j}\alpha.$$

By the definition (2.3) of $H(\mathscr{Q})$ in terms of holonomies, it is sufficient to show that $\int_{\bar\Gamma_0}\alpha=-H^s_{\mathcal{S}}(\bar x_0,\bar x_1)$, $\int_{\bar\Gamma_1}\alpha=-H^u_{\mathcal{S}}(\bar x_1,\bar x_2)$, $\int_{\bar\Gamma_2}\alpha=-H^s_{\mathcal{S}}(\bar x_2,\bar x_3)$, and $\int_{\bar\Gamma_3}\alpha=-H^u_{\mathcal{S}}(\bar x_3,\bar x_0)$. Let us prove the formula for $\bar\Gamma_0$, the others are proved similarly.

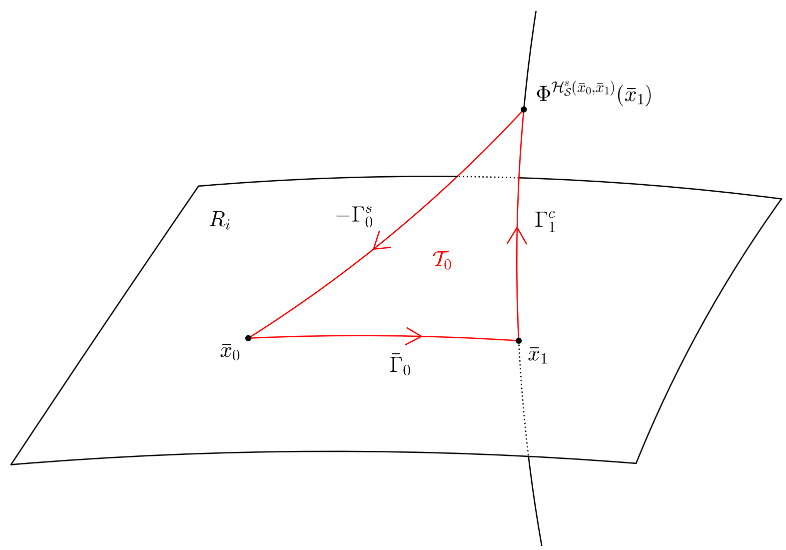


FIGURE 5. $\mathcal{T}_0$ is the closed region bounded by the arcs $\bar\Gamma_0,-\Gamma^s_0,\Gamma^c_1$.

Let Γ^s_0 be the arc of the stable manifold $\mathcal{W}^s_{\Phi,\mathrm{loc}}(x_0)$ connecting x_0 to x_1, and let Γ^c_1 be the orbit segment $\Gamma^c_1:=\{\Phi^t(x_1)\}_{t\in[0,\mathcal{H}^s_{\mathcal{S}}(\bar x_0,\bar x_1)]}\subset\mathcal{W}^c_{\Phi,\mathrm{loc}}(x_1)$. We define $\mathcal{T}_0\subset M$ as the set of all points $x\in\mathcal{W}^{cs}_{\Phi,\mathrm{loc}}(x_0)$ in the closed region bounded by the arcs $\bar\Gamma_0,\Gamma^s_0,\Gamma^c_1$, see Figure 5. By Stokes theorem, we have

$$\int_{\mathcal{T}_0}d\alpha=\int_{\bar\Gamma_0}\alpha-\int_{\Gamma^s_0}\alpha+\int_{\Gamma^c_1}\alpha.$$

Since $X|_{\mathcal{W}^{cs}_\Phi(\Lambda)}\in\ker d\alpha|_{\mathcal{W}^{cs}_\Phi(\Lambda)}$, it holds that $\int_{\mathcal{T}_0}d\alpha=0$. By Lemma 2.16, we have $\int_{\Gamma^s_0}\alpha=0$, hence,

$$\int_{\bar\Gamma_0}\alpha=-\int_{\Gamma^c_1}\alpha.$$

Moreover,

$$\int_{\Gamma^c_1}\alpha=\int_0^{H^s_{\mathcal{S}}(\bar x_0,\bar x_1)}\alpha(X(\Phi^t(\bar x_1)))dt.$$

Since $\imath_X\alpha|_\Lambda \equiv 1$, we also have $\int_0^{H^s_{\mathcal{S}}(\bar{x}_0,\bar{x}_1)} \alpha(X(\Phi^t(\bar{x}_1)))dt = H^s_{\mathcal{S}}(\bar{x}_0,\bar{x}_1)$, which concludes. $\square$

As an immediate consequence of Proposition 2.11 and Proposition 2.17, we thus obtain:

Corollary 2.18. *For any small quadrilateral $\mathscr{Q} = (x_0,x_1,x_2,x_3) \in \Lambda^4$, the quantity* $\mathrm{Area}(\mathscr{Q})$ *is determined by the lengths of periodic orbits.*

Corollary 2.19. *Fix $k \geq 2$. For $i = 1,2$, let $\Phi_i = (\Phi_i^t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^k$ Axiom A flow defined on a 3-manifold M_i. Let Λ_i be a basic set for Φ_i, and let α_i be a smooth contact form adapted to Λ_i. If there exists a flow conjugacy $\Psi\colon \Lambda_1 \to \Lambda_2$ between $\Phi_1|_{\Lambda_1}$ and $\Phi_2|_{\Lambda_2}$, then, for any point $x_0 \in \Lambda_1$, and for any small quadrilateral $\mathscr{Q} = (x_0,x_1,x_2,x_3) \in \Lambda_1^4$, it holds*

$$\mathrm{Area}(\mathscr{Q}) = \mathrm{Area}(\Psi(\mathscr{Q})), \tag{2.13}$$

where $\Psi(\mathscr{Q})$ is the quadrilateral $\Psi(\mathscr{Q}) := (\Psi(x_0),\Psi(x_1),\Psi(x_2),\Psi(x_3)) \in \Lambda_2^4$.

Remark 2.20. Let us give another proof of Corollary 2.19. As in Corollary 2.19, let Φ_1, Φ_2 be two (contact) Axiom A flows whose restrictions $\Phi_1|_{\Lambda_1}$, $\Phi_2|_{\Lambda_2}$ to certain hyperbolic sets Λ_1,Λ_2 are conjugate by $\Psi\colon \Lambda_1 \to \Lambda_2$. In order to show that (2.13) holds for any small quadrilateral $\mathscr{Q} = (x_0,x_1,x_2,x_3) \in \Lambda_1^4$, instead of invoking Corollary 2.18, we may argue as follows. By Proposition 2.17, it amounts to showing that the temporal displacements of the quadrilaterals $\mathscr{Q},\Psi(\mathscr{Q})$ are equal, i.e., $H(\mathscr{Q}) = H(\Psi(\mathscr{Q}))$. Let us consider the points $x_4 := x_4(\mathscr{Q})$ and $x_4' := x_4(\Psi(\mathscr{Q}))$ (recall Definition 2.6). Since $x_4(\mathscr{Q}), x_4(\Psi(\mathscr{Q}))$ are defined dynamically, and Ψ is a flow conjugacy between $\Phi_1|_{\Lambda_1}$, $\Phi_2|_{\Lambda_2}$, we have $x_4' = x_4(\Psi(\mathscr{Q})) = \Psi(x_4)$. In particular,

$$x_4 = \Phi_1^{H(\mathscr{Q})}(x_1), \quad \text{and} \quad \Psi(x_4) = \Phi_2^{H(\Psi(\mathscr{Q}))}(\Psi(x_1)). \tag{2.14}$$

Since Ψ is a flow conjugacy between $\Phi_1|_{\Lambda_1}$, $\Phi_2|_{\Lambda_2}$, it preserves time, hence the temporal distance between the points x_1,x_4 is the same as the one between their images $\Psi(x_1),\Psi(x_4)$. Therefore, by (2.14), we conclude that $H(\mathscr{Q}) = H(\Psi(\mathscr{Q}))$.

2.6. **Smoothness of the conjugacy.** In the following, we fix a point $x_0 \in \Lambda \cap R_i$, for some $i \in \{1,\dots,m\}$. Let Q_0 be the set of all sufficiently small quadrilaterals $\mathscr{Q} = (x_0,x_1,x_2,x_3) \in \Lambda^4$ based at x_0. The goal of this part is to show that the set of areas $\{\mathrm{Area}(\mathscr{Q})\}_{\mathscr{Q}\in Q_0}$ determines the "infinitesimal" shape of the set $\Lambda \cap \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0)$, resp. $\Lambda \cap \mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0)$.

In particular, given another Axiom A flow whose restriction to some basic set is conjugate to $\Phi|_\Lambda$ by some homeomorphism Ψ, and such that, for any small quadrilateral $\mathscr{Q}$, it holds $\mathrm{Area}(\mathscr{Q}) = \mathrm{Area}(\Psi(\mathscr{Q}))$, we show that Ψ is differentiable at any point of Λ, with Hölder continuous differential.

We take a chart $\mathcal{R} = \mathcal{R}_{x_0}\colon \mathcal{U}_0 \to \mathcal{V}_0$ from a neighborhood $\mathcal{U}_0 \subset R_i$ of x_0 to a neighborhood $\mathcal{V}_0 \subset \mathbb{R}^2$ of $\{0_{\mathbb{R}^2}\}$ such that $\mathcal{R}(\mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0) \cap \mathcal{U}_0) \subset (\mathbb{R}\times\{0\}) \cap \mathcal{V}_0$ and $\mathcal{R}(\mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0) \cap \mathcal{U}_0) \subset (\{0\}\times\mathbb{R}) \cap \mathcal{V}_0$. In the following, we thus identify $\mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0)$, resp. $\mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0)$ with the horizontal, resp. vertical coordinate axis of $\mathbb{R}^2$. Moreover, for any point $v = \mathcal{R}(u) \in \mathcal{V}_0$, we denote by $\rho(v)d\xi\wedge d\eta := \mathcal{R}_*(d\alpha_u)$ the corresponding area form. For each point $y_0 \in \mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0)$, we see $\mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(y_0)$ as the graph of some function $\gamma^s_{y_0}$ over the horizontal axis. By an abuse of notation, in the following, we identify an object and its image in the chart $\mathcal{R}$. For instance, a point $x_1 \in$

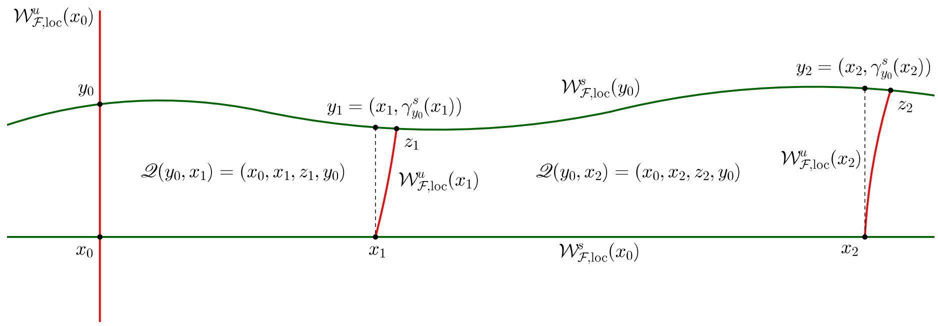


FIGURE 6. Small quadrilaterals.

$\mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0)$ will be identified with the point $(x_1, 0) \in \mathbb{R}^2$; besides, we will denote by $V_{x_1} := \big(\{x_1\} \times \mathbb{R}\big) \cap \mathcal{V}_0$ the vertical segment in $\mathcal{V}_0$ passing through $x_1 \simeq (x_1, 0)$.

Definition 2.21 (Holonomy maps for $\mathcal{W}^s_{\mathcal{F},\mathrm{loc}}$)**.** For any points $y_0 \in \mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0) \cap \Lambda$ and $x_1 \in \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0)$, we define the point $\mathcal{H}^s_{x_0,x_1}(y_0) \in V_{x_1}$ as

$$\{\mathcal{H}^s_{x_0,x_1}(y_0)\} := \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(y_0) \cap V_{x_1} = (x_1, \gamma^s_{y_0}(x_1)).$$

In other words, the map $\mathcal{H}^s_{x_0,x_1}$ is the holonomy map along $\mathcal{W}^s_{\mathcal{F},\mathrm{loc}}$ from $\mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0) \cap \Lambda \simeq V_{x_0}$ to V_{x_1}. To ease the notation, we also abbreviate $y_1 = y_1(y_0) := \mathcal{H}^s_{x_0,x_1}(y_0)$. Note that, *a priori*, $y_1 \notin \Lambda$.

Lemma 2.22. *There exists a continuous function $C = C_{x_0} \colon \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0) \to \mathbb{R}$ such that for any $x_1 \in \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0)$, it holds*

$$\lim_{\mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0)\cap\Lambda\ni y_0\to x_0} \frac{d(y_1(y_0), x_1)}{d(y_0, x_0)} = \lim_{\mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0)\cap\Lambda\ni y_0\to x_0} \frac{\gamma^s_{y_0}(x_1)}{\gamma^s_{y_0}(x_0)} = C(x_1). \tag{2.15}$$

Moreover, it holds

$$\lim_{\mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0)\ni x_1\to x_0} C(x_1) = 1.$$

Proof. According to Remark 1.6, stable holonomy maps are $\mathcal{C}^{1,\beta}$, for some $\beta \in (0,1)$. For any $y_0 \in \mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0) \cap \Lambda$, we let $y_1 = y_1(y_0) \in V_{x_1}$ be defined as above. As $\mathcal{H}^s_{x_0,x_1}(x_0) = x_1$ and $\mathcal{H}^s_{x_0,x_1}(y_0) = y_1$, the quotient in (2.15) can be written as $\frac{d(\mathcal{H}^s_{x_0,x_1}(y_0), \mathcal{H}^s_{x_0,x_1}(x_0))}{d(y_0,x_0)}$. From the definition of $\gamma^s_{y_0}$, this quantity is also equal to $\frac{\gamma^s_{y_0}(x_1)}{\gamma^s_{y_0}(x_0)}$. Moreover, it has a limit as $y_0 \to x_0$, which we denote by $C(x_1) \in \mathbb{R}$. We thus get a continuous map $C = C_{x_0} \colon \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0) \to \mathbb{R}$. Moreover, the holonomy map $\mathcal{H}^s_{x_0,x_1}$ converges to the identity in the $\mathcal{C}^1$ topology as $x_1 \to x_0$, hence $\lim_{x_1\to x_0} C(x_1) = C(x_0) = 1$. □

For any points $y_0 \in \mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0) \cap \Lambda$, $x_1 \in \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0) \cap \Lambda$ close to x_0, we also abbreviate $z_1 = z_1(y_0) := [y_0, x_1]_{R_i}$. Recall that by local product structure, we have $z_1 \in \Lambda$. We denote by $\mathscr{Q}(y_0, x_1) := (x_0, x_1, z_1, y_0) \in \Lambda^4$ the associated quadrilateral.

Lemma 2.23. *For any points $y_0 \in \mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0)\cap\Lambda$, $x_1 \in \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0)\cap\Lambda$ close to x_0, the area of the quadrilateral $\mathscr{Q}(y_0,x_1)$ is equal to*

$$\mathrm{Area}(\mathscr{Q}(y_0,x_1)) = (y_0-x_0)(x_1-x_0)[\rho(x_0)+o(1)], \tag{2.16}$$

where ρ is the density function of $\mathcal{R}_*(d\alpha)$ introduced above. Therefore, for any points $y_0 \in \mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0)\cap\Lambda$, $x_1,x_2 \in \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0)\cap\Lambda$ close to x_0, we have[5]

$$\frac{\mathrm{Area}(\mathscr{Q}(y_0,x_2))}{\mathrm{Area}(\mathscr{Q}(y_0,x_1))} = \frac{x_2-x_0}{x_1-x_0}+o(1). \tag{2.17}$$

Proof. For any $y_0 \in \mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0)\cap\Lambda$, $x_1 \in \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0)\cap\Lambda$ close to x_0, we have

$$\mathrm{Area}(\mathscr{Q}(y_0,x_1)) = \int_{x_0}^{x_1}\left(\int_0^{\gamma^s_{y_0}(\xi)} \rho(\xi,\eta)\,d\eta\right)d\xi + o\big((y_1-x_1)^2\big),$$

where $y_1 = y_1(y_0)$. Here, we use the fact that the unstable lamination $\mathcal{W}^u_{\mathcal{F}}(y_0)$ is $\mathcal{C}^1$, so that the angle between $\mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_1)$ and V_{x_1} is going to 0 as $x_1 \to x_0$, and hence, the area of the missing "triangle" bounded by $\mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(y_0)$, $\mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_1)$ and V_{x_1} is a $o((y_1-x_1)^2)$, noting that $\rho = O(1)$ on the quadrilateral. Since the argument is a local one, (2.15) guarantees that $y_1 - x_1 = O(y_0-x_0)$. In the following, we will always assume that $y_0-x_0 \le x_1-x_0$, so that $o\big((y_1-x_1)^2\big) = o\big((y_0-x_0)(x_1-x_0)\big)$. Therefore, we obtain

$$\begin{aligned}
\mathrm{Area}(\mathscr{Q}(y_0,x_1)) &= \int_{x_0}^{x_1} \gamma^s_{y_0}(\xi)\big(\rho(\xi,0)+O(y_0-x_0)\big)\,d\xi + o\big((y_0-x_0)(x_1-x_0)\big)\\
&= \int_{x_0}^{x_1} (C(\xi)(y_0-x_0)+o(y_0-x_0))\big(\rho(\xi,0)+O(y_0-x_0)\big)\,d\xi\\
&\quad + o\big((y_0-x_0)(x_1-x_0)\big)\\
&= (y_0-x_0)\int_{x_0}^{x_1}\big(C(\xi)\rho(\xi,0)+o(1)\big)\,d\xi + o\big((y_0-x_0)(x_1-x_0)\big)\\
&= (y_0-x_0)(x_1-x_0)[\rho(x_0)+o(1)],
\end{aligned}$$

since $C(\xi) = C(x_0)+o(1) = 1+o(1)$, when $\xi \to x_0$. Observe now that (2.17) follows immediately by taking the quotient. □

For $i=1,2$, let $\Phi_i = (\Phi_i^t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^k$ Axiom A flow defined on a smooth 3-manifold M_i. Let Λ_i be a basic set for Φ_i, and let α_i be a smooth contact form adapted to Λ_i. Assume that there exists a flow conjugacy $\Psi\colon \Lambda_1 \to \Lambda_2$ between $\Phi_1|_{\Lambda_1}$ and $\Phi_2|_{\Lambda_2}$. For any point $x_0 \in \Lambda_1$, and for $* = s,u$, without loss of generality, because of Lemma 2.16 and up to translating along the flow direction, we can assume that $\mathcal{W}^*_{\Phi_1,\mathrm{loc}}(x_0)$, resp. $\mathcal{W}^*_{\Phi_2,\mathrm{loc}}(\Psi(x_0))$ belongs to some rectangle $R^{(1)}$ of a Markov family for Φ_1, resp. to some rectangle $R^{(2)}$ of a Markov family for Φ_2, so that $\mathcal{W}^*_{\Phi_1,\mathrm{loc}}(x_0) = \mathcal{W}^*_{R^{(1)}}(x_0)$, and $\mathcal{W}^*_{\Phi_2,\mathrm{loc}}(\Psi(x_0)) = \mathcal{W}^*_{R^{(2)}}(\Psi(x_0))$. Moreover, by using some chart as above, we see $\Psi|_{\mathcal{W}^*_{\Phi_1,\mathrm{loc}}(x_0)}$ as a map from $S_1 \subset \mathbb{R}$ to $S_2 \subset \mathbb{R}$, with $x_0 \simeq 0 \simeq \Psi(x_0)$.

[5] We thank Disheng Xu for the idea to use three points x_0, x_1, x_2 in the same leaf and consider the ratio of areas to get rid of the "width" of quadrilaterals.

Proposition 2.24. *Assume that the flow conjugacy Ψ is iso-length-spectral. Then, for any point $x_0 \in \Lambda_1$, and for $* = s, u$, the following limit exists:*

$$\partial_* \Psi(x_0) := \lim_{\mathcal{W}^*_{\Phi_1,\mathrm{loc}}(x_0)\cap\Lambda \ni x_1 \to x_0} \frac{\Psi(x_1)-\Psi(x_0)}{x_1-x_0}.$$

Moreover, the associated map $\partial_*\Psi$ is Hölder continuous on Λ_1. In other words, for some $\beta \in (0,1)$, the conjugacy Ψ is $\mathcal{C}^{1,\beta}$ along $\mathcal{W}^s_{\Phi_1,\mathrm{loc}}, \mathcal{W}^u_{\Phi_1,\mathrm{loc}}$ in the sense of Whitney.

Proof. Let us consider the case where $* = s$; the other case is analogous. Fix $x_0 \in \Lambda_1$. Take $y_0 \in \mathcal{W}^u_{\Phi_1,\mathrm{loc}}(x_0)\cap\Lambda_1$, $x_1, x_2 \in \mathcal{W}^s_{\Phi_1,\mathrm{loc}}(x_0)\cap\Lambda_1$ close to x_0. Without loss of generality, we assume that $d(x_0,x_1) \le d(x_0,x_2)$. By Corollary 2.19, for $i = 1,2$, the quadrilaterals $\mathscr{Q}(y_0,x_i) = (x_0,x_i,z_i,y_0) \in \Lambda_1^4$ and $\Psi(\mathscr{Q})(y_0,x_i) := (\Psi(x_0),\Psi(x_i),\Psi(z_i),\Psi(y_0)) \in \Lambda_2^4$ have the same area; hence,

$$\frac{\mathrm{Area}(\mathscr{Q}(y_0,x_2))}{\mathrm{Area}(\mathscr{Q}(y_0,x_1))} = \frac{\mathrm{Area}(\Psi(\mathscr{Q})(y_0,x_2))}{\mathrm{Area}(\Psi(\mathscr{Q})(y_0,x_1))}.$$

We deduce from formula (2.17) that

$$\frac{\Psi(x_2)-\Psi(x_0)}{x_2-x_0} = \frac{\Psi(x_1)-\Psi(x_0)}{x_1-x_0} + o\left(\frac{\Psi(x_1)-\Psi(x_0)}{x_2-x_0}\right).$$

For any $x \in \mathcal{W}^s_{\Phi_1,\mathrm{loc}}(x_0)\cap\Lambda_1$ close to x_0, we denote $q(x) := \frac{\Psi(x)-\Psi(x_0)}{x-x_0}$. Recall that $d(x_0,x_1) \le d(x_0,x_2)$; thus, the previous identity can be written as

$$q(x_1) = q(x_2) + o\big(\max(q(x_1),q(x_2))\big).$$

Now, let us fix a sequence of points $(u_n)_{n\in\mathbb{N}} \in (\mathcal{W}^s_{\Phi_1,\mathrm{loc}}(x_0)\cap\Lambda_1)^{\mathbb{N}}$ going to x_0 as $n \to +\infty$. It is easy to see that that $(q(u_n))_{n\in\mathbb{N}}$ is bounded. Consequently, for any $n \ge 0$, $p \ge 0$, the previous identity gives

$$q(u_{n+p}) - q(u_n) = o(1).$$

We deduce that $(q(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence, hence it converges to some limit $\ell \in \mathbb{R}$. Therefore, for any sequence $(v_n)_{n\in\mathbb{N}} \in (\mathcal{W}^s_{\Phi_1,\mathrm{loc}}(x_0)\cap\Lambda_1)^{\mathbb{N}}$ converging to x_0, it holds that $q(v_n) \to \ell$ as $n \to +\infty$. This shows that Ψ is differentiable at x_0 along $\mathcal{W}^s_{\Phi_1,\mathrm{loc}}(x_0)$, thus at any point in Λ_1, along $\mathcal{W}^s_{\Phi_1,\mathrm{loc}}$.

In order to show that the map $\partial_s\Psi$ is Hölder continuous on Λ_1 along $\mathcal{W}^s_{\Phi_1,\mathrm{loc}}$, we argue as follows. Fix $x_0 \in \Lambda_1 \cap R^{(1)}$, and let $x_0' \in \mathcal{W}^s_{R^{(1)}}(x_0)\cap\Lambda_1$ be close to x_0. Let $(u_n)_{n\in\mathbb{N}} \in (\mathcal{W}^s_{R^{(1)}}(x_0)\cap\Lambda_1)^{\mathbb{N}}$, resp. $(u_n')_{n\in\mathbb{N}} \in (\mathcal{W}^s_{R^{(1)}}(x_0')\cap\Lambda_1)^{\mathbb{N}}$, be a sequence of points in Λ_1 converging to x_0, resp. x_0' along $\mathcal{W}^s_{R^{(1)}}(x_0) = \mathcal{W}^s_{R^{(1)}}(x_0')$. For any point $y_0 \in \mathcal{W}^u_{R^{(1)}}(x_0)\cap\Lambda_1$ close to x_0, and for each integer $n \in \mathbb{N}$, we let $\overline{\mathscr{Q}}_n(y_0) = (x_0,u_n,z_n,y_0) \in (\Lambda_1\cap R^{(1)})^4$ and $\overline{\mathscr{Q}}'_n(y_0) = (x_0',u_n',z_n',y_0') \in (\Lambda_1\cap R^{(1)})^4$, where $z_n = [y_0,u_n]_{R^{(1)}}$, $y_0' = [y_0,x_0']_{R^{(1)}}$ and $z_n' = [y_0',u_n']_{R^{(1)}}$. Let $\mathscr{Q}_n(y_0)$, resp. $\mathscr{Q}_n'(y_0)$, be the lift of $\overline{\mathscr{Q}}_n(y_0)$, resp. $\overline{\mathscr{Q}}'_n(y_0)$, as in the proof of Lemma 2.12. We deduce from (2.16) that

$$\begin{aligned}
\mathrm{Area}(\mathscr{Q}_n(y_0)) &= (y_0-x_0)(u_n-x_0)[\rho(x_0)+o(1)],\\
\mathrm{Area}(\mathscr{Q}_n'(y_0)) &= (y_0'-x_0')(u_n'-x_0')[\rho(x_0')+o(1)]\\
&= C_{x_0}(x_0')(y_0-x_0)(u_n'-x_0')[\rho(x_0')+o(1)],
\end{aligned}$$

so that

$$\frac{\mathrm{Area}(\mathscr{Q}'_n(y_0))}{\mathrm{Area}(\mathscr{Q}_n(y_0))} = C_{x_0}(x'_0)\frac{u'_n - x'_0}{u_n - x_0}\left(1 + O(x'_0 - x_0) + o(1)\right).$$

As the images of the quadrilaterals $\mathscr{Q}_n(y_0)$ and $\mathscr{Q}'_n(y_0)$ by Ψ have the same area, we deduce that

$$\begin{aligned}&C_{\Psi(x_0)}(\Psi(x'_0))\frac{\Psi(u'_n) - \Psi(x'_0)}{\Psi(u_n) - \Psi(x_0)}\left(1 + O(\Psi(x'_0) - \Psi(x_0)) + o(1)\right)\\ &= C_{x_0}(x'_0)\frac{u'_n - x'_0}{u_n - x_0}\left(1 + O(x'_0 - x_0) + o(1)\right).\end{aligned}$$

Observe that

$$\begin{aligned}C_{x_0}(x'_0) &= 1 + O(x'_0 - x_0),\\ C_{\Psi(x_0)}(\Psi(x'_0)) &= 1 + O(\Psi(x'_0) - \Psi(x_0)) = 1 + O(|x'_0 - x_0|^\beta),\end{aligned}$$

for some $\beta \in (0,1)$ since Ψ is Hölder continuous. Thus, for $y_0 \to x_0$ we obtain

$$\frac{\Psi(u'_n) - \Psi(x'_0)}{u'_n - x'_0} = \frac{\Psi(u_n) - \Psi(x_0)}{u_n - x_0}\left(1 + O(|x'_0 - x_0|^\beta)\right).$$

Letting $n \to +\infty$, we deduce that $|\partial_s\Psi(x'_0) - \partial_s\Psi(x_0)| = O(|x'_0 - x_0|^\beta)$. Thus, applying Whitney's theorem, we conclude that Ψ is $\mathcal{C}^{1,\beta}$ in the sense of Whitney along $\mathcal{W}^s_{\Phi_1,\mathrm{loc}}$, for $\beta \in (0,1)$. □

Recall that roughly speaking, Journé's lemma (see [37]) says that once a function is regular along the leaves of two transverse foliations, then it is regular globally. It has been generalized by Nicol-Török [50] in the case of laminations on Cantor sets (see Theorem 1.5 and Remark 1.6 in [50]). In our case, it reads as follows.

Theorem 2.25 (Theorem 1.5 in [50])**.** *Let $\Lambda \subset \mathbb{R}^2$ be a closed, hyperbolic basic set, and for $\beta \in (0,1)$, let $\mathcal{W}^s$, $\mathcal{W}^u$ be two transverse uniformly $\mathcal{C}^{1,\beta}$ laminations of Λ. Suppose that $\Theta\colon \Lambda \to \mathbb{R}^2$ is uniformly $\mathcal{C}^{1,\beta}$ in the sense of Whitney when restricted to the leaves of $\mathcal{W}^s$, $\mathcal{W}^u$. Then Θ is $\mathcal{C}^{1,\beta}$ in the sense of Whitney on Λ.*

From Proposition 2.24 and Theorem 2.25, we then deduce the following

Corollary 2.26. *Assume that there exists an iso-length-spectral flow conjugacy $\Psi\colon \Lambda_1 \to \Lambda_2$ between $\Phi_1|_{\Lambda_1}$ and $\Phi_2|_{\Lambda_2}$. Then Ψ is $\mathcal{C}^{1,\beta}$ in the sense of Whitney on Λ, for some $\beta \in (0,1)$.*

Proof. By Proposition 2.24, we know that Ψ is $\mathcal{C}^{1,\beta}$ in the sense of Whitney along stable/unstable leaves. For $i = 1,2$, let us fix a Markov family $\mathcal{R}^{(i)} = \{R^{(i)}_1, \dots, R^{(i)}_{m(i)}\}$ with a cross-section $\mathcal{S}^{(i)}$ as given by Theorem 2.5. By projecting Λ_1, Λ_2 along flow lines on $\mathcal{S}^{(1)}, \mathcal{S}^{(2)}$, and applying Theorem 2.25 to the projected sets, we deduce that the map $\widetilde{\Psi}$ induced by Ψ between $\Lambda_1 \cap \mathcal{S}^{(1)}$ and $\Lambda_2 \cap \mathcal{S}^{(2)}$ is $\mathcal{C}^{1,\beta}$ in the sense of Whitney, for some $\beta \in (0,1)$. Since the projection along the flow direction is $\mathcal{C}^k$, and since we can describe Ψ in terms of $\widetilde{\Psi}$ and the two projections along X_1, X_2, we conclude that Ψ is $\mathcal{C}^{1,\beta}$ in the sense of Whitney. □

2.7. **Upgraded regularity of the conjugation.** As previously, let us fix a homeomorphism $\Psi\colon \Lambda_1 \to \Lambda_2$ that is $\mathcal{C}^{1,\beta}$ in Whitney sense, for some $\beta \in (0,1)$, which satisfies

$$\Psi \circ \Phi_1^t(x) = \Phi_2^t \circ \Psi(x), \quad \text{for all } (x,t) \in \Lambda_1 \times \mathbb{R}. \tag{2.18}$$

We will show that the conjugacy map is even more regular:

Proposition 2.27. *The conjugacy map $\Psi|_{\Lambda_1}$ is $\mathcal{C}^k$ in Whitney sense.*

Proof. Recall that for $i = 1,2$ and $\star = s,u$, there exists $\delta_i^{(\star)} > 0$ such that for any $x \in \Lambda_i$, we have

$$\delta_i^{(\star)} = \dim_H(\mathcal{W}^\star_{\Phi_i,\mathrm{loc}}(x) \cap \Lambda_i).$$

As Ψ is $\mathcal{C}^{1,\beta}$, we also have $\delta_1^{(\star)} = \delta_2^{(\star)} =: \delta^{(\star)}$.

Fix some small $\varepsilon > 0$. By Theorem 2.5, for $i = 1,2$, there exists a proper Markov family $\mathcal{R}^{(i)} = \{R_1^{(i)}, \dots, R_{m(i)}^{(i)}\}$ for $\Phi_{i|\Lambda_i}$ of size ε, for some integer $m(i) \geq 1$. Let $\mathcal{S}^{(i)} := R_1^{(i)} \cup \dots \cup R_{m(i)}^{(i)}$, resp. $\mathcal{F}_i$, be the associated cross-section, resp. Poincaré map. We also denote by $\overline{\Lambda}_i := \Lambda_i \cap \mathcal{S}^{(i)}$ the trace of Λ_i on $\mathcal{S}^{(i)}$. The map $\widetilde{\Psi}$ induced by Ψ between $\overline{\Lambda}_1$ and $\overline{\Lambda}_2$ is $\mathcal{C}^{1,\beta}$ in the sense of Whitney. Recall that $\dim_H(\overline{\Lambda}_1) = \dim_H(\overline{\Lambda}_2) = \delta^{(s)} + \delta^{(u)}$ (see [46] for a reference).

By [56, Theorem 22.1], for $i = 1,2$ and $\star = s,u$, there exists a (unique) *equilibrium state*[6] $\mu_i^\star$ such that for every $x \in \overline{\Lambda}_i$, the conditional measure $m_{i,x}^\star$ of $\mu_i^\star$ on $\mathcal{W}^\star_{\mathcal{F}_i}(x) \cap \overline{\Lambda}_i$ is equivalent to the $\delta^{(\star)}$-Hausdorff measure $H^{\delta^{(\star)}}$. More precisely, μ_i^s is the equilibrium state for the *potential*[6] $p_i^{(s)} := \delta^{(s)} \log \|D\mathcal{F}_i|_{E^s_{\mathcal{F}_i}}\|$, and μ_i^u is the equilibrium state for the potential $p_i^{(u)} := -\delta^{(u)} \log \|D\mathcal{F}_i|_{E^u_{\mathcal{F}_i}}\|$; besides, the *pressure*[6] $P(p_i^{(\star)})$ vanishes, for $\star = s,u$.

By (2.18), for any periodic point $x \in \Lambda_1$ of period $q(x) \geq 1$, the differentials $D\mathcal{F}_1^{q(x)}(x)$ and $D\mathcal{F}_2^{q(x)}(\widetilde{\Psi}(x))$ are conjugate, hence have the same eigenvalues, i.e.,

$$\sum_{k=0}^{q(x)-1} \left(\log \|D\mathcal{F}_1^k(x)|_{E^{(\star)}_{\mathcal{F}_1}}\| - \log \|D\mathcal{F}_2^k\big(\widetilde{\Psi}(x)\big)|_{E^{(\star)}_{\mathcal{F}_2}}\| \right) = 0, \quad \star = s,u.$$

By Livsic's Theorem, we deduce that the potentials $\widetilde{\Psi}^* p_2^{(\star)}$ and $p_1^{(\star)}$ are cohomologous, and by [8, Proposition 4.5], we thus have $\widetilde{\Psi}^* \mu_2^{(\star)}|_{\overline{\Lambda}_1} = \mu_1^{(\star)}|_{\overline{\Lambda}_1}$. Consequently, $\widetilde{\Psi}^* m^\star_{2,\widetilde{\Psi}(x)} = m^\star_{1,x}$, for $\star = s,u$, and for a.e. $x \in \overline{\Lambda}_1$.

In the following we deal with the unstable case; the stable one is analogous. To ease the notation, we abbreviate $\delta := \delta^{(u)}$. For $i = 1,2$, and $x_i \in \overline{\Lambda}_i$, the conditional measure m^u_{i,x_i} is equivalent to H^δ, hence we can introduce the density function $\rho^u_{i,x_i}\colon \mathcal{W}^u_{\mathcal{F}_i}(x_i) \to \mathbb{R}^*$, so that $dm^u_{i,x_i} = \rho^u_{i,x_i} dH^\delta$. Recall that the conditional measure m^u_{i,x_i} depends only on the leaf $\mathcal{W}^u_{\mathcal{F}_i}(x_i)$. Our goal in the following paragraph is to show that the function $\rho^u_{i,x_i}(\cdot)/\rho^u_{i,x_i}(x_i)$ is $\mathcal{C}^{k-1}$ in the sense of Whitney.

[6]See for instance [56] for more details about equilibrium states, potentials, pressure etc.

As $P(p_i^u) = 0$, for any integer $n \geq 0$, and for any $y_i \in \mathcal{W}^u_{\mathcal{F}_i}(x_i)$, we have (see for instance [13, Section 3.2])

$$\frac{d((\mathcal{F}_i^{-n})_* m^u_{i,x_i})}{dm^u_{i,\mathcal{F}_i^{-n}(x_i)}}(\mathcal{F}_i^{-n}(y_i)) = e^{-S_n p_i^u(\mathcal{F}_i^{-n}(y_i))}, \tag{2.19}$$

where $S_n p_i^u$ is the n^{th} Birkhoff sum of p_i^u, i.e.,

$$S_n p_i^u(\mathcal{F}_i^{-n}(y_i)) := \sum_{k=1}^n p_i^u(\mathcal{F}^{-k}(y_i)) = -\sum_{k=1}^n \log \left\| D\mathcal{F}_i^{-1}(\mathcal{F}^{-k}(y_i))|_{E^u_{\mathcal{F}_i}} \right\|^\delta.$$

In terms of densities, (2.19) thus yields:

$$\frac{(\mathcal{F}_i^{-n})_* \rho^u_{i,x_i}}{\rho^u_{i,\mathcal{F}_i^{-n}(x_i)}}(\mathcal{F}_i^{-n}(y_i)) = \frac{\rho^u_{i,x_i}(y_i)}{\rho^u_{i,\mathcal{F}_i^{-n}(x_i)}(\mathcal{F}_i^{-n}(y_i))} = \prod_{k=1}^n \left\| D\mathcal{F}_i^{-1}(\mathcal{F}^{-k}(y_i))|_{E^u_{\mathcal{F}_i}} \right\|^\delta.$$

Let us consider the ratio of the above quantity and the corresponding one at x_i. As the distance $d(\mathcal{F}^{-n}(x_i), \mathcal{F}^{-n}(y_i))$ decays exponentially fast with respect to n, and assuming that $\mathcal{F}^{-n}(x_i), \mathcal{F}^{-n}(y_i)$ converge to a point x_i^∞ (up to taking subsequences), letting $n \to +\infty$, we obtain

$$\rho_i^u(x_i, y_i) := \frac{\rho^u_{i,x_i}(y_i)}{\rho^u_{i,x_i}(x_i)} = \prod_{k=1}^{+\infty} \left(\frac{\|D\mathcal{F}_i^{-1}(\mathcal{F}^{-k}(y_i))|_{E^u_{\mathcal{F}_i}}\|}{\|D\mathcal{F}_i^{-1}(\mathcal{F}^{-k}(x_i))|_{E^u_{\mathcal{F}_i}}\|} \right)^\delta. \tag{2.20}$$

In particular, based on that expression, and arguing as in [18, Lemma 4.3], we deduce that the function $\rho_i^u(x_i, \cdot)$ is $\mathcal{C}^{k-1}$ in the sense of Whitney.

In the rest of this section, we follow the proof of [18, Lemma 4.5]. Fix a point $x_1 \in \overline{\Lambda}_1$ and let $x_2 := \widetilde{\Psi}(x_1) \in \overline{\Lambda}_2$. Since the foliations $\mathcal{W}^u_{\mathcal{F}_1}, \mathcal{W}^u_{\mathcal{F}_2}$ have one dimensional leaves, we can parametrize patches of the unstable leaves by Riemannian length. Recall that $\widetilde{\Psi}^* m^u_{2,x_2} = m^u_{1,x_1}$; we deduce that for any point $y_1 \in \mathcal{W}^u_{\mathcal{F}_1}(x_1)$, it holds (taking charts for $\mathcal{W}^u_{\mathcal{F}_1}(x_1), \mathcal{W}^u_{\mathcal{F}_2}(x_2)$, identifying functions on the leaves and functions of the coordinates, and seeing the Whitney extension of $\widetilde{\Psi}|_{\mathcal{W}^u_{\mathcal{F}_1}(x_1)}$ as a map from $\mathbb{R}$ to $\mathbb{R}$):

$$\int_{x_1}^{y_1} \rho^u_{1,x_1}(s)\, dH^\delta(s) = \int_{\widetilde{\Psi}(x_1)}^{\widetilde{\Psi}(y_1)} \rho^u_{2,\widetilde{\Psi}(x_1)}(s)\, dH^\delta(s).$$

By (2.20), we have

$$\rho^u_{1,x_1}(x_1) \int_{x_1}^{y_1} \rho_1^u(x_1, s)\, dH^\delta(s) = \rho^u_{2,x_2}(x_2) \int_{\widetilde{\Psi}(x_1)}^{\widetilde{\Psi}(y_1)} \rho_2^u(x_2, s)\, dH^\delta(s).$$

For y_1 very close to x_1, we thus obtain

$$\rho^u_{1,x_1}(x_1) \int_{x_1}^{y_1} (1 + o(1))\, dH^\delta(s) = \rho^u_{2,x_2}(x_2) \int_{\widetilde{\Psi}(x_1)}^{\widetilde{\Psi}(y_1)} (1 + o(1))\, dH^\delta(s),$$

that is

$$\frac{\rho^u_{1,x_1}(x_1)}{\rho^u_{2,x_2}(x_2)} = \frac{\int_{\widetilde{\Psi}(x_1)}^{\widetilde{\Psi}(y_1)} dH^\delta(s)}{\int_{x_1}^{y_1} dH^\delta(s)} + o(1).$$

Consequently,

$$\log\left(\frac{\rho^u_{1,x_1}}{\rho^u_{2,x_2}\circ\widetilde{\Psi}}\right)(x_1)=\log|\widetilde{\Psi}(y_1)-\widetilde{\Psi}(x_1)|\times\frac{\log\left|\int_{\widetilde{\Psi}(x_1)}^{\widetilde{\Psi}(y_1)}dH^\delta\right|}{\log|\widetilde{\Psi}(y_1)-\widetilde{\Psi}(x_1)|}-$$
$$-\log|y_1-x_1|\times\frac{\log\left|\int_{x_1}^{y_1}dH^\delta\right|}{\log|y_1-x_1|}+o(1).$$

When $\mathcal{W}^u_{\mathcal{F}_1}(x_1)\ni y_1\to x_1$, both $\frac{\log\left|\int_{\widetilde{\Psi}(x_1)}^{\widetilde{\Psi}(y_1)}dH^\delta\right|}{\log|\widetilde{\Psi}(y_1)-\widetilde{\Psi}(x_1)|}$ and $\frac{\log\left|\int_{x_1}^{y_1}dH^\delta\right|}{\log|y_1-x_1|}$ tend to the dimension of the measure H^δ, namely, δ. We deduce that

$$\log\left(\frac{\rho^u_{1,x_1}}{\rho^u_{2,x_2}\circ\widetilde{\Psi}}\right)(x_1)=\delta\log\left(\frac{\widetilde{\Psi}(y_1)-\widetilde{\Psi}(x_1)}{y_1-x_1}\right)+o(1).$$

As $\widetilde{\Psi}|_{\overline{\Lambda}_1}$ is $\mathcal{C}^{1,\beta}$ in the sense of Whitney, letting $\mathcal{W}^u_{\mathcal{F}_1}(x_1)\ni y_1\to x_1$, we get

$$\frac{\rho^u_{1,x_1}}{\rho^u_{2,x_2}\circ\widetilde{\Psi}}(x_1)=(\partial_u\widetilde{\Psi}(x_1))^\delta.$$

In other words, on Λ_1, the map $\widetilde{\Psi}$ satisfies

$$\partial_u\widetilde{\Psi}(\cdot)=\left(\frac{\rho^u_{1,(\cdot)}(\cdot)}{\rho^u_{2,\widetilde{\Psi}(\cdot)}\circ\widetilde{\Psi}(\cdot)}\right)^{\frac{1}{\delta}}. \tag{2.21}$$

We have seen that the functions $\rho^u_{1,(\cdot)},\rho^u_{2,\widetilde{\Psi}(\cdot)}$ are $\mathcal{C}^{k-1}$ in Whitney sense. As $\widetilde{\Psi}$ is $\mathcal{C}^{1,\beta}$ on Λ_1 along $\mathcal{W}^u_{\mathcal{F}_1}$, the right hand side of (2.21) is $\mathcal{C}^{1,\beta}$ on Λ_1 along $\mathcal{W}^u_{\mathcal{F}_1}$. We deduce that $\widetilde{\Psi}$ is $\mathcal{C}^2$ on Λ_1 along $\mathcal{W}^u_{\mathcal{F}_1}$ in Whitney sense. By repeating the argument, we conclude that $\widetilde{\Psi}$ is $\mathcal{C}^k$ on Λ_1 along $\mathcal{W}^u_{\mathcal{F}_1}$ in Whitney sense. The same arguments applied at stable leaves imply that $\widetilde{\Psi}$ restricted to the leaves of $\mathcal{W}^s_{\mathcal{F}_1}$ is also $\mathcal{C}^k$ in Whitney sense. By using the version of Journé's Lemma in [50, Theorem 1.5] for laminations on hyperbolic sets, and arguing as in the proof of Corollary 2.26, we conclude that the conjugacy map $\Psi|_{\Lambda_1}$ is $\mathcal{C}^k$ in Whitney sense, as desired. □

2.8. **Preservation of contact forms: end of the proof of Theorem A.** We have just seen that the flow conjugacy Ψ is $\mathcal{C}^k$ in the sense of Whitney on Λ_1. In this subsection, we show that it implies that Ψ respects the contact structures. See Feldman-Ornstein [23] for related results in the case of contact Anosov flows on 3-manifolds.

Lemma 2.28. *We have $\Psi^*\alpha_2|_{\Lambda_1}=\alpha_1|_{\Lambda_1}$.*

Proof. By Lemma 2.16, for $i=1,2$, and for any $x_i\in\Lambda_i$, it holds

$$E^s_{\Phi_i}(x_i)\oplus E^u_{\Phi_i}(x_i)=\ker\alpha(x_i).$$

Recall that Ψ is a flow conjugacy, i.e.,

$$\Psi\circ\Phi_1^t(x_1)=\Phi_2^t\circ\Psi(x_1),\quad\forall\, t\in\mathbb{R},\ x_1\in\Lambda_1. \tag{2.22}$$

Therefore, for $* = s, u$, it holds

$$D\Psi(x_1)E^*_{\Phi_1}(x_1) = E^*_{\Phi_2}(\Psi(x_1)).$$

In particular, $\ker \Psi^*\alpha_1(x_1) = \ker \alpha_2(\Psi(x_1))$. Moreover, differentiating (2.22) with respect to t, we obtain $D\Psi(x_1)X_1(x_1) = X_2(\Psi(x_1))$.

Let us show how this implies the result. We want to show that for any $x \in \Lambda_1$, it holds $\Psi^*\alpha_2(x) = \alpha_1(x)$. For any $v \in T_xM_1$, we decompose it as $v = v^s + v^u + cX_1(x)$, with $v^s \in E^s_{\Phi_1}(x)$, $v^u \in E^u_{\Phi_1}(x)$, $c \in \mathbb{R}$. We obtain

$$\begin{aligned}\Psi^*\alpha_2(x)(v) &= \alpha_2(\Psi(x))\big(D\Psi(x)v^s + D\Psi(x)v^u + cD\Psi(x)X_1(x)\big)\\ &= c\alpha_2(\Psi(x))(D\Psi(x)X_1(x)) = c\alpha_2(\Psi(x))(X_2(\Psi(x)))\\ &= c\imath_{X_2}\alpha_2(\Psi(x)) = c = c\imath_{X_1}\alpha_1(x)\\ &= c\alpha_1(x)(X_1(x)) = \alpha_1(x)(v),\end{aligned}$$

which concludes. □

Together with Proposition 2.1 and Corollary 2.26, this concludes the proof of Theorem A.

3. Smooth conjugacy of billiard maps of hyperbolic billiards

In the following, we give the proof of Theorem C. Let us consider two billiards $\mathcal{D}_1, \mathcal{D}_2$ with $\mathcal{C}^k$ boundaries, $k \geq 3$, that are iso-length-spectral on two basic sets $\Lambda_1^{\tau_1}$, $\Lambda_2^{\tau_2}$. For $i = 1, 2$, we denote by Φ_i, resp. $\mathcal{F}_i$, the associated billiard flow, resp. billiard map. Recall that Φ_i preserves the contact form $\alpha_i := \lambda_i + dt_i$, where $\lambda_i := -r_i ds_i$ is the Liouville one-form, and that $\mathcal{F}_i$ preserves the symplectic form $ds_i \wedge dr_i$. We let Λ_1, Λ_2 be the respective projections of $\Lambda_1^{\tau_1}, \Lambda_2^{\tau_2}$ onto the first two coordinates, i.e.,

$$\Lambda_i := \{(s_i, r_i) : (s_i, r_i, t_i) \in \Lambda_i^{\tau_i} \text{ for some } t_i \in \mathbb{R}\}, \quad i = 1, 2. \tag{3.1}$$

By Proposition 2.1, there exists a flow conjugacy $\widetilde{\Psi}\colon (s_1, r_1, t_1) \mapsto (s_2, r_2, t_2)$ between the billiard flows $\Phi_1|_{\Lambda_1^{\tau_1}}$ and $\Phi_2|_{\Lambda_2^{\tau_2}}$. The map $\widetilde{\Psi}$ induces a conjugacy $\Psi\colon (s_1, r_1) \mapsto (s_2, r_2)$ between the billiard maps $\mathcal{F}_1|_{\Lambda_1}, \mathcal{F}_2|_{\Lambda_2}$.

Lemma 3.1. *The conjugacy map Ψ is $\mathcal{C}^{k-1}$ in Whitney sense.*

Proof. We argue as in Proposition 2.24, Corollary 2.26 and Proposition 2.27. The main point is that since the flows Φ_1, Φ_2 have the same periodic data, quadrilaterals formed by points in $\Lambda_1^{\tau_1}$, $\Lambda_2^{\tau_2}$ in correspondence have the same areas. Let us give more details. Given a point $x_0^{(1)} \in \Lambda_1^{\tau_1}$, we consider a small quadrilateral $\mathscr{Q}^{(1)} = (x_0^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}) \in (\Lambda_1^{\tau_1})^4$ and the associated quadrilateral $\mathscr{Q}^{(2)} := (x_0^{(2)}, x_1^{(2)}, x_2^{(2)}, x_3^{(2)}) \in (\Lambda_2^{\tau_2})^4$, with $x_j^{(2)} := \widetilde{\Psi}(x_j^{(1)})$, for $j = 0, \dots, 3$. For $i = 1, 2$ and $j = 0, \dots, 3$, we denote by $(s_j^{(i)}, r_j^{(i)}, t_j^{(i)})$ the coordinates of the point $x_j^{(i)}$ and we let $\bar{x}_j^{(i)} := (s_j^{(i)}, r_j^{(i)})$ be the projection of $x_j^{(i)}$ on the first two coordinates. Since the flows $\Phi_1|_{\Lambda_1^{\tau_1}}$ and $\Phi_2|_{\Lambda_2^{\tau_2}}$ are conjugated, by Corollary 2.19, we have

$$\mathrm{Area}(\mathscr{Q}^{(1)}) = \mathrm{Area}(\mathscr{Q}^{(2)}),$$

where for $i = 1, 2$, $\mathrm{Area}(\mathscr{Q}^{(i)})$ is the area of the region bounded by the points $(\bar{x}_0^{(i)}, \bar{x}_1^{(i)}, \bar{x}_2^{(i)}, \bar{x}_3^{(i)}) \in \Lambda_i^4$. Considering smaller and smaller quadrilaterals, and arguing as in Proposition 2.24, we deduce that the map Ψ is $\mathcal{C}^{1,\beta}$ in Whitney sense at $x_0^{(1)}$ along stable and unstable leaves, for some $\beta > 0$. It follows from Corollary 2.26 that Ψ is $\mathcal{C}^{1,\beta}$ in Whitney sense on Λ_1. As $\mathcal{F}_1, \mathcal{F}_2$ are $\mathcal{C}^{k-1}$, arguing as in Proposition 2.27, we can upgrade the regularity and show that Ψ is actually $\mathcal{C}^{k-1}$ in Whitney sense on Λ_1. □

Recall that for $i = 1, 2$, we denote by $\tau_i(s_i, r_i) = h_i(s_i, s_i') > 0$ the length of the segment between consecutive bounces $(s_i, r_i) \in \Lambda_i$ and $(s_i', r_i') = \mathcal{F}_i(s_i, r_i) \in \Lambda_i$, so that $\mathcal{F}_i^* \lambda_i - \lambda_i = d\tau_i$. By the fact that $\mathcal{D}_1, \mathcal{D}_2$ have the same periodic length data on Λ_1 and Λ_2, it follows from Livsic's theorem that the restriction of $\tau_2 \circ \Psi - \tau_1$ to Λ_1 is a coboundary, i.e., for some continuous function $\chi \colon \Lambda_1 \to \mathbb{R}$, we have

$$\tau_2 \circ \Psi - \tau_1 = \chi \circ \mathcal{F}_1 - \chi \quad \text{on } \Lambda_1. \tag{3.2}$$

Actually, as Ψ is $\mathcal{C}^{k-1}$ in Whitney sense, by the results of Nicol-Török [50, Theorem 3.2], the function χ is also $\mathcal{C}^{k-1}$ in Whitney sense.

Lemma 3.2. *It holds*

$$\Psi^* \lambda_2 - \lambda_1 = d\chi \quad \textit{on } \Lambda_1. \tag{3.3}$$

In particular, by differentiating (3.3), it holds $\Psi^*(ds_2 \wedge dr_2) = ds_1 \wedge dr_1$ on Λ_1.

Proof. Since $\mathcal{F}_i^* \lambda_i - \lambda_i = d\tau_i$, for $i = 1, 2$, and as $\Psi \circ \mathcal{F}_1|_{\Lambda_1} = \mathcal{F}_2 \circ \Psi|_{\Lambda_1}$, we deduce from (3.2) that on Λ_1,

$$\begin{aligned} \mathcal{F}_1^*\big(\Psi^* \lambda_2 - \lambda_1 - d\chi\big) &= \Psi^*(\lambda_2 + d\tau_2) - \lambda_1 - d\tau_1 - \mathcal{F}_1^* d\chi \\ &= \Psi^* \lambda_2 - \lambda_1 + d(\tau_2 \circ \Psi - \tau_1 - \chi \circ \mathcal{F}_1) = \Psi^* \lambda_2 - \lambda_1 - d\chi. \end{aligned}$$

Let ϖ be the one-form $(\Psi^* \lambda_2 - \lambda_1 - d\chi)|_{\Lambda_1}$. By the above identity, for any q-periodic point $p_1 \in \Lambda_1$, $q \geq 2$, we have

$$\varpi(p_1) = (\mathcal{F}_1^q)^* \varpi(p_1) = \varpi(p_1) \circ D\mathcal{F}_1^q(p_1). \tag{3.4}$$

By the hyperbolicity, we have a splitting $E^s_{\mathcal{F}_1}(p_1) \oplus E^u_{\mathcal{F}_1}(p_1)$ of the tangent space at p_1 into stable and unstable spaces. Let us choose a basis $(e^s(p_1), e^u(p_1)) \in E^s_{\mathcal{F}_1}(p_1) \times E^u_{\mathcal{F}_1}(p_1)$. We denote by $(d\pi_1^s, d\pi_1^u)$ the dual basis, i.e., for any tangent vector v, $d\pi_1^s(v)$, resp. $d\pi_1^u(v)$ denotes the component of v along $e^s(p_1)$, resp. $e^u(p_1)$. In particular, there exist $\alpha^s(p_1), \alpha^u(p_1) \in \mathbb{R}$ such that

$$\varpi(p_1) = \alpha^s(p_1)\, d\pi_1^s + \alpha^u(p_1)\, d\pi_1^u.$$

Letting $0 < \mu(p_1) < 1 < \mu^{-1}(p_1)$ be the eigenvalues of $D\mathcal{F}_1^q(p_1)$, we thus have

$$\varpi(p_1) \circ D\mathcal{F}_1^q(p_1) = \mu(p_1)\alpha^s(p_1)\, d\pi_1^s + \mu^{-1}(p_1)\alpha^u(p_1)\, d\pi_1^u.$$

We deduce from (3.4) that

$$\big(1 - \mu(p_1)\big)\alpha^s(p_1)\, d\pi_1^s + \big(1 - \mu^{-1}(p_1)\big)\alpha^u(p_1)\, d\pi_1^u = 0.$$

As $\mu(p_1) \neq 1$, and since $d\pi_1^s, d\pi_1^u$ is a basis of the cotangent space at p_1, it follows that $\alpha^s(p_1) = \alpha^u(p_1) = 0$. In other words, $\varpi(p_1) = 0$, for any periodic point $p_1 \in \Lambda_1$. By the continuity of ϖ on Λ_1, and since periodic points are dense in Λ_1, we deduce that $\varpi|_{\Lambda_1} = 0$, as desired. □

In order to complete the proof of Theorem C, we still need to show (1.8), which is done in the next subsection.

3.1. **Image of the time-reversal involution by the conjugacy.** We consider the same framework and keep the same notation as above. The conjugacy Ψ is not unique, as we may pre-compose, resp. post-compose it with any fixed iterate of $\mathcal{F}_1$, resp. $\mathcal{F}_2$. Yet, in some cases, there is a canonical way to choose the conjugacy in such a way that it preserves the time-reversal symmetry of the billiard dynamics; this is what we discuss in this subsection. Recall that for $i = 1, 2$, $\mathcal{I}_i \colon (s_i, r_i) \mapsto (s_i, -r_i)$ is the time-reversal involution, so that $\mathcal{F}_i \circ \mathcal{I}_i = \mathcal{I}_i \circ \mathcal{F}_i^{-1}$. In the following, we investigate when it is actually possible to normalize the conjugacy such that it conjugates the time-reversal involutions of $\mathcal{F}_1$ and $\mathcal{F}_2$, i.e.,

$$\Psi \circ \mathcal{I}_1 = \mathcal{I}_2 \circ \Psi \quad \text{on } \Lambda_1. \tag{3.5}$$

Let us denote by $\hat{\mathcal{I}}_1 := \Psi^{-1} \circ \mathcal{I}_2 \circ \Psi|_{\Lambda_1}$ the image of $\mathcal{I}_2$ after conjugating by Ψ. Clearly, $\hat{\mathcal{I}}_1$ is involutive, and it conjugates $\mathcal{F}_1$ to its inverse $\mathcal{F}_1^{-1}$. In particular, the map $\Gamma := \hat{\mathcal{I}}_1 \circ \mathcal{I}_1$ belongs to the centralizer of the map $\mathcal{F}_1$ on the basic set Λ_1, i.e.,

$$\Gamma \circ \mathcal{F}_1 = \mathcal{F}_1 \circ \Gamma \quad \text{on } \Lambda_1.$$

The centralizer of Axiom A diffeomorphisms at basic pieces is typically trivial (see [24, 61]), hence we expect Γ to be an iterate of $\mathcal{F}_1$. It is actually the case, by [61, Theorem A], as long as the map Γ fixes the orbits of $\mathcal{F}_1$, i.e., assuming that

$$\forall\, x_1 \in \Lambda_1, \quad \Gamma(x_1) = \mathcal{F}_1^{\ell}(x_1), \quad \text{for some } \ell = \ell(x_1) \in \mathbb{Z}. \tag{3.6}$$

Actually, we can prove directly:

Lemma 3.3. *If (3.6) holds, then there exists an integer $m \in \mathbb{Z}$ such that*

$$\mathcal{I}_2 \circ \Psi|_{\Lambda_1} = \Psi \circ \mathcal{I}_1 \circ \mathcal{F}_1^m|_{\Lambda_1}. \tag{3.7}$$

Proof. Let $x_1 \in \Lambda_1$, and take $\ell = \ell(x_1) \in \mathbb{Z}$ such that (3.6) holds for x_1. We have

$$\begin{aligned}\Gamma(\mathcal{F}_1(x_1)) &= \Psi^{-1} \circ \mathcal{I}_2 \circ \Psi \circ \mathcal{I}_1 \circ \mathcal{F}_1(x_1) = \Psi^{-1} \circ \mathcal{I}_2 \circ \Psi \circ \mathcal{F}_1^{-1} \circ \mathcal{I}_1(x_1) = \cdots = \\ &= \mathcal{F}_1 \circ \Psi^{-1} \circ \mathcal{I}_2 \circ \Psi \circ \mathcal{I}_1(x_1) = \mathcal{F}_1 \circ \Gamma(x_1) = \mathcal{F}_1^{\ell}(\mathcal{F}_1(x_1)),\end{aligned}$$

hence the integer ℓ in (3.6) is constant along the orbits. As $\mathcal{F}_1|_{\Lambda_1}$ is transitive, considering $x_1 \in \Lambda_1$ with a dense orbit, and by continuity, this finishes the proof. □

Besides, in the case where $\mathcal{D}_1, \mathcal{D}_2$ are open dispersing billiards, after changing the conjugacy, it is possible to verify (3.5):

Lemma 3.4. *If, furthermore, $\mathcal{F}_1, \mathcal{F}_2$ have a periodic point of period 2 (in particular, when $\mathcal{D}_1, \mathcal{D}_2 \in \mathbf{B}$), then, based on (3.7), we can redefine Ψ so that (3.5) holds.*

Proof. Let us show that the integer m in (3.7) is even. Indeed, let x_1 be a periodic point of period 2 for $\mathcal{F}_1$. Thus, both $\Psi(x_1)$ and $\mathcal{F}_1^m(x_1)$ are 2-periodic, for $\mathcal{F}_2$ and $\mathcal{F}_1$ respectively. In particular, the point $\Psi(x_1)$, resp. $\mathcal{F}_1^m(x_1)$, is fixed under $\mathcal{I}_2$, resp. $\mathcal{I}_1$. Therefore, by (3.7), $\Psi(x_1) = \Psi(\mathcal{F}_1^m(x_1))$; by the injectivity of Ψ, we conclude that $\mathcal{F}_1^m(x_1) = x_1$, hence $m = 2\ell$, for some $\ell \in \mathbb{Z}$. Let us consider the map $\widehat{\Psi} := \Psi \circ \mathcal{F}_1^{-\ell}$. By (3.7), equation (3.5) is satisfied for $\widehat{\Psi}$ in place of Ψ, as

$$\mathcal{I}_2 \circ \widehat{\Psi}|_{\Lambda_1} = \Psi \circ \mathcal{I}_1 \circ \mathcal{F}_1^{m-\ell}|_{\Lambda_1} = \Psi \circ \mathcal{F}_1^{\ell - m} \circ \mathcal{I}_1|_{\Lambda_1} = \widehat{\Psi} \circ \mathcal{I}_1|_{\Lambda_1}.$$

Besides, the map $\widehat{\Psi}$ still conjugates $\mathcal{F}_1|_{\Lambda_1}$ to $\mathcal{F}_2|_{\Lambda_2}$, and it is also $\mathcal{C}^{k-1}$ in Whitney sense, which concludes. □

Alternatively, we have:

Lemma 3.5. *If there exists $x_1 \in \Lambda_1 \cap \{r_1 = 0\}$ whose orbit is dense in Λ_1, such that $\mathcal{F}_2^m \circ \Psi(x_1) \in \{r_2 = 0\}$, for $m \in \mathbb{Z}$, then we can redefine Ψ so that (3.5) holds.*

Proof. Let us assume that there exists a point $x_1 \in \Lambda_1 \cap \{r_1 = 0\}$ whose orbit is dense in Λ_1, and such that $\mathcal{F}_2^m \circ \Psi(x_1) \in \{r_2 = 0\}$ for some $m \in \mathbb{Z}$. Let us consider the map $\widehat{\Psi} := \mathcal{F}_2^m \circ \Psi|_{\Lambda_1} = \Psi \circ \mathcal{F}_1^m|_{\Lambda_1}$. For any integer $\ell \in \mathbb{Z}$, we have

$$\begin{aligned}\mathcal{I}_2 \circ \widehat{\Psi}(\mathcal{F}_1^\ell(x_1)) &= \mathcal{I}_2 \circ \mathcal{F}_2^m \circ \Psi \circ \mathcal{F}_1^\ell(x_1) = \cdots = \mathcal{F}_2^{-\ell} \circ \mathcal{I}_2 \circ \mathcal{F}_2^m \circ \Psi(x_1)\\ &= \mathcal{F}_2^{-\ell} \circ \mathcal{F}_2^m \circ \Psi(x_1) = \mathcal{F}_2^m \circ \Psi \circ \mathcal{F}_1^{-\ell}(x_1) = \widehat{\Psi} \circ \mathcal{I}_1(\mathcal{F}_1^\ell(x_1)).\end{aligned}$$

In other words, (3.5) is satisfied for $\widehat{\Psi}$ in place of Ψ on the orbit of x_1; as the latter is dense, and by continuity, it is satisfied everywhere on Λ_1, which concludes. □

Remark 3.6. Note that if there exists a conjugacy map Ψ which satisfies (3.5), then it is unique in the following sense: if $\widehat{\Psi}$ is another conjugacy map which satisfies (3.5) and such that $\Psi^{-1} \circ \widehat{\Psi}$ fixes $\mathcal{F}_1$-orbits, then $\widehat{\Psi} = \Psi$. Indeed, in this case, $\Psi^{-1} \circ \widehat{\Psi}$ commutes with $\mathcal{F}_1$; arguing as above, we see that it is equal to $\mathcal{F}_1^m$, for some $m \in \mathbb{Z}$. Since $\Psi^{-1} \circ \widehat{\Psi}$ is also in the centralizer of $\mathcal{I}_1$, we deduce that $\mathcal{F}_1^m$ commutes with $\mathcal{I}_1$. But we also have $\mathcal{F}_1^m \circ \mathcal{I}_1 = \mathcal{I}_1 \circ \mathcal{F}_1^{-m}$, and hence, $\mathcal{F}_1^{2m} = \mathrm{id}$ on Λ_1; in other words, each point in Λ_1 is $2m$-periodic. As $\mathcal{F}_1|_{\Lambda_1}$ is transitive, it is possible only if $m = 0$, i.e., $\widehat{\Psi} = \Psi$ on Λ_1.

Assuming that (3.5) holds, we also have the following result.

Lemma 3.7. *The function χ in (3.2) can be chosen such that $\chi \circ \mathcal{I}_1 = -\chi$, i.e.,*

$$\chi(s_1, -r_1) = -\chi(s_1, r_1), \quad \forall\,(s_1, r_1) \in \Lambda_1. \tag{3.8}$$

Proof. For $i = 1, 2$, we have $\tau_i = \tau_i \circ \mathcal{I}_i \circ \mathcal{F}_i$ (see Figure 7) and $\mathcal{F}_i \circ \mathcal{I}_i = \mathcal{I}_i \circ \mathcal{F}_i^{-1}$. Thus, by (3.5), we deduce that on Λ_1, it holds

$$\begin{aligned}\chi \circ \mathcal{F}_1 - \chi &= \tau_2 \circ \Psi - \tau_1 = \tau_2 \circ \mathcal{I}_2 \circ \mathcal{F}_2 \circ \Psi - \tau_1 \circ \mathcal{I}_1 \circ \mathcal{F}_1\\ &= \tau_2 \circ \mathcal{I}_2 \circ \Psi \circ \mathcal{F}_1 - \tau_1 \circ \mathcal{I}_1 \circ \mathcal{F}_1 = \big(\tau_2 \circ \Psi - \tau_1\big) \circ \mathcal{I}_1 \circ \mathcal{F}_1\\ &= \chi \circ \mathcal{F}_1 \circ \mathcal{I}_1 \circ \mathcal{F}_1 - \chi \circ \mathcal{I}_1 \circ \mathcal{F}_1 = \chi \circ \mathcal{I}_1 - \chi \circ \mathcal{I}_1 \circ \mathcal{F}_1,\end{aligned}$$

hence

$$(\chi + \chi \circ \mathcal{I}_1) \circ \mathcal{F}_1 = \chi + \chi \circ \mathcal{I}_1.$$

Therefore, the function $\chi + \chi \circ \mathcal{I}_1$ on Λ_1 is $\mathcal{F}_1$-invariant, hence constant, as $\mathcal{F}_1|_{\Lambda_1}$ is transitive and χ is continuous. Since χ is defined up to constant (for any $c \in \mathbb{R}$, (3.2) also holds for $\chi + c$ in place of χ), we can assume that this constant vanishes, which concludes. □

In particular, for any point $x_1 = (s_1, 0) \in \Lambda_1 \cap \{r_1 = 0\}$, (3.8) gives $\chi(x_1) = 0$, while (3.3) gives $d\chi(x_1) = 0$, as $\Psi(\Lambda_1 \cap \{r_1 = 0\}) = \Lambda_2 \cap \{r_2 = 0\}$. The proof of Theorem C is complete.

Similarly, the conjugacy $\widetilde{\Psi}$ between the billiard flows $\Phi_1|_{\Lambda_1^{\tau_1}}, \Phi_2|_{\Lambda_2^{\tau_2}}$ is not unique, as we can pre-, resp. post-compose it with any Φ_1^t, resp. Φ_2^t, $t \in \mathbb{R}$. Yet, there is

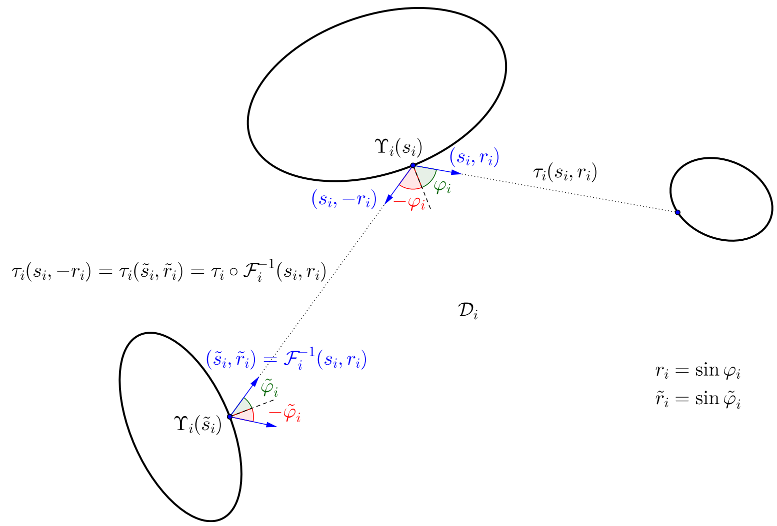


FIGURE 7. Time-reversal symmetry and generating functions.

also a canonical way to choose it, which we now explain. For $i = 1, 2$, we denote by $\widetilde{\mathcal{I}}_i : (x_i, y_i, \omega_i) \mapsto (x_i, y_i, \omega_i + \pi)$ the time reversal involution in (x_i, y_i, ω_i)-coordinates. Let us for instance assume that for $i = 1, 2$, there exists $X_i \in \Lambda_i^{\tau_i}$ associated to a point on $\partial\mathcal{D}_i$ with a perpendicular bounce, whose orbit is dense, and such that X_2, $\widetilde{\Psi}(X_1)$ are in the same orbit. After time-translation, $\widetilde{\Psi}(X_1) = X_2$, and then,

$$\widetilde{\Psi} \circ \widetilde{\mathcal{I}}_1|_{\Lambda_1^{\tau_1}} = \widetilde{\mathcal{I}}_2 \circ \widetilde{\Psi}|_{\Lambda_1^{\tau_1}}. \tag{3.9}$$

To show this, we argue as in Lemma 3.5: indeed, as $\widetilde{\Psi}(X_1) = X_2$, we see that (3.9) holds on the orbit of X_1, hence everywhere, by the transitivity of $\Phi_1|_{\Lambda_1^{\tau_1}}$.

Although it is not clear *a priori* that $\widetilde{\Psi}$ sends points associated to bounces on $\partial\mathcal{D}_1$ to points associated to bounces on $\partial\mathcal{D}_2$, we will show that it is indeed the case when the point on $\partial\mathcal{D}_1$ has a perpendicular bounce. For $i = 1, 2$, we denote by $\widetilde{\Pi}_i : (x_i, y_i, \omega_i) \mapsto (x_i, y_i)$ the projection on the table $\mathcal{D}_i$.

Lemma 3.8. *Assume that* (3.9) *holds. Then, for any* $Y_1 \in \Lambda_1^{\tau_1}$ *associated to a point* $\widetilde{\Pi}_1(Y_1) \in \partial\mathcal{D}_1$ *with a perpendicular bounce, its image* $Y_2 := \widetilde{\Psi}(Y_1) \in \Lambda_2^{\tau_2}$ *under* $\widetilde{\Psi}$ *is also associated to a point* $\widetilde{\Pi}_2(Y_2) \in \partial\mathcal{D}_2$ *with a perpendicular bounce on an obstacle.*

Proof. Let Y_1, $Y_2 := \widetilde{\Psi}(Y_1)$ be as in the lemma. As Y_1 has a perpendicular bounce, we have $\widetilde{\mathcal{I}}_1 \circ \Phi_1^{-t}(Y_1) = \Phi_1^t(Y_1)$, for all $t \in \mathbb{R}$. Since $\widetilde{\Psi}$ conjugates $\Phi_1|_{\Lambda_1^{\tau_1}}$ to $\Phi_2|_{\Lambda_2^{\tau_2}}$, by (3.9), and as $\widetilde{\Pi}_i \circ \widetilde{\mathcal{I}}_i = \widetilde{\Pi}_i$, $i = 1, 2$, we deduce that for any $t \in \mathbb{R}$, it holds

$$\begin{aligned}\widetilde{\Pi}_2 \circ \Phi_2^{-t}(Y_2) = \widetilde{\Pi}_2 \circ \widetilde{\mathcal{I}}_2 \circ \widetilde{\Psi} \circ \Phi_1^{-t}(Y_1) &= \widetilde{\Pi}_2 \circ \widetilde{\Psi} \circ \widetilde{\mathcal{I}}_1 \circ \Phi_1^{-t}(Y_1) \\ &= \widetilde{\Pi}_2 \circ \widetilde{\Psi} \circ \Phi_1^t(Y_1) = \widetilde{\Pi}_2 \circ \Phi_2^t(Y_2).\end{aligned}$$

But $\widetilde{\Pi}_2 \circ \Phi_2^{-t}(Y_2) = \widetilde{\Pi}_2 \circ \Phi_2^t(Y_2)$, for all $t \in \mathbb{R}$, if and only if Y_2 is associated to a point on $\partial\mathcal{D}_2$ with a perpendicular bounce, which concludes. □

3.2. **Proof of Corollary D.** As in Corollary D, fix $\ell \geq 3$, and let $\mathcal{D}_1, \mathcal{D}_2 \in \mathbf{B}_{ne}(\ell)$ with $\mathcal{C}^k$ boundaries, for some $k \geq 3$, such that $\mathcal{D}_1, \mathcal{D}_2$ have the same marked length spectrum. Then, according to Theorem C, the respective billiards maps $\mathcal{F}_1, \mathcal{F}_2$ are conjugated on $\Omega(\mathcal{F}_1), \Omega(\mathcal{F}_2)$ by a map $\Psi\colon \Omega(\mathcal{F}_1) \to \Omega(\mathcal{F}_2)$ that is $\mathcal{C}^{k-1}$ in Whitney sense and such that $\Psi^*(ds_2 \wedge dr_2) = ds_1 \wedge dr_1$ on $\Omega(\mathcal{F}_1)$. In the following, we let $\Omega_i := \Omega(\mathcal{F}_i)$, for $i = 1, 2$.

Let us recall that $\mathcal{F}_1|_{\Omega_1}$, $\mathcal{F}_2|_{\Omega_2}$ are conjugated to the same subshift of finite type on the alphabet $\mathscr{A} = \{1, \ldots, \ell\}$ associated with the transition matrix $(1-\delta_{i,j})_{1 \leq i,j \leq \ell}$, where $\delta_{i,j} = 1$, when $i = j$, and $\delta_{i,j} = 0$ otherwise. We say that a word $\varsigma = (\varsigma_j)_j \in \mathscr{A}^{\mathbb{Z}}$ is *admissible*, if $\varsigma_{j+1} \neq \varsigma_j$, for all $j \in \mathbb{Z}$. We also let $\mathrm{Adm} \subset \cup_{j \geq 2} \mathscr{A}^j$ be the set of all finite words $\sigma = \sigma_1 \ldots \sigma_j$, $j \geq 2$, such that $\sigma^\infty := \cdots \sigma\sigma\sigma \cdots \in \mathrm{Adm}_\infty$. We normalize the conjugacy Ψ by requiring that for each $y_1 \in \Omega_1$, the points y_1 and $\Psi(y_1) \in \Omega_2$ are coded by the same admissible word. Symbolically, the actions of $\mathcal{I}_1, \mathcal{I}_2$ amount to switching the symbolic past and future. In particular, by our choice that Ψ preserves the symbolic coding, we have $\Psi \circ \mathcal{I}_1 = \mathcal{I}_2 \circ \Psi$ on Ω_1, where $\mathcal{I}_i\colon (s_i, r_i) \mapsto (s_i, -r_i)$ is the time-reversal involution, for $i = 1, 2$.

Due to the equality of the marked length spectra, the respective generating functions τ_1, τ_2 of $\mathcal{F}_1, \mathcal{F}_2$ satisfy

$$\tau_2 \circ \Psi - \tau_1 = \chi \circ \mathcal{F}_1 - \chi \quad \text{on } \Omega_1,$$

for some coboundary $\chi\colon \Omega_1 \to \mathbb{R}$. Arguing as above, we see that χ is $\mathcal{C}^{k-1}$ in Whitney sense; moreover, properties (1.7)-(1.8) about χ follow from the previous results (see Lemma 3.2 and Lemma 3.7), which concludes the proof of Corollary D.

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