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Dynamische Systeme

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Abstracts

Spectral determination of a class of open dispersing billiards

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We consider billiard tables $\mathcal{D} \subset \mathbb{R}^2$ given by $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^m \mathcal{O}_i$, for some integer $m \geq 3$, where each $\mathcal{O}_i$ is a convex domain with sufficiently smooth boundary $\partial\mathcal{O}_i$ (at least of class C^3; in some places, we will actually assume the boundary to be analytic). We refer to each of the $\mathcal{O}_i$'s as *obstacle*, and parametrize $\partial\mathcal{O}_i$ in arclength. We assume that the *non-eclipse condition* holds, i.e., that the convex hull of any two obstacles is disjoint from the other $m-2$ obstacles. The set of all billiard tables obtained by removing from the plane m strictly convex obstacles with C^3, resp. analytic boundary satisfying the non-eclipse condition will be denoted by $\mathbf{B}(m)$, resp. $\mathbf{B}^\omega(m) \subset \mathbf{B}(m)$.

Fix $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^m \mathcal{O}_i \in \mathbf{B}(m)$. We denote the collision space by

$$\mathcal{M} = \cup_i \mathcal{M}_i, \qquad \mathcal{M}_i = \big\{(q,v),\ q \in \partial\mathcal{O}_i,\ v \in \mathbb{R}^2,\ \|v\| = 1,\ \langle v, n\rangle \geq 0\big\},$$

where n is the unit normal vector to $\partial\mathcal{O}_i$ pointing inside $\mathcal{D}$. Each $x = (q,v) \in \mathcal{M}$ can be identified with a pair $(s,r) \in \mathbb{R} \times [-1,1]$, where s is the associated arclength parameter, φ is the oriented angle between n and v, and $r := \sin(\varphi)$. Whenever it is well-defined, the image by the billiard map $\mathcal{F}$ of a pair (s, r) of parameters is the new pair (s', r') associated to the next collision of the billiard trajectory with $\partial\mathcal{D}$; the map $\mathcal{F}$ is symplectic for the form $ds \wedge dr$ (in fact, exact symplectic).

It is clear that a lot of trajectories will escape to infinity. In fact, due to the convexity of the obstacles, the set of points $x = (s, r)$ whose iterates $\mathcal{F}^n(x)$ under the billiard map are well-defined for any $n \in \mathbb{Z}$ is homeomorphic to a Cantor set $\mathcal{NE}$ (see e.g. [4, 7]). The restriction of the dynamics to $\mathcal{NE}$ is conjugated to a subshift of finite type associated to the transition matrix $A := (1 - \delta_{ij})_{1 \leq i,j \leq m} \in \mathfrak{M}_m(\mathbb{R})$. In other words, any *admissible* word $(\varsigma_j)_j \in \{1, \cdots, m\}^{\mathbb{Z}}$ – i.e., such that $\varsigma_{j+1} \neq \varsigma_j$ for all $j \in \mathbb{Z}$ – can be realized by a unique orbit.[1] In particular, any periodic orbit of period $p \geq 2$ can be represented by an admissible word $\sigma^\infty := \ldots \sigma\sigma\sigma \ldots$ for some *finite admissible word* $\sigma = (\sigma_1 \sigma_2 \ldots \sigma_p) \in \{1, \cdots, m\}^p$. We denote by Adm the set of finite admissible words $\sigma \in \cup_{p \geq 2} \{1, \cdots, m\}^p$.

The *Marked Length Spectrum* $\mathcal{MLS}(\mathcal{D})$ of $\mathcal{D}$ is defined as the function

$$\mathcal{L} \colon \mathrm{Adm} \to \mathbb{R}_+, \quad \sigma \mapsto \mathcal{L}(\sigma),$$

where $\mathcal{L}(\sigma)$ is the length of the closed trajectory labeled by σ. In the following, an object is said to be a $\mathcal{MLS}$*-invariant* if it can be obtained by the sole knowledge of the Marked Length Spectrum. We are interested in the following *inverse problem*:

$$\mathcal{MLS}(\mathcal{D}) \quad \rightsquigarrow \quad \text{“Geometry” of } \mathcal{D}?$$

Note that in finite regularity, the information given by $\mathcal{MLS}(\mathcal{D})$ is insufficient to reconstruct the geometry of the whole table; at best, we can hope to describe the geometry near points associated to an arclength parameter s such that $(s, r) \in \mathcal{NE}$ for some $r \in [-1, 1]$. In a first work [1], we show that it is indeed possible to extract some information from $\mathcal{MLS}(\mathcal{D})$ on the local geometry near very specific points. In [3], we assume that the boundary of the obstacles is analytic, i.e. $\mathcal{D} \in \mathbf{B}^\omega(m)$, in such a way that local geometric information may determine the whole table; in this case, under some symmetry and (mild) non-degeneracy assumptions, we show that $\mathcal{MLS}(\mathcal{D})$ does determine $\mathcal{D}$ up to isometries.

0.1. **Results in finite regularity.** Let us fix $m \geq 3$ and $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^m \mathcal{O}_i \in \mathbf{B}(m)$. Given a periodic point $x = (s, r) \in \mathcal{NE}$, the basic idea is to combine the information given by a sequence $(x_n)_{n \geq 0}$ of periodic points $x_n \in \mathcal{NE}$ accumulating x in order to extract some geometric quantities at the point of arclength parameter s. Thanks to the symbolic coding recalled above, this amounts to considering periodic orbits encoded by longer and longer finite admissible words obtained by truncating the coding of x.

One major issue is that *a priori*, the information obtained in this way is “averaged” over the different points in the orbit of x; yet, in [1], we found out some mechanism which allows us to distinguish between the two points in 2-periodic orbits. More precisely, let us consider a 2-periodic orbit encoded by a word $\sigma = (\sigma_1 \sigma_0) \in \{1, \cdots, m\}^2$, $\sigma_0 \neq \sigma_1$. Let $\tau_1 \in \{1, \cdots, m\} \setminus \{\sigma_0, \sigma_1\}$, and set $\tau := (\tau_1 \sigma_0)$. We consider the sequence of periodic orbits encoded by the words

[1] Each symbol $\varsigma_j \in \{1, \cdots, m\}$ corresponds to the obstacle $\mathcal{O}_{\varsigma_j}$ where the bounce happens.

$h_n := \tau\sigma^n \in \mathrm{Adm}$, $n \geq 0$; as $n \to +\infty$, their points accumulate the points of some orbit h_∞ that is homoclinic to the orbit encoded by σ.

Theorem 0.1 (Bálint-De Simoi-Kaloshin-Leguil [1]). *We denote by $R_0, R_1 > 0$ the respective radii of curvature at the points with symbols σ_0, σ_1, and let $\lambda < 1$ be the smallest eigenvalue of $D\mathcal{F}^2$ at the points of σ. For $n \gg 1$, it holds:*

(1) $\mathcal{L}(\tau\sigma^n) - (n+1)\mathcal{L}(\sigma) - \mathcal{L}^\infty = -C \cdot \mathcal{Q}\left(\frac{2R_0}{\mathcal{L}(\sigma)}, \frac{2R_1}{\mathcal{L}(\sigma)}\right)\lambda^n + O(\lambda^{\frac{3n}{2}})$, *$n$ even,*

(2) $\mathcal{L}(\tau\sigma^n) - (n+1)\mathcal{L}(\sigma) - \mathcal{L}^\infty = -C \cdot \mathcal{Q}\left(\frac{2R_1}{\mathcal{L}(\sigma)}, \frac{2R_0}{\mathcal{L}(\sigma)}\right)\lambda^n + O(\lambda^{\frac{3n}{2}})$, *$n$ odd,*

for some real number $\mathcal{L}^\infty = \mathcal{L}^\infty(\sigma, \tau) \in \mathbb{R}$, some constant $C = C(\sigma, \tau) > 0$, and some explicit quadratic form $\mathcal{Q}\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

The reason why the parity of n affects the estimates is due to the "palindromic" symmetry of h_n: indeed, each point in $\partial\mathcal{D}$ with arclength parameter s such that (s, r) belongs to the orbit h_n for some $r \in [-1, 1]$ is seen twice – as $(s, -r)$ also belongs to h_n – except when $r = 0$; this is the case for exactly two points in the orbit h_n, associated to perpendicular bounces. Among those two points, only one contributes to the first order term in the above estimates, and it is either on the boundary of the obstacle $\mathcal{O}_{\sigma_0}$ or of the obstacle $\mathcal{O}_{\sigma_1}$ depending on the parity of n. Theorem 0.1 has the following geometric consequence:

Corollary 0.2 (Bálint-De Simoi-Kaloshin-Leguil [1]). *The radii of curvature at the bouncing points of periodic orbits of period two are $\mathcal{MLS}$-invariants.*

Moreover, by Theorem 0.1, the *Lyapunov exponent* $-\frac{1}{2}\log\lambda$ of σ is also a $\mathcal{MLS}$-invariant. More generally, let us consider a periodic orbit of period $p \geq 2$, encoded by some finite admissible word $\widetilde{\sigma} \in \{1, \cdots, m\}^p$. The *Lyapunov exponent* $\mathrm{LE}(\widetilde{\sigma})$ of $\widetilde{\sigma}$ is equal to $-\frac{1}{p}\log\widetilde{\lambda}$, where $\widetilde{\lambda} = \widetilde{\lambda}(\widetilde{\sigma}) < 1$ is the smallest eigenvalue of $D\mathcal{F}^p$ at the points in the orbit. By adapting the construction explained above, we get:

Theorem 0.3 (Bálint-De Simoi-Kaloshin-Leguil [1]). *The Lyapunov exponent of each periodic orbit is a $\mathcal{MLS}$-invariant.*

0.2. **$\mathcal{MLS}$-determination of analytic billiard tables.** Let us now consider the case where the boundary of the table is analytic. Fix $m \geq 3$, and let $\mathbf{B}^\omega_{\mathrm{sym}}(m) \subset \mathbf{B}^\omega(m)$ be the subset of all billiard tables $\mathcal{D} \in \mathbf{B}^\omega(m)$ such that:

- the jets of the curvature function $\mathcal{K}$ are the same at the endpoints of the 2-periodic orbit (12);
- the jets of $\mathcal{K}|_{\partial\mathcal{O}_1}$, $\mathcal{K}|_{\partial\mathcal{O}_2}$ are even, assuming that $0_1 \in \partial\mathcal{O}_1$, $0_2 \in \partial\mathcal{O}_2$ are the arclength parameters of the endpoints of the orbit (12).

In the analytic setting, and modulo the partial $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry assumption introduced above, we can show:

Theorem 0.4 (De Simoi-Kaloshin-Leguil [3]). *There exists an open and dense set of billiard tables $\mathbf{B}^*_{\mathrm{sym}}(m) \subset \mathbf{B}^\omega_{\mathrm{sym}}(m)$ so that if $\mathcal{D} \in \mathbf{B}^*_{\mathrm{sym}}(m)$, then the geometry of $\mathcal{D}$ is entirely determined (modulo isometries) by $\mathcal{MLS}(\mathcal{D})$.*

The open and dense condition we require is actually a *non-degeneracy condition*: roughly speaking, it means that after a change of coordinates, the first coefficient in the expansion of the dynamics does not vanish.[2]

It is a standard fact that any continuous deformation of smooth domains which preserves the (unmarked) Length Spectrum $\mathcal{LS}(\mathcal{D})$ automatically preserves $\mathcal{MLS}(\mathcal{D})$ (see e.g. [8, Proposition 3.2.2]). A family $(\mathcal{D}_t)_{t\in(-1,1)}$ is an *iso-length-spectral family of billiards in* $\mathbf{B}^*_{\mathrm{sym}}(m)$ if each $\mathcal{D}_t$ is in $\mathbf{B}^*_{\mathrm{sym}}(m)$, the map $(-1,1)\ni t\mapsto \mathcal{D}_t$ is continuous, and $\mathcal{LS}(\mathcal{D}_t)=\mathcal{LS}(\mathcal{D}_0)$, for all $t\in(-1,1)$. Therefore, we obtain:

Corollary 0.5 ([3])**.** *Any iso-length-spectral deformation in* $\mathbf{B}^*_{\mathrm{sym}}(m)$ *is isometric.*

Our results are an analog of the result of Colin de Verdière [2] for the class of chaotic billiards under consideration, or an analog in terms of the Marked Length Spectrum of the results of Zelditch [9, 10, 11] (see also [5]).

Let us give some ideas of the proof. Fix $\mathcal{D}\ \in \mathbf{B}^*_{\mathrm{sym}}(m)$. For the 2-periodic $\sigma=(12)$, we consider the same sequence $(h_n)_{n\geq 0}$ of periodic orbits accumulating some orbit h_∞ homoclinic to σ. In a first time, we show that after a *canonical*[3] symplectic change of coordinates, the dynamics of the square $\mathcal{F}^2$ of the billiard map in a neighbourhood of the trajectory h_∞ can be replaced with two maps: the *Birkhoff Normal Form* $N=N(\sigma)$ of $\mathcal{F}^2$ associated to σ, and some gluing map $\mathcal{G}=\mathcal{G}(\sigma,\tau)$. Working with this new system of coordinates, we show that for each integer $n\geq 0$, the Lyapunov exponent of h_n can be expanded as a series (reminiscent of the asymptotic expansion of the lengths obtained in [6]):

$$2\lambda^n\cosh(2(n+1)\mathrm{LE}(h_n))=\sum_{p=0}^{+\infty}\sum_{q=0}^{p}L_{q,p}n^q\lambda^{np},$$

for some sequence $(L_{q,p})_{\substack{p=0,\cdots,+\infty\\ q=0,\cdots,p}}$, and where $\lambda=\lambda(\sigma)<1$. In particular, each coefficient $L_{q,p}$ is a $\mathcal{MLS}$-invariant. Then, we show that modulo the non-degeneracy condition mentioned previously, it is possible to extract enough information from $(L_{q,p})_{p,q}$ to recover N and the differential $D\mathcal{G}$ at some points in h_n. In fact, the $\mathcal{MLS}$-determination of N does not require any symmetry assumption, and the same procedure can be carried out for more general palindromic periodic orbits. Following [2], and thanks to the $\mathbb{Z}_2\times\mathbb{Z}_2$-symmetry of $\{\mathcal{O}_1,\mathcal{O}_2\}$, we then show that the jet of $\mathcal{K}$ can be read off from the coefficients of N, which by analyticity determines entirely the geometry of $\mathcal{O}_1,\mathcal{O}_2$. Finally, we explain how the information given by the differential of the gluing map $\mathcal{G}$ can be utilized in order to recover the geometry of the other obstacles (note that no symmetry assumption is needed for this last step).

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