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Mini-Workshop: Reflectionless Operators: The Deift and Simon Conjectures

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Abstracts

Exponential estimates on the size of spectral gaps for quasi-periodic Schrödinger operators

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(joint work with Jiangong You†, Zhiyan Zhao‡ & Qi Zhou♯)

In the following, we consider one-dimensional discrete Schrödinger operators on $\ell^2(\mathbb{Z})$:

$$(H_{V,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n, \quad \forall\, n \in \mathbb{Z},$$

for some phase $\theta \in \mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$, some analytic potential $V\colon \mathbb{T}^d \to \mathbb{R}$, and where the (multi-)frequency $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{T}^d$ is chosen in such a way that $(1, \alpha_1, \dots, \alpha_d)$ is rationally independent. In this case, the spectrum of $H_{V,\alpha,\theta}$ is a compact subset of $\mathbb{R}$, independent of θ, denoted by $\Sigma_{V,\alpha}$. By the Gap-Labelling Theorem, it is of the form $\Sigma_{V,\alpha} = [\underline{E}, \overline{E}] \setminus \cup_{k \in \mathbb{Z}^d \setminus \{0\}} G_k(V)$ for some spectral gaps

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$G_k(V) = (E_k^-, E_k^+)$ labelled by integer vectors. For all $k \neq 0$, the restriction of the integrated density of states (IDS) $N_{V,\alpha}\colon \mathbb{R} \to [0,1]$ of $H_{V,\alpha,\theta}$ to the associate gap satisfies $N_{V,\alpha}|_{G_k(V)} = \langle k, \alpha\rangle \mod \mathbb{Z}$.

One particularly important example is given by almost Mathieu operators (AMO) $H_{\lambda,\alpha,\theta}$, in the case where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and the potential has the form $V = 2\lambda\cos 2\pi(\cdot)$ for some coupling constant $\lambda \in \mathbb{R}$. For simplicity, we denote by $G_k(\lambda) = (E_k^-, E_k^+)$ the gap with label k.

0.1. **Estimates on spectral gaps.** In what follows, given $d \geq 1$, $\gamma > 0$, $\tau \geq d-1$, we let

$$\mathrm{DC}_d(\gamma,\tau) := \left\{\varphi \in \mathbb{R}^d : \inf_{j\in\mathbb{Z}} |\langle n,\varphi\rangle - j| > \frac{\gamma}{|n|^\tau}, \quad \forall\, n \in \mathbb{Z}^d\backslash\{0\}\right\}$$

and we set $\mathrm{DC}_d := \bigcup_{\gamma>0,\,\tau>d-1} \mathrm{DC}_d(\gamma,\tau)$. Our first result is about exponential asymptotics on the size of spectral gaps for non-critical AMO with a Diophantine frequency.

Theorem 0.1 (L.-You-Zhao-Zhou, [17])**.** *For $\alpha \in \mathrm{DC}_1$, and for any $0 < \xi_0 < 1$, there exist $C_0 = C_0(\lambda,\alpha,\xi_0) > 0$, $C_1 = C_1(\lambda,\alpha) > 0$, and a numerical constant $\xi_1 > 1$ such that*

$$\begin{array}{rcccll} C_1\lambda^{\xi_1|k|} & \leq & |G_k(\lambda)| & \leq & C_0\lambda^{\xi_0|k|}, & \text{if } 0<\lambda<1,\\ C_1\lambda^{-\xi_1|k|} & \leq & |G_k(\lambda)| & \leq & C_0\lambda^{-\xi_0|k|}, & \text{if } 1<\lambda<\infty, \end{array}$$

for all $k \in \mathbb{Z}\backslash\{0\}$, where $|G_k(\lambda)|$ denotes the length of $G_k(\lambda)$.

The study of lower bounds dates back to the long-standing conjecture of the "Ten Martini Problem" [20] (finally solved by Avila-Jitomirskaya [4], after partial results due to Puig [19] etc.) and the so-called "Dry Ten Martini Problem" (see [6] for recent progress on this question), which is a further elaboration asking whether for any $\lambda \neq 0$ and irrational α, all possible spectral gaps of $H_{\lambda,\alpha,\theta}$ predicted by the Gap-Labelling Theorem are non-collapsed.

Regarding the question of upper bounds, the first result is due to Moser-Pöschel. In [18], given an analytic potential $V\colon \mathbb{T}^d \to \mathbb{R}$, $d \geq 2$, and $\varpi \in \mathrm{DC}_d$, they consider the continuous quasi-periodic Schrödinger operator on $L^2(\mathbb{R})$:

$$(\mathcal{L}_{V,\varpi}y)(t) = -y''(t) + V(\varpi t)y(t).$$

Thanks to KAM techniques, Moser-Pöschel proved that if V is small enough, then $|G_k(V)|$ is exponentially small with respect to $|k|$ provided that $|k|$ is sufficiently large and $\langle k,\varpi\rangle$ is not too close to the other $\langle m,\varpi\rangle$. Later on, Amor [16] proved that in the same setting, the spectral gaps have sub-exponential decay for all $k \in \mathbb{Z}^d\backslash\{0\}$. Damanik-Goldstein [9] gave a stronger result: $|G_k(V)| \leq \varepsilon e^{-\frac{r_0}{2}|k|}$ if $V\colon \mathbb{T}^d \to \mathbb{R}$ has a bounded analytic extension to the strip $\{z \in \mathbb{C}/\mathbb{Z} : |\Im z| < r_0\}$ and $\varepsilon := \sup_{|\Im z|<r_0} |V(z)|$ is sufficiently small. We obtain:

Theorem 0.2 (L.-You-Zhao-Zhou, [17])**.** *Let $\alpha \in \mathrm{DC}_d$ and let $V\colon \mathbb{T}^d \to \mathbb{R}$ be an analytic potential with a bounded analytic extension to the strip $\{z \in \mathbb{C}/\mathbb{Z} :$*

$|\Im z| < r_0\}$. *For any* $r \in (0, r_0)$, *there exists* $\varepsilon_0 = \varepsilon_0(V, \alpha, r_0, r) > 0$ *such that if* $\sup_{|\Im z|<r_0} |V(z)| < \varepsilon_0$, *then*

$$|G_k(V)| \le \varepsilon_0^{\frac{2}{3}} e^{-r|k|}, \quad \forall\, k \in \mathbb{Z}^d \backslash \{0\}.$$

0.2. **Homogeneous spectrum.** The exponential upper bounds on the size of spectral gaps in Theorem 0.2 can be used to prove homogeneity of the spectrum. Recall that a closed set $\mathcal{S} \subset \mathbb{R}$ is called homogeneous in the sense of Carleson if there exists $\mu > 0$ such that

$$|\mathcal{S} \cap (E - \epsilon, E + \epsilon)| > \mu\epsilon, \quad \forall\, E \in \mathcal{S},\ \forall\, 0 < \epsilon \le \operatorname{diam}\mathcal{S}.$$

Homogeneity of the spectrum plays an essential role in the inverse spectral theory of almost periodic potentials (for instance in the fundamental work of Sodin-Yuditskii [21, 22]).

Let us recall some recent results on the homogeneity of the spectrum. Building on the localization estimates developed in [9], Damanik-Goldstein-Lukic [11] proved that the spectrum of continuous Schrödinger operators $\mathcal{L}_{V,\varpi}$ with Diophantine ϖ and sufficiently small analytic potential V is homogeneous. For the discrete operator $H_{V,\alpha}$ in the positive Lyapunov exponent regime, Damanik-Goldstein-Schlag-Voda [12] proved that the spectrum is homogeneous for any $\alpha \in$ SDC, i.e., such that for some $\gamma, \tau > 0$, $\inf_{j\in\mathbb{Z}} |n\alpha - j| \ge \frac{\gamma}{|n|(\log|n|)^\tau}$, for all $n \in \mathbb{Z}\backslash\{0\}$. We show the following result:

Theorem 0.3 (L.-You-Zhao-Zhou, [17])**.** *Let* $\alpha \in$ SDC. *For a (measure-theoretically) typical analytic potential* $V\colon \mathbb{T} \to \mathbb{R}$, *the spectrum* $\Sigma_{V,\alpha}$ *is homogeneous.*

Given $E \in \mathbb{R}$, we set $S_E^V(\cdot) := \begin{pmatrix} E - V(\cdot) & -1 \\ 1 & 0 \end{pmatrix}$ and we let (α, S_E^V) be the associate *Schrödinger cocycle*. The energy $E \in \Sigma_{V,\alpha}$ is called *supercritical* (resp. *subcritical*) if the Lyapunov exponent is positive, i.e., $L(\alpha, S_E^V) > 0$ (resp. $L(\alpha, S_E^V(\cdot + \mathrm{i}\epsilon)) = 0$ for $|\epsilon| < \delta$). By Avila's global theory of one-frequency quasi-periodic Schrödinger operators [3], for a (measure-theoretically) typical analytic potential $V\colon \mathbb{T} \to \mathbb{R}$, any $E \in \Sigma_{V,\alpha}$ is either subcritical or supercritical. In particular, he shows that typically, there exists a finite number of intervals $(I_i)_i$ such that the set of all subcritical energies in the spectrum is $\Sigma_{V,\alpha}^{\mathrm{sub}} = \cup_i(\Sigma_{V,\alpha} \cap I_i)$. Since the supercritical regime was already handled in [12], we focus on energies E in the subcritical part of the spectrum. If $(p_n/q_n)_n$ denotes the sequence of best approximants of α, we let $\beta(\alpha) := \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}$. We obtain the following description of the subcritical spectrum and of the spectral gaps one of whose edge points is in $\Sigma_{V,\alpha}^{\mathrm{sub}}$:

Theorem 0.4 (L.-You-Zhao-Zhou, [17])**.** *Let* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ *satisfy* $\beta(\alpha) = 0$. *For typical analytic potentials* $V\colon \mathbb{T} \to \mathbb{R}$, *the following assertions hold.*

(1) *There exist constants* $C, \vartheta > 0$ *depending on* V, α, *such that*

$$|G_k(V)| \le Ce^{-\vartheta|k|}, \quad \forall\, k \in \mathbb{Z}\backslash\{0\} \text{ with } \overline{G_k(V)} \cap \Sigma_{V,\alpha}^{\mathrm{sub}} \ne \emptyset.$$

(2) *For any $\eta > 0$, there exists $D = D(V, \alpha, \eta) > 0$ such that*

$$\mathrm{dist}(G_k(V), G_{k'}(V)) \geq De^{-\eta|k'-k|},$$

if $k \neq k' \in \mathbb{Z}$ satisfy $\overline{G_k(V)} \cap I_i \neq \emptyset$ and $\overline{G_{k'}(V)} \cap I_i \neq \emptyset$ for some i.

(3) *There exists $\mu_0 \in (0, 1)$ such that*

$$|\Sigma_{V,\alpha} \cap (E - \epsilon, E + \epsilon)| > \mu_0 \epsilon, \quad \forall\ E \in \Sigma_{V,\alpha}^{\mathrm{sub}},\ \forall\ 0 < \epsilon \leq \mathrm{diam}\Sigma_{V,\alpha}.$$

0.3. **Deift's conjecture.** Deift's conjecture (Problem 1 of [13, 14]) asks whether for almost periodic initial data, the solutions to the KdV equation are almost periodic in the time variable. Tsugawa [24] proved local existence and uniqueness of solutions to the KdV equation when the frequency is Diophantine and the Fourier coefficients of the potential decay at a sufficiently fast polynomial rate. Damanik-Goldstein [10] then proved global existence and uniqueness for a Diophantine frequency and small quasi-periodic analytic initial datum. Recently, Binder-Damanik-Goldstein-Lukic [7] showed that in the same setting, the solution is in fact almost periodic in time, thus proving Deift's conjecture in this case. In our work, we consider the discrete version of Deift's conjecture, namely that for almost periodic initial data, the Toda flow is almost periodic in the time variable. Recall the Toda lattice equation

$$\begin{cases} a_n'(t) &= a_n(t)\,(b_{n+1}(t) - b_n(t))\,, \\ b_n'(t) &= 2(a_n^2(t) - a_{n-1}^2(t)), \end{cases} \qquad n \in \mathbb{Z}. \tag{1}$$

In view of Theorem 12.6 in [23], given an initial condition $(a(0), b(0)) \in \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z})$, there is a unique solution $(a, b) \in \mathcal{C}^\infty(\mathbb{R}, \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z}))$ to (1). We can identify $(a(t), b(t))$ with a doubly infinite Jacobi matrix $J(t)$:

$$(J(t)u)_n := a_{n-1}(t)\,u_{n-1} + b_n(t)\,u_n + a_n(t)\,u_{n+1}. \tag{2}$$

As a consequence of homogeneity (Theorem 0.4) and purely absolutely continuous spectrum of subcritical Schrödinger operators [2], we prove a discrete version of Deift's conjecture for almost periodic initial data, building on an previous result of Vinnikov-Yuditskii [25]. We show the following generalization of the result of Binder-Damanik-Goldstein-Lukic [7] to Avila's subcritical regime (see also the recent paper [8] for related advance on this problem).

Theorem 0.5 (L.-You-Zhao-Zhou, [17])**.** *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ with $\beta(\alpha) = 0$. Let $V\colon \mathbb{T} \to \mathbb{R}$ be a subcritical analytic potential, i.e., such that (α, S_E^V) is subcritical for all $E \in \Sigma_{V,\alpha}$. We consider the Toda flow (1) with initial condition $(a_n, b_n)(0) = (1, V(\theta + n\alpha))$, $n \in \mathbb{Z}$. Then*

(1) *For any $\theta \in \mathbb{T}$, (1) admits a unique solution $(a(t), b(t))$ defined for all $t \in \mathbb{R}$.*
(2) *For every t, the Jacobi matrix $J(t)$ given by (2) is almost periodic and has constant spectrum $\Sigma_{V,\alpha}$.*
(3) *The solution $(a(t), b(t))$ is almost periodic in t in the following sense: there is a continuous map $\mathcal{M}\colon \mathbb{T}^{\mathbb{Z}} \to \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z})$, a point $\varphi \in \mathbb{T}^{\mathbb{Z}}$ and a direction $\varpi \in \mathbb{R}^{\mathbb{Z}}$, such that $(a(t), b(t)) = \mathcal{M}(\varphi + \varpi t)$.*

In particular, the above conclusion holds for $V = 2\lambda \cos 2\pi(\cdot)$ *with* $0 < \lambda < 1$.

0.4. **Some ideas of the proofs.** Our approach is from the perspective of dynamical systems, and is based on quantitative (strong) almost reducibility. To obtain bounds on the size of spectral gaps, we analyze the behavior of Schrödinger cocycles close to the boundary of some spectral gap. At the edge points, the cocycles are reducible to constant parabolic cocycles. The key points in our proof are the exponential decay of the off-diagonal coefficient of the parabolic matrix, and the subexponential growth of the conjugacy (in restriction to $\mathbb{T}$) with respect to the label k. We first consider the case of small analytic potentials, and we distinguish between two cases in the proof. If the frequency is Diophantine, we develop a new KAM scheme to show almost reducibility with nice estimates (this result works for multifrequencies, and for both continuous and discrete cocycles). Moreover, in order to get a sharp decay on the size of spectral gaps (Theorem 0.2), we prove almost reducibility of the cocycle in a fixed band, arbitrarily close to the initial band. On the other hand, for a one-dimensional frequency α satisfying $\beta(\alpha) = 0$, we use the almost localization argument (via Aubry duality) given by Avila [1] (initially developed by Avila-Jitomirskaya [5]); one key ingredient in the proof is the Corona Theorem. The generalization to the global subcritical regime is based on Avila's global theory of analytic $\mathrm{SL}(2,\mathbb{R})-$cocycles [3], especially his proof of the Almost Reducibility Conjecture [2, 3].

Homogeneity of the spectrum in the subcritical regime is derived from the upper bounds on the size of spectral gaps, together with Hölder continuity of the IDS. Thanks to Avila's global theory of one-frequency Schrödinger operators [3], one can then prove Theorem 0.3 by combining our results in the subcritical case with previous work of Damanik-Goldstein-Schlag-Voda [12] in the supercritical regime.

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