Birkhoff attractors of dissipative billiards

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Abstract

We study the dynamics of dissipative billiard maps within planar convex domains. Such maps have a global attractor. We are interested in the topological and dynamical complexity of the attractor, in terms both of the geometry of the billiard table and of the strength of the dissipation. We focus on the study of an invariant subset of the attractor, the so-called Birkhoff attractor. On the one hand, we show that for a generic convex table with "pinched" curvature, the Birkhoff attractor is a normally contracted manifold when the dissipation is strong. On the other hand, for a mild dissipation, we prove that generically the Birkhoff attractor is complicated, both from the topological and the dynamical point of view.

1 Introduction

In the present paper, given a convex planar domain, we consider a variant of the usual billiard map in order to model some dissipative phenomena, which result in the existence of a global attractor. For such dissipative maps, Birkhoff [Bir32] introduced an invariant subset of the attractor, the so-called Birkhoff attractor; as we shall see, it is minimal in some sense among all invariant sets which separate phase-space, and it is essentially the place where interesting dynamics occurs. We investigate the properties of the Birkhoff attractor, in particular, how they change as the dissipation parameter is varied.

Loosely speaking, like for conservative billiards, we consider a massless particle moving with unit velocity inside the billiard table $\Omega \subset \mathbb{R}^2$ according to the usual law except at collisions with the boundary $\partial\Omega$, which we now assume to be *inelastic*. More precisely (see Fig. 1),

– the motion happens along straight lines between two collisions;

– at each orthogonal collision, the velocity vector is replaced with its opposite, while at a non-orthogonal collision, it is changed in such a way that the (unoriented) outgoing angle of reflection is strictly smaller than the incoming angle of incidence, both being measured with respect to the normal to $\partial\Omega$.

In other words, the reflected angle bends toward the inner normal at the incidence point. We refer to Definition A here below for more details, and to Section 3.1 for further properties of these billiard maps.

Billiards exhibiting some form of dissipation have already been considered in previous works. To the best of the authors' knowledge, for outer billiards, dissipation was first introduced in [Day47, Lemma 2.2]; see also [Tab93, Page 83]. Subsequently, dynamical properties of dissipative polygonal outer billiards have been studied in [DMGaG15]. Regarding standard billiards, the paper [MPS10] by Markarian-Pujals-Sambarino is dedicated to the study of limit sets of dissipative billiards (called here *pinball billiards*) for various types of tables (close to a circle, with semi-dispersing walls, which possess some hyperbolicity...), through the existence of a dominated splitting. Motivated by these rigorous results, the paper [AMS09] numerically investigates and characterizes the bifurcations of the resulting attractors as the contraction parameter is varied. In [MOKPdC12] the authors construct simple examples of non elastic convex billiards with dominated splitting and attractors supporting a rational or irrational rotation. Let us conclude this brief overview by mentioning

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some works about dissipative billiards for tables with flat walls. The paper [AMS12] concentrates on inelastic billiard dynamics in an equilateral triangular table and studies –mainly numerically– the structure of fractal strange attractors and their evolution as the contraction parameter changes. Finally, in a series of works [DMLDD+12, DMLDD+14, DMLDDGa18, DGaS17], Del Magno-Lopes Dias-Duarte-Gaivão-Pinheiro and Soufi investigate dissipative billiards within various types of polygonal tables; in particular, they study the structure of the nonwandering sets of such billiards, the existence of hyperbolic attractors, and prove the existence of countably many SRB measures on these attractors under suitable conditions.

Let us now move on to the formal definition of dissipative billiard maps considered in the present work. Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain with C^k boundary $\partial\Omega$, $k \geq 2$. We say that $\Omega \subset \mathbb{R}^2$ is *strongly convex* if –additionally– its curvature never vanishes. We assume that the perimeter of $\partial\Omega$ is normalized to one. We fix an orientation of $\partial\Omega$ and parametrize $\partial\Omega$ in arclength by some map $\Upsilon\colon \mathbb{T} \to \mathbb{R}^2$, where $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. The phase-space is the set of pairs (x, v) consisting of a point x on $\partial\Omega$, and a unit vector $v \in T_x\Omega$ pointing inward, or tangent to $\partial\Omega$. It is naturally identified with the cylinder $\mathbb{A} := \mathbb{T} \times [-1, 1]$; indeed, any point (x, v) in phase-space corresponds to a pair $(s, r) \in \mathbb{T} \times [-1, 1]$, where $x = \Upsilon(s) \in \partial\Omega$, and $r = \sin\varphi$ is the sine of the oriented angle $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ from the vector v to the inward normal to $\partial\Omega$ at x. The usual conservative billiard map is then defined as

$$f = f_1\colon \left\{ \begin{array}{ccc} \mathbb{A} & \to & \mathbb{A}\,, \\ (s, r) & \mapsto & f(s, r) = (s', r'_1)\,, \end{array} \right. \tag{1.1}$$

where $\Upsilon(s')$ represents the point where the trajectory starting at $\Upsilon(s)$ along the direction making an angle $\arcsin r$ with the normal at $\Upsilon(s)$, hits the boundary again, and r'_1 is the sine of the reflected angle at $\Upsilon(s')$, according to the standard reflection law (angle of incidence = angle of reflection). Let us now fix a dissipation parameter $\lambda \in (0, 1)$.

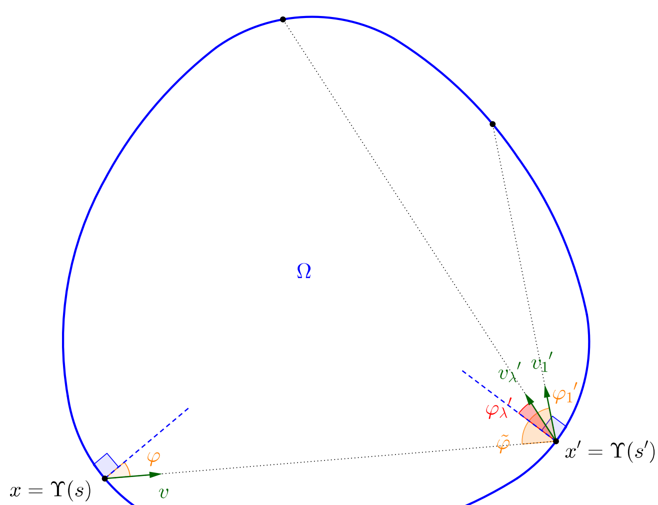

Figure 1: The standard billiard map and its dissipative counterpart.

Definition A. *Given a domain Ω as above, let us fix a C^{k-1} function $\lambda\colon \mathbb{A} \to (0, 1)$ such that*

$$0 < \partial_r\lambda(s, r)r + \lambda(s, r) < 1, \quad \forall\,(s, r) \in \mathrm{int}(\mathbb{A}), \tag{1.2}$$

and let $\mathcal{H}_\lambda\colon (s, r) \mapsto (s, \lambda(s, r)r)$. The dissipative billiard map f_λ associated to λ is then defined as the map

$$f_\lambda := \mathcal{H}_\lambda \circ f\colon \left\{ \begin{array}{ccc} \mathbb{A} & \to & \mathbb{A}\,, \\ (s, r) & \mapsto & f_\lambda(s, r) = (s', r'_\lambda)\,. \end{array} \right.$$

where

$$r'_\lambda = r'_\lambda(s,r) := \lambda(s', r'_1) r'_1,$$

for $r'_1 = r'_1(s,r)$ *as in* (1.1). *Note that for any* $(s,r) \in \mathrm{int}(\mathbb{A})$, *we have* $\det D\mathcal{H}_\lambda(f_1(s,r)) = \partial_r \lambda(s', r'_1) r'_1 + \lambda(s', r'_1)$. *In particular, since* $\det Df_1(s,r) = 1$, *and by* (1.2), *we obtain*

$$0 < \det Df_\lambda(s,r) = \det D\mathcal{H}_\lambda(f_1(s,r)) < 1, \quad \forall\, (s,r) \in \mathrm{int}(\mathbb{A}). \tag{1.3}$$

By (1.3), the resulting billiard map f_λ is no longer conservative; actually, it turns out to be a dissipative map in the sense of [LC88] (see Definition 2.1). In particular, f_λ contracts the standard area form $\omega = dr \wedge ds$. We refer to Section 3 for a few general facts about dissipative billiards.

Remark 1.1. For the purpose of this article, it is sufficient to consider C^2 billiard tables. Indeed, due to dissipation, trajectories which are close to the boundary drift further away from the boundary, and pathological phenomena such as in [Hal77] do not occur.

Remark 1.2. For simplicity, in most of what follows, we will restrict ourselves to the case where λ is actually a constant function, i.e., $\lambda \equiv \lambda_* \in (0,1)$. In that case, we will say that f_λ has constant dissipation. Then, the dissipative billiard map associated to λ simply becomes

$$f_\lambda \colon \left\{ \begin{array}{ccc} \mathbb{A} & \to & \mathbb{A}\,, \\ (s,r) & \mapsto & f_\lambda(s,r) = (s', r'_\lambda)\,, \end{array} \right.$$

where

$$r'_\lambda = r'_\lambda(s,r) := \lambda r'_1.$$

In the following, when it is clear from the context, we will abbreviate $r'_\lambda = r'$.

For constant dissipation, there is a natural one-parameter family of dissipative billiard maps $\{f_\lambda\}_{\lambda \in (0,1)}$; in particular, we will study transitions in the behavior of the Birkhoff attractor as λ changes. However, the simplifying hypothesis that λ is constant is not essential. Indeed, as we will explain, most results shown in the present work hold under the more general assumption that $\lambda \colon \mathbb{A} \to (0,1)$ is a C^1 function as in Definition A that is close enough to being constant, namely $\|D\lambda\| \ll 1$.

Due to the dissipative character of f_λ, there is a contraction of the phase-space which results in the existence of attractors. Indeed, as $f_\lambda(\mathbb{A}) \subset \mathrm{int}(\mathbb{A})$, there exists a *global attractor*

$$\Lambda^0_\lambda := \bigcap_{k \geq 0} f^k_\lambda(\mathbb{A})\,. \tag{1.4}$$

The attractor Λ^0_λ is f_λ-invariant, non-empty, compact and connected. Moreover, Λ^0_λ separates $\mathbb{A}$, i.e., $\mathbb{A} \setminus \Lambda^0_\lambda$ is the disjoint union of two connected open sets U_λ, V_λ. However, we can find a smaller invariant set –the so-called *Birkhoff attractor*– by "removing the hairs" from Λ^0_λ (see e.g. [LC90, Page 91]). The Birkhoff attractor, here denoted Λ_λ, is then defined as

$$\Lambda_\lambda := \overline{U}_\lambda \cap \overline{V}_\lambda\,. \tag{1.5}$$

We remark that, even if Λ_λ is compact and f_λ-invariant, it may no longer be an attractor in the usual sense. Actually, Λ_λ can also be characterized as the minimal element (with respect to inclusion) among all sets which are compact, connected, f_λ-invariant and separate $\mathbb{A}$. We refer to Section 2 for more details about the Birkhoff attractor and its properties.

The notion of Birkhoff attractor was first introduced by Birkhoff in [Bir32]. In the framework of dissipative twist maps of the annulus, further properties of the Birkhoff attractor have been investigated by the works of Charpentier [Cha34] and of Le Calvez [LC88]. The Birkhoff attractor of the thickened Arnol'd family has been studied by Crovisier in [Cro02]. Different authors have derived criteria to guarantee the existence of chaotic behaviors for invariant annular continua, see [BG91b, BG91a, HH86, Kor17, Cas88, PPS18, PT23]. Recently, the notion of Birkhoff attractor has been generalised to higher dimensions for conformally symplectic maps of some symplectic manifolds by Arnaud, Humilière and Viterbo, see [AHV24, Vit22].

Notation 1.3. Fix some dissipative map $f\colon \mathbb{A} \to \mathbb{A}$, with a hyperbolic periodic point $p \in \mathbb{A}$, of period $q \geq 1$. If p is of saddle type, we will denote its 1-dimensional stable, resp. unstable manifold as

$$\mathcal{W}^s(p; f^q) := \{x \in \mathbb{A} : \lim_{n\to+\infty} d(f^{qn}(x), p) = 0\},$$
$$\mathcal{W}^u(p; f^q) := \{x \in \mathbb{A} : \lim_{n\to+\infty} d(f^{-qn}(x), p) = 0\}.$$

If p is a sink, we will denote its 2-dimensional stable manifold as

$$\mathcal{W}^s(p; f^q) := \{x \in \mathbb{A} : \lim_{n\to+\infty} d(f^{qn}(x), p) = 0\}.$$

In either case, for $* = s/u$, we will sometimes abbreviate $\mathcal{W}^*(\mathcal{O}_f(p)) := \cup_{i=0}^{q-1}\mathcal{W}^*(f^i(p); f^q)$, or simply $\mathcal{W}^*(\mathcal{O}_\lambda(p))$, when $f = f_\lambda$ is some dissipative billiard map.

Considering the crucial role of elliptic tables in the conservative case, it is natural to start our study with dissipative billiard maps within ellipses. The detailed study of the corresponding dynamics is contained in Section 4, whose main result is the next theorem.

Theorem B. *Given an ellipse $\mathcal{E}$ of eccentricity $e \in (0,1)$, let $f_\lambda\colon \mathbb{A} \to f_\lambda(\mathbb{A}) \subset \mathrm{int}(\mathbb{A})$ be a dissipative billiard map within $\mathcal{E}$ in the sense of Definition A (we allow non-constant dissipation). Then, the 2-periodic orbits $\{H, f_\lambda(H)\}$ and $\{E, f_\lambda(E)\}$ corresponding to the trajectories along the major and minor axes are hyperbolic of saddle and sink type, respectively, and the Birkhoff attractor satisfies*

$$\Lambda_\lambda^0 = \Lambda_\lambda = \mathcal{W}^u(\mathcal{O}_\lambda(H)) \cup \{E, f_\lambda(E)\} = \overline{\mathcal{W}^u(\mathcal{O}_\lambda(H))}.$$

Moreover, for $i = 0, 1$, $\mathcal{W}^u(f_\lambda^i(H); f_\lambda^2) \setminus \{f_\lambda^i(H)\}$ is the disjoint union of two branches $\mathscr{C}_i^1, \mathscr{C}_i^2$, with $\mathscr{C}_i^j \subset \mathcal{W}^s(f_\lambda^j(E); f_\lambda^2)$, $j = 0, 1$.

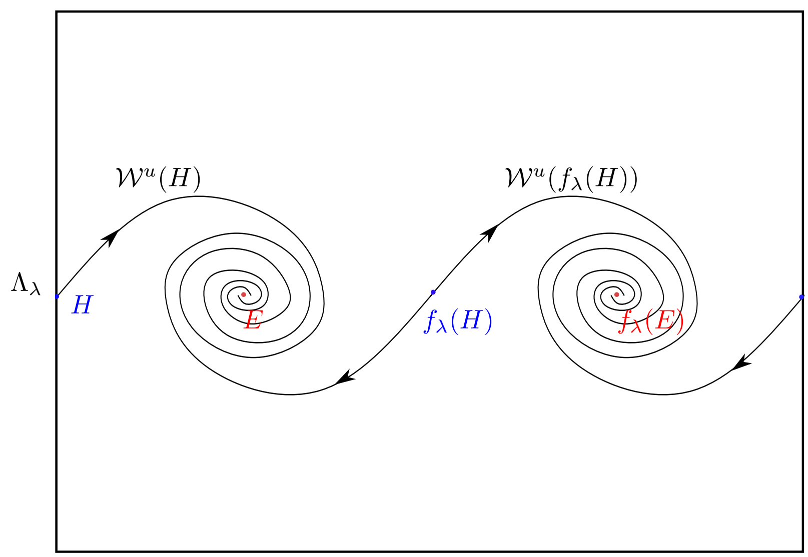

Figure 2: Birkhoff attractor of a dissipative billiard map f_λ within an ellipse of non-zero eccentricity when the dissipation is mild, i.e., λ is close to 1.

At the end of Section 4, we prove that the conclusion of Theorem B remains true also for strictly convex domains whose boundary is sufficiently C^2-close to an ellipse, as stated in the next corollary. For simplicity, we state it in the case where the dissipation function λ is a constant in $(0,1)$.

Corollary C. *Let $\mathcal{E}$ be an ellipse of eccentricity $e \in (0,1)$. Let $\lambda \in (0,1)$. There exists $\epsilon = \epsilon(\mathcal{E}, \lambda) > 0$ such that for any C^k $(k \geq 2)$ domain $\Omega \subset \mathbb{R}^2$ satisfying $d_{C^2}(\partial\Omega, \mathcal{E}) < \epsilon$, the following holds. Denoting by*

f_λ^Ω *the dissipative billiard map within* Ω*, there exist* 2*-periodic orbits* $\mathcal{O}_{f_\lambda^\Omega}(H) = \{H, f_\lambda^\Omega(H)\}$ *and* $\mathcal{O}_{f_\lambda^\Omega}(E) = \{E, f_\lambda^\Omega(E)\}$ *of saddle and sink type, respectively, and the Birkhoff attractor is equal to*

$$\Lambda_\lambda = \mathcal{W}^u(\mathcal{O}_{f_\lambda^\Omega}(H)) \cup \mathcal{O}_{f_\lambda^\Omega}(E) .$$

Moreover, the function $(\mathcal{E}, \lambda) \mapsto \epsilon(\mathcal{E}, \lambda)$ *can be chosen to be continuous.*

The first examples of Birkhoff attractors for a dissipative billiard map f_λ within a circle or an ellipse (see Fig. 2 illustrating the Birkhoff attractor in the case of an ellipse when the dissipation is mild, i.e., λ close to 1) naturally lead us to consider topological properties of Birkhoff attractors, in particular investigate when Λ_λ is topologically as simple as it can be, namely, a graph. The main results in this direction are contained in Section 5. Through the following definition, we introduce the class of billiards for which we can guarantee such a simple behavior of the Birkhoff attractor.

Definition D. *For any* $k \geq 2$*, let* $\mathcal{D}^k$ *be the set of strongly convex domains* Ω *with* C^k *boundary* $\partial\Omega$ *such that, given a parametrization* $\Upsilon \colon \mathbb{T} \to \mathbb{R}^2$ *of* $\partial\Omega$*, the following geometric condition holds (see Fig. 3):*

$$\max_{s \in \mathbb{T}} \tau(s)\mathcal{K}(s) < -1 , \tag{1.6}$$

where $\mathcal{K}(s) < 0$ *denotes the curvature of* $\partial\Omega$ *at the point* $\Upsilon(s)$*, and* $\tau(s) > 0$ *is the length of the first segment of the* f_1*-orbit starting at* $\Upsilon(s)$ *perpendicularly to* $\partial\Omega$*. Alternatively, condition (1.6) amounts to asking that the centers of the osculating circles at the points of* $\partial\Omega$ *remain in* Ω*.*

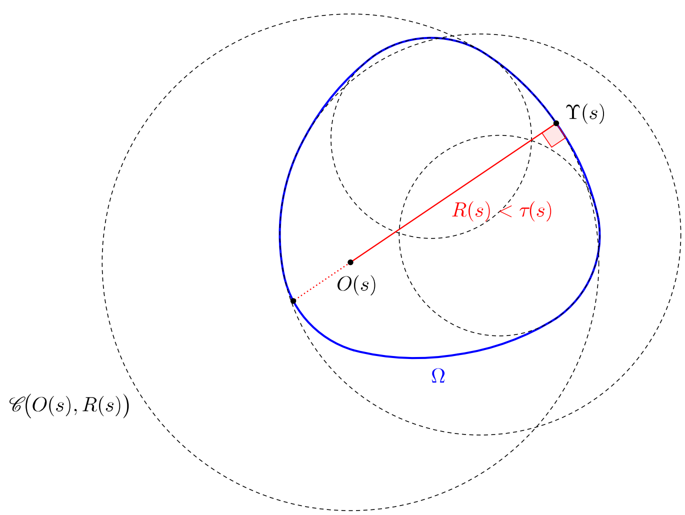

Figure 3: The geometric condition $\max_{s\in\mathbb{T}} \tau(s)\mathcal{K}(s) < -1$ in Definition D. Here, $R(s) := -\frac{1}{\mathcal{K}(s)}$ is the radius of curvature, and $\mathscr{C}\big(O(s), R(s)\big)$ is the osculating circle at $\Upsilon(s)$.

Clearly, the set $\mathcal{D}^k$ is C^k-open. More precisely, for any $\Omega \in \mathcal{D}^k$, there exists a C^2-open neighborhood $\mathcal{U}$ of Ω such that for any C^k domain $\Omega' \in \mathcal{U}$, we have $\Omega' \in \mathcal{D}^k$.
The main result of Section 5 is proving that the geometric condition contained in Definition D together with strong dissipation (λ close to 0) suffice to guarantee that the corresponding Birkhoff attractor is a graph. Our result is also about the dynamics on the attractor and the graph's regularity. We use the notions of dominated splitting for an invariant set and of normally contracted manifold. We refer to Section 5 for more details about such definitions.

Theorem E. *Let* $f_\lambda \colon \mathbb{A} \to f_\lambda(\mathbb{A}) \subset \mathrm{int}(\mathbb{A})$ *be a dissipative billiard map with constant dissipation* $\lambda \in (0,1)$ *within some domain* $\Omega \in \mathcal{D}^k$*,* $k \geq 2$*. Then the following assertions hold.*

1. *There exists $\lambda(\Omega) \in (0,1)$ such that, for any $\lambda \in (0, \lambda(\Omega))$, the Birkhoff attractor Λ_λ coincides with Λ^0_λ and has a dominated splitting $E^s \oplus E^c$, where E^s is uniformly contracted by Df_λ. Moreover, Λ_λ is a normally contracted C^1 graph over $\mathbb{T} \times \{0\}$ which is tangent to E^c.*
2. *There exists $\lambda'(\Omega) \in (0, \lambda(\Omega))$ such that for any $\lambda \in (0, \lambda'(\Omega))$, Λ_λ is actually a C^{k-1} graph.*
3. *There exists an open and dense set $\mathscr{U}$ of C^k domains[1] such that if, moreover, $\Omega \in \mathcal{D}^k \cap \mathscr{U}$, then there exists $\lambda''(\Omega) \in (0, \lambda'(\Omega))$ such that, for any $\lambda \in (0, \lambda''(\Omega))$, Λ_λ is a C^{k-1} normally contracted graph of rotation number $\frac{1}{2}$. Moreover,*
$$\Lambda_\lambda = \bigcup_{i=1}^{\ell} \overline{\mathcal{W}^u(\mathcal{O}_\lambda(H_i))},$$
for some finite collection $\{\mathcal{O}_\lambda(H_i)\}_{i=1,\cdots,\ell} = \{H_i, f_\lambda(H_i)\}_{i=1,\cdots,\ell}$ of 2-periodic orbits of saddle type.

Remark 1.4. Given $k \geq 2$ and a domain $\Omega \in \mathcal{D}^k \cap \mathscr{U}$ as in the above statement, the conclusion of Theorem E holds for general dissipative billiard maps f_λ in the sense of Definition A, provided that the dissipation function $\lambda\colon \mathbb{A} \to (0,1)$ satisfies $\|\lambda\|_{C^1} \ll 1$. See e.g. Remark 5.17 for more details.

Remark 1.5. A consequence of Theorem E is that if $\partial\Omega$ is an ellipse $\mathcal{E}$ of eccentricity $e \in (0, \frac{\sqrt{2}}{2})$, then $\mathcal{E} \in \mathcal{D}^\infty$ and, for any $\lambda \in (0, \lambda(\mathcal{E}))$, the corresponding Birkhoff attractor Λ_λ is a normally contracted C^1 graph, which is actually C^∞ except possibly at the 2-periodic sink $\{E, f_\lambda(E)\}$, where Λ_λ is tangent to the weak stable space of the sink, see Corollary 5.11. We will also see that, when the eccentricity e is larger than $\frac{\sqrt{2}}{2}$, then for $\lambda \in (0,1)$ small, the Birkhoff attractor Λ_λ is no longer a graph (see Proposition 5.16).

We may wonder if Birkhoff attractors of dissipative billiards may exhibit more complex topological properties than the examples described in Sections 4 and 5. In fact, following a result by M. Charpentier [Cha34], a Birkhoff attractor for a dissipative diffeomorphism can be an "indecomposable continuum", and a sufficient condition for this to occur is that the Birkhoff attractor contains points with different rotation numbers. The aim of Section 6 is exploring this direction and discussing some topological and dynamical implications of such a phenomenon.

In order to state the main results, we need to premise the notion of upper and lower rotation number for Λ_λ. Denote by V_Λ (resp. U_Λ) the connected component of $\mathbb{A} \setminus \Lambda_\lambda$ containing $\{(s,1) \in \mathbb{A}: \ s \in \mathbb{T}\}$ (resp. $\{(s,-1)) \in \mathbb{A}: \ s \in \mathbb{T}\}$). For any $(s,r) \in \mathbb{A}$ the upper and lower vertical lines are, respectively

$$V^+(s,r) := \{(s,y) \in \mathbb{A}: \ y \geq r\} \qquad \text{and} \qquad V^-(s,r) := \{(s,y) \in \mathbb{A}: \ y \leq r\}.$$

Let us now define

$$\Lambda^+_\lambda := \{x \in \Lambda_\lambda: \ V^+(x) \setminus \{x\} \subset V_\Lambda\} \qquad \text{and} \qquad \Lambda^-_\lambda := \{x \in \Lambda_\lambda: \ V^-(x) \setminus \{x\} \subset U_\Lambda\}.$$

Given a covering $\pi\colon \mathbb{R} \times [-1,1] \to \mathbb{T} \times [-1,1]$ of $\mathbb{A}$, let $\tilde{\Lambda}^\pm_\lambda := \pi^{-1}(\Lambda^\pm_\lambda)$. Moreover, let $\tilde{\pi}_1\colon \mathbb{R} \times [-1,1] \to \mathbb{R}$ be the first coordinate projection, and $F_\lambda\colon \mathbb{R} \times [-1,1] \to \mathbb{R} \times [-1,1]$ a continuous lift of f_λ. Then, by a result due to G.D. Birkhoff [Bir32] and rephrased in all details by P. Le Calvez [LC88], the sequence

$$\left(\frac{\tilde{\pi}_1 \circ F_\lambda^{-n} - \tilde{\pi}_1}{n}\right)_{n \in \mathbb{N}}$$

converges uniformly on $\tilde{\Lambda}^+_\lambda$ (resp. $\tilde{\Lambda}^-_\lambda$) to a constant ρ^+_λ (resp. ρ^-_λ). The constants ρ^+_λ and ρ^-_λ – called upper and lower rotation numbers – do depend on the chosen lift, but not their difference. We refer the reader to Subsection 6.2 for more details.

For the conservative billiard map $f = f_1$, let denote by $\mathscr{V}(f)$ the union of all f-invariant essential curves in $\mathbb{A}$, i.e., f-invariant homotopically non-trivial curves. We recall that an instability region for f is an open bounded connected component of $\mathbb{A} \setminus \mathscr{V}(f)$ that contains in its interior an essential curve. The main result of Section 6 is the next Theorem, whose proof is mainly based on an adaptation of some arguments of the work [LC88]. Let us recall that a continuum is a compact connected topological space.

[1] We refer the reader e.g. to [DCOKPdC07, Section 2] for more details on the topology on the space of C^k convex billiards.

Theorem F. *Let $\Omega \subset \mathbb{R}^2$ be a strongly convex domain with C^k boundary, $k \geq 2$. Let $f = f_1$ be the associated conservative billiard map. If f admits an instability region that contains the zero section $\mathbb{T} \times \{0\}$, then there exists $\lambda_0(\Omega) \in (0,1)$ such that, for any $\lambda \in [\lambda_0(\Omega), 1)$, the Birkhoff attractor Λ_λ of the dissipative billiard map f_λ with constant dissipation λ has $\rho_\lambda^+ - \rho_\lambda^- > 0$, with $\frac{1}{2} \in (\rho_\lambda^-, \rho_\lambda^+) \mod \mathbb{Z}$.*

Remark 1.6. Theorem F was stated for a dissipative map f_λ with constant dissipation. However, the result holds for general dissipative billiard maps in the sense of Definition A, as long as the dissipation function $\lambda\colon \mathbb{A} \to (0,1)$ is sufficiently close to the constant function 1 in the C^1-topology, i.e., $\|1 - \lambda\|_{C^1} \ll 1$. See e.g. Proposition 6.10 for more details in this direction.

The above theorem has several interesting consequences for Λ_λ. In fact, in the case where $\rho_\lambda^+ - \rho_\lambda^- > 0$, the corresponding Birkhoff attractor turns out to be complicated both topologically and dynamically. In particular,

- Λ_λ is an indecomposable continuum, i.e., it cannot be written as the union of two proper continua (directly from the work [Cha34] of M. Charpentier; see also [BG91a]).
- Each rational $\frac{p}{q} \in (\rho_\lambda^-, \rho_\lambda^+)$ is the rotation number of a periodic orbit in Λ_λ (as a straightforward application of [BG91b]).
- If x is a saddle periodic point of type (p,q), with $\frac{p}{q} \in (\rho_\lambda^-, \rho_\lambda^+)$, then its unstable manifold $\mathcal{W}^u(x; f^q)$ satisfies $\overline{\mathcal{W}^u(x; f^q)} \subset \Lambda_\lambda$ (by [LC88, Proposition 14.3]).
- There exists $n_0 \in \mathbb{N}$ so that $f_\lambda^{n_0}$ has a rotational horseshoe (by [PPS18, Theorem A]). In particular, $f_\lambda|_{\Lambda_\lambda}$ has positive topological entropy.

Applying essentially [DCOKPdC07], we prove that the conclusions of Theorem F hold generically for C^k strongly convex domains, $k \geq 3$, as explained in the next corollary.

Corollary G. *For $k \geq 3$, there exists an open and dense subset $\mathscr{U}$ of the set of C^k strongly convex domains such that for every $\Omega \in \mathscr{U}$, the following assertions hold.*

1. *There exists $\lambda_0(\Omega) \in (0,1)$ such that, for any $\lambda \in [\lambda_0(\Omega), 1)$, the Birkhoff attractor Λ_λ of the corresponding dissipative billiard map f_λ has $\rho_\lambda^+ - \rho_\lambda^- > 0$, with $\frac{1}{2} \in (\rho_\lambda^-, \rho_\lambda^+) \mod \mathbb{Z}$.*
2. *There exists $\lambda_1(\Omega) \in [\lambda_0(\Omega), 1)$ such that, for any $\lambda \in [\lambda_1(\Omega), 1)$ and any 2-periodic point p of saddle type*[2]*, there exists a horseshoe $K_\lambda(p) \subset \Lambda_\lambda$ in the homoclinic class of the 2-periodic point p.*

Finally, as a consequence of [Mat82], we emphasize that the conclusion of point (1) in Corollary G also holds for **any** convex domain Ω whose boundary is C^2 and contains some point at which the curvature vanishes. In this case (see Corollary 6.16), for any $\epsilon > 0$, there exists $\lambda_0 = \lambda_0(\Omega, \epsilon) \in (0,1)$ such that for any $\lambda \in [\lambda_0, 1)$, the corresponding Birkhoff attractor Λ_λ has $\rho_\lambda^+ - \rho_\lambda^- \in (1 - \epsilon, 1)$.

From the results presented above, it is possible to highlight a phase transition for Birkhof attractors of dissipative billiards when the parameter λ varies. We would like to emphasize how the topological and dynamical properties of the Birkhoff attractor change in terms of the dissipative parameter. From Corollary C and Corollary G, we obtain the following conclusion.

Corollary H. *Let $\mathcal{E}$ be an ellipse of eccentricity $e \in (0,1)$. Fix $k \geq 3$. There exists an open and dense set $\mathscr{G}$ of C^k domains such that the following holds. For any $0 < \lambda_1 < \lambda_2 < 1$ there exists $\delta > 0$ so that if $\Omega \in \mathscr{G}$ and $d_{C^2}(\partial\Omega, \mathcal{E}) < \delta$, then:*

1. *there are 2-periodic orbits $\{H, f_\lambda(H)\}$ and $\{E, f_\lambda(E)\}$ of saddle and sink type, respectively;*
2. *for any $\lambda \in [\lambda_1, \lambda_2]$,*

$$\Lambda_\lambda = \overline{\mathcal{W}^u(\mathcal{O}_\lambda(H))} = \mathcal{W}^u(\mathcal{O}_\lambda(H)) \cup \{E, f_\lambda(E)\}\,. \tag{1.7}$$

   In particular, Λ_λ has rotation number $\frac{1}{2}$;

[2] e.g., when the 2-periodic orbit $\{p, f_\lambda(p)\}$ corresponds to a diameter of the table

3. *there exists $\lambda_0(\Omega) > \lambda_2$ such that, if $\lambda \in [\lambda_0(\Omega), 1)$, then $\rho_\lambda^+ - \rho_\lambda^- > 0$, with $\frac{1}{2} \in (\rho_\lambda^-, \rho_\lambda^+) \mod \mathbb{Z}$. In particular, Λ_λ is an indecomposable continuum that contains a horseshoe.*

Proof. Let $\lambda \in (0,1) \mapsto \epsilon(\mathcal{E}, \lambda) > 0$ be the continuous function given by Corollary C; let

$$\delta := \frac{1}{2} \min_{\lambda \in [\lambda_1, \lambda_2]} \epsilon(\mathcal{E}, \lambda) > 0 .$$

Fix $k \geq 2$. Then, for any C^k domain Ω with $d_{C^2}(\partial\Omega, \mathcal{E}) < \delta$, and for any $\lambda \in [\lambda_1, \lambda_2]$, we have $d_{C^2}(\partial\Omega, \mathcal{E}) < \epsilon(\mathcal{E}, \lambda)$, hence (1.7) holds for 2-periodic orbits $\{H, f_\lambda(H)\}$ and $\{E, f_\lambda(E)\}$ of saddle and sink type, respectively. Now, by Corollary G, we obtain point (3) of the Corollary if, moreover, Ω is chosen within an open and dense set of C^k domains. □

We refer to Fig. 4. The phase transition described for perturbations of elliptic tables also holds for domains in $\mathcal{D}^k$, $k \geq 3$. The following corollary is a consequence of Theorem E and Corollary G.

Corollary I. *For any $k \geq 3$, there exists an open and dense set $\mathscr{U}$ of C^k domains such that for every $\Omega \in \mathcal{D}^k \cap \mathscr{U}$, the following holds. There exist $0 < \lambda''(\Omega) < \lambda_0(\Omega) < 1$ such that:*

1. *if $\lambda \in (0, \lambda''(\Omega))$, then Λ_λ is equal to the attractor Λ_λ^0 and it is a C^{k-1} normally contracted graph of rotation number $\frac{1}{2}$ satisfying*
$$\Lambda_\lambda = \bigcup_{i=1}^{\ell} \overline{\mathcal{W}^u(\mathcal{O}_\lambda(H_i))} ,$$
for some finite collection $\{\mathcal{O}_\lambda(H_i)\}_{i=1,\cdots,\ell} = \{H_i, f_\lambda(H_i)\}_{i=1,\cdots,\ell}$ of 2-periodic orbits of saddle type;

2. *if $\lambda \in [\lambda_0(\Omega), 1)$, then $\rho_\lambda^+ - \rho_\lambda^- > 0$, with $\frac{1}{2} \in (\rho_\lambda^-, \rho_\lambda^+) \mod \mathbb{Z}$. In particular, Λ_λ is an indecomposable continuum that contains a horseshoe.*

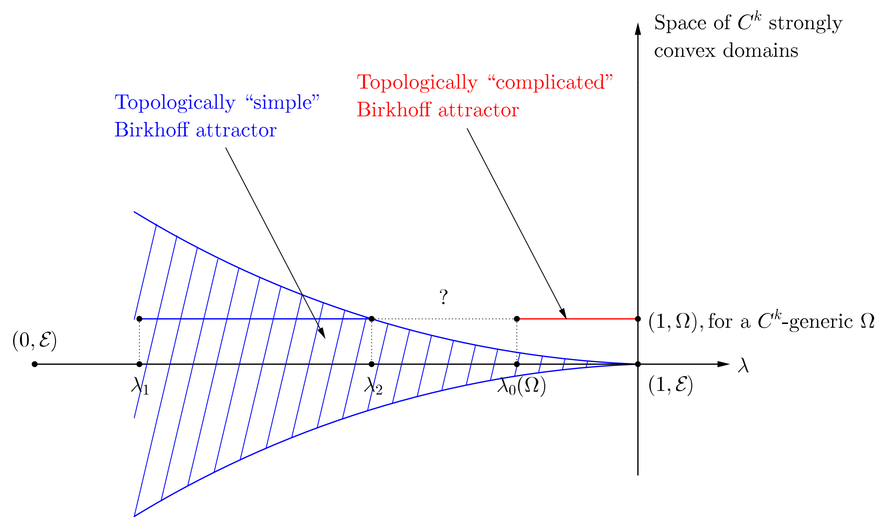

Figure 4: Phase transition for a C^k-generic domain near an ellipse of non-zero eccentricity, $k \geq 3$.

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2 Dissipative maps and Birkhoff attractors

Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and $\mathbb{A} := \mathbb{T} \times [-1, 1]$, with coordinates $(s, r) \in \mathbb{A}$. Endow $\mathbb{A}$ with the standard 2-form $\omega = dr \wedge ds = d\alpha$, where $\alpha = rds$ is the standard Liouville 1-form. The form ω induces then an orientation on $\mathbb{A}$ and the Lebesgue measure denoted by m. For the following definition of dissipative map we refer to [LC88, Page 245].

Definition 2.1. *For two continuous maps $\phi^-, \phi^+ : \mathbb{T} \to \mathbb{R}$ such that $\phi^- < \phi^+$, let us denote*

$$C := \{(s, r) \in \mathbb{A} : \ \phi^-(s) \leq r \leq \phi^+(s)\}.$$

A map $f : C \to \mathrm{int}(C)$ is a **dissipative map** *if:*

1. *f is a homeomorphism of C into its image, homotopic to the identity;*
2. *f is a C^1 diffeomorphism of $\mathrm{int}(C)$ into its image;*
3. *there exists $\lambda \in (0, 1)$ such that for any Borel set $Y \subset C$ it holds $m(f(Y)) \leq \lambda m(Y)$.*

Observe that the following condition is equivalent to condition 3 in Definition 2.1:

3'. there exists $\lambda \in (0, 1)$ such that for every $(s, r) \in \mathrm{int}(C)$ it holds $0 < \det Df(s, r) \leq \lambda$.

When considering dynamical systems with dissipation, it is natural to mention the notion of conformally symplectic maps, which is stated here in the more general framework of symplectic manifolds. See [MS17, AF24, AFR22].

Definition 2.2. *Let $(\mathcal{M}, \omega)$ be a symplectic manifold. A diffeomorphism f from $\mathcal{M}$ into its image (contained in $\mathcal{M}$) is* **conformally symplectic** *if there exists a smooth function $a : \mathcal{M} \to \mathbb{R}$ such that $f^*\omega = a\omega$.*

As shown by Libermann in [Lib59, Page 210], if the dimension of $\mathcal{M}$ is greater than or equal to 4, then the smooth function a is a constant, called the *conformality ratio*. In our case, i.e., for $\dim \mathcal{M} = 2$, the function a is not *a priori* constant. This motivates the next definition.

Definition 2.3. *Let $(\mathcal{M}, \omega)$ be a symplectic manifold with $\dim \mathcal{M} = 2$. A diffeomorphism f from $\mathcal{M}$ into its image (contained in $\mathcal{M}$) is constant conformally symplectic if there exists a constant $a > 0$ such that $f^*\omega = a\omega$.*

Observe that conformally symplectic maps are stable under symplectic changes of coordinates. This is not true anymore for constant conformally symplectic maps.

Let $\mathcal{X}$ be the set of compact sets of $\mathrm{int}(C)$, endowed with the Hausdorff distance. We say that an element $X \in \mathcal{X}$ *separates* C if its complement has two connected components: a lower one $U_X \supset \{(s, \phi^-(s)) \in \mathbb{A} : s \in \mathbb{T}\}$, and an upper one $V_X \supset \{(s, \phi^+(s)) \in \mathbb{A} : \ s \in \mathbb{T}\}$. We denote by $\mathcal{X}(f) \subset \mathcal{X}$ the subset of $\mathcal{X}$ consisting of the sets which are compact, connected, f-invariant and separate C.

Let f be a dissipative map according to Definition 2.1. We can define the **Birkhoff attractor** of f as follows, see [Bir32], [LC88, Section 2] and [LC90, Chapter 6] for an exhaustive treatment of the argument. First, observe that, by the dissipative character of f and as $f(C) \subset \mathrm{int}(C)$, there exists an attractor

$$\Lambda^0 := \bigcap_{k \geq 0} f^k(C) \tag{2.1}$$

which is f-invariant, non-empty, compact and connected. Moreover, it separates C, i.e., $C \setminus \Lambda^0$ is the disjoint union of two connected open sets $U_{\Lambda^0}, V_{\Lambda^0}$ as above. In other words, $\Lambda^0 \in \mathcal{X}(f)$.

Definition 2.4. *Let f be a dissipative map and let Λ^0 be its corresponding attractor, see (2.1). Let $U_{\Lambda^0}, V_{\Lambda^0}$ be the two connected components of $C \setminus \Lambda^0$. Then, the Birkhoff attractor Λ is defined as*

$$\Lambda := \overline{U}_{\Lambda^0} \cap \overline{V}_{\Lambda^0} \, . \tag{2.2}$$

The Birkhoff attractor of a dissipative map can also be described as the minimal set, with respect to the inclusion, among the elements of $\mathcal{X}(f)$, as we will state immediately.

Proposition 2.5. *[LC90, Proposition 6.1]*

1. *The set $\mathcal{X}(f)$ contains a minimal element with respect to the inclusion, which is the Birkhoff attractor Λ for f.*
2. *If $X \in \mathcal{X}(f)$ then $\Lambda = \overline{U}_X \cap \overline{V}_X$; in particular, $\Lambda = \mathrm{Fr}\, U_\Lambda = \mathrm{Fr}\, V_\Lambda$.*

From the properties of the Birkhoff attractor, we can deduce the following two lemmas.

Lemma 2.6. *Let $X \in \mathcal{X}(f)$. Let $x \in X$ be such that $C \setminus (X \setminus \{x\})$ is connected. Then, $x \in \Lambda$.*

Proof. By Proposition 2.5(1), it holds that $\Lambda \subset X$; in particular, $U_X \subset U_\Lambda$ and $V_X \subset V_\Lambda$. By hypothesis, $U_X \cup V_X \cup \{x\}$ is connected. Assume by contradiction that $x \in X \setminus \Lambda \subset C \setminus \Lambda = U_\Lambda \cup V_\Lambda$. Without loss of generality, we can assume that $x \in U_\Lambda$. Therefore

$$U_X \cup \{x\} \cup V_X \subset U_\Lambda \cup V_\Lambda \,.$$

The set on the left hand side is connected, while the set on the right hand side is the disjoint union of two open sets. We then conclude that the left term is contained in one of the two open sets. Without loss of generality, we suppose that $U_X \cup \{x\} \cup V_X \subset U_\Lambda$. In particular, $V_X \subset U_\Lambda \cap V_\Lambda = \emptyset$, which is the required contradiction. □

Lemma 2.7. *Let $X \subset \mathrm{int}(\mathbb{A})$ be f-invariant and such that $\Lambda \subset X$. Then X separates the annulus.*

Proof. Observe that, since $X \subset \mathrm{int}(\mathbb{A})$, clearly both $\mathbb{T} \times \{1\}$ and $\mathbb{T} \times \{-1\}$ are contained in $\mathbb{A} \setminus X$. By contradiction, assume that X does not separate the annulus, i.e. $\mathbb{A} \setminus X$ is connected.[3] Moreover, since $\Lambda \subset X$, it holds that $\mathbb{A} \setminus X$ is contained in either U_Λ or V_Λ, the two connected components of $\mathbb{A} \setminus \Lambda$. Without loss of generality, assume that $\mathbb{A} \setminus X \subset U_\Lambda$. Thus, since $\mathbb{T} \times \{1\} \subset \mathbb{A} \setminus X$, we have $\mathbb{T} \times \{1\} \subset U_\Lambda \cap V_\Lambda = \emptyset$. This provides the required contradiction and concludes the proof. □

Given a dissipative map $f\colon C \to \mathrm{int}(C)$ and a hyperbolic periodic point $p \in C$ of f of period $q \geq 1$, we will denote by $\mathcal{W}^s(p; f^q)$ its stable manifold and by $\mathcal{W}^u(p; f^q)$ its unstable manifold for the iterate f^q. Note that p cannot be a source by the dissipative character of f. We will also sometimes abbreviate $\mathcal{W}^*(p; f^q)$ simply as $\mathcal{W}^*(p)$ when the information about the map and the period are clear from the context. Observe that, for any hyperbolic point in the attractor Λ^0, its unstable manifold is also contained in Λ^0. Similarly, the next proposition guarantees that for any periodic point of saddle type in the Birkhoff attractor Λ, certain branches of its stable/unstable manifold have to belong to the Birkhoff attractor. See also [LC88, Section 14.3] for related results.

Proposition 2.8. *Let $f\colon C \to \mathrm{int}(C)$ be a dissipative map. Assume that $p \in \Lambda$ is a hyperbolic periodic point of saddle type, with period $q \geq 1$. Then at least one the two branches of $\mathcal{W}^u(p; f^q) \setminus \{p\}$ is contained in Λ, unless Λ locally coincides near p with the local stable manifold $\mathcal{W}^s_{\mathrm{loc}}(p; f^q)$ of p; in the latter case, $\mathcal{W}^s(p; f^q) \subset \Lambda$. In particular, at least one of the following non-exclusive properties holds:*

- $\overline{\mathcal{W}^s(p; f^q)} \subset \Lambda$;
- $\overline{\mathcal{W}^u(p; f^q)} \subset \Lambda$;
- $\exists\, \delta > 0$ *such that* $\Lambda \cap B(p, \delta) \setminus \{p\} = \mathcal{B}^u \cup \mathcal{B}^s$, *where* $\mathcal{B}^*$ *is a branch of* $\mathcal{W}^*_{\mathrm{loc}}(p; f^q) \setminus \{p\}$, $* = s, u$.

Proof. Let $p \in \Lambda$ be a hyperbolic periodic point of saddle type, with period $q \geq 1$. We denote by $0 < \mu_1 < 1 < \mu_2$ the eigenvalues of $Df^q(p)$. By Hartman-Grobman Theorem, the dynamics can be linearized near p. Let then U be an open neighborhood of p, and $\Psi\colon U \to \mathbb{R}^2$ be a homeomorphism such that $\Psi(p) = 0$ and $\Psi \circ f^q|_U = A \circ \Psi|_U$, where

$$A = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, \quad 0 < \mu_1 < 1 < \mu_2 \,.$$

[3] Indeed, it cannot disconnect $\mathbb{A}$ into more than 2 connected components, by the dissipative character of f.

Let us assume that Λ does not locally coincide with the local stable manifold of p, that is, for any neighborhood V of p there exists a point in $\Lambda \setminus \mathcal{W}^s_{\text{loc}}(p; f^q)$. Let $\epsilon > 0$ be such that the ball of radius ϵ centred at the origin $B(0,\epsilon)$ is contained in $\Psi(U)$. Then there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists a point $q_n \in \Lambda \cap U$ such that

- $\Psi(q_n) = (x_n, y_n) \in B(0,\epsilon)$;
- $|y_n| \in (\frac{\epsilon}{2}\mu_2^{-n}, \epsilon\mu_2^{-n})$.

Observe that each q_n does not belong to the local stable manifold of p and that $f^{qk}(q_n) \in U$ for all $0 \leq k \leq n$. Thus, we can consider the sequence $(f^{qn}(q_n))_n$ of points in Λ. Denote $\Psi \circ f^{qn}(q_n) =: (X_n, Y_n)$; in particular, for every $n \in \mathbb{N}$ it holds $|X_n| < \epsilon\mu_1^n$ and $|Y_n| \in (\frac{\epsilon}{2}, \epsilon)$. Therefore –up to passing to a subsequence– the sequence $(f^{qn}(q_n))_n$ converges to a point $Q \neq p$ belonging to the local unstable manifold of p. Since Λ is closed and invariant, we conclude that

$$f^{qn}(Q) \in (\Lambda \cap \mathcal{W}^u(p; f^q)) \setminus \{p\}, \quad \forall n \in \mathbb{Z}. \tag{2.3}$$

Let us now prove that the branch $\mathcal{B}^u(p,Q)$ of $\mathcal{W}^u_{\text{loc}}(p; f^q) \setminus \{p\}$ containing Q is contained in Λ, and therefore, by the invariance of Λ, that the whole branch of the unstable manifold of p also does. Assume this is not the case: since Λ is closed, and since, by (2.3), in any neighborhood of p there exists a point in $\Lambda \cap \mathcal{W}^u_{\text{loc}}(p; f^q)$, for every neighborhood $V \subset U$ of p, we can find two points $q_1 = \Psi^{-1}(0, a_1), q_2 = \Psi^{-1}(0, a_2) \in (\Lambda \cap U) \setminus \{p\}$ belonging to the local unstable manifold of p such that

$$\Gamma_0 := \Psi^{-1}(\{0\} \times (a_1, a_2)) \subset V, \quad \text{and} \quad \Gamma_0 \cap \Lambda = \emptyset.$$

Let $\mathcal{U}$ be a bounded neighborhood of Λ such that $f^{-1}|_{\mathcal{U}}$ is a diffeomorphism onto its image. Up to considering V small enough, we have that

$$\bigcup_{n \in \mathbb{N}} f^{-qn}(\Lambda \cup \Gamma_0) \subset \mathcal{U}\,.$$

Let $O_0 \subset \mathcal{U}$ be a bounded open set among the connected components of $C \setminus (\Lambda \cup \Gamma_0)$, with $\Gamma_0 \subset \partial O_0$. Then, the family $(f^{-qn}(O_0))_{n \in \mathbb{N}}$ is uniformly bounded in measure, which contradicts the dissipative character of f. □

With the notations of Proposition 2.8, we can show that, in some cases, Λ contains both the stable manifold $\mathcal{W}^s(p; f^q)$ and the unstable manifold $\mathcal{W}^u(p; f^q)$ of the point p. In the following, for $* = s, u$, we denote $\mathcal{W}^*(\mathcal{O}_f(p)) = \bigcup_{k=0}^{q-1} \mathcal{W}^*(f^k(p); f^q)$.

Lemma 2.9. *Let p be a periodic hyperbolic point belonging to Λ. Let us assume that one connected component $\mathcal{B}$ of $\mathcal{W}^s(p; f^q) \setminus \{p\}$ belongs to Λ, and that $\mathcal{B}$ intersects $\mathcal{W}^u(p; f^q)$ transversally. Then*

$$\overline{\mathcal{W}^u(\mathcal{O}_f(p)) \cup \mathcal{W}^s(\mathcal{O}_f(p))} \subset \Lambda\,.$$

Proof. Let z be a point of transverse intersection between $\mathcal{B}$ and $\mathcal{W}^u(p; f^q)$. The standard λ-lemma (see e.g. [PdM82, Chapter 2.7]) guarantees that there exists a 1-dimensional disk $D \subset \mathcal{B}$ whose past iterates $(f^{-qn}(D))_{n \geq 0}$ accumulate $\mathcal{W}^s(p; f^q)$. As $f^{-qn}(D) \subset \Lambda$ for every $n \geq 0$ and since Λ is closed, we deduce that $\overline{\mathcal{W}^s(p; f^q)} \subset \Lambda$. Let us denote by $\mathcal{B}'$ the branch of $\mathcal{W}^u(p; f^q) \setminus \{p\}$ containing z. By invariance of Λ, all the points $(f^{-qn}(z))_{n \geq 0}$ belong to $\mathcal{W}^u(p; f^q) \cap \Lambda$; in particular, for any $\delta > 0$, $\Lambda \cap B(p,\delta)$ contains points of the branch $\mathcal{B}' \subset \mathcal{W}^u(p; f^q) \setminus \{p\}$. Thus, arguing as in the proof of Proposition 2.8, we deduce that $\mathcal{B}' \subset \Lambda$. Another application of the λ-lemma (now for a small disk $D' \subset \mathcal{B}'$) gives that in fact, $\overline{\mathcal{W}^u(p; f^q)} \subset \Lambda$. By f-invariance of Λ, we conclude that $\overline{\mathcal{W}^u(\mathcal{O}_f(p)) \cup \mathcal{W}^s(\mathcal{O}_f(p))} \subset \Lambda$. □

Corollary 2.10. *Let p be a periodic hyperbolic point belonging to Λ. Let us assume that for each pair of branches $\mathcal{B}^u \subset \mathcal{W}^u(p; f^q) \setminus \{p\}$ and $\mathcal{B}^s \subset \mathcal{W}^s(p; f^q) \setminus \{p\}$, $\mathcal{B}^u$ and $\mathcal{B}^s$ intersect transversally. Then, Λ contains a horseshoe $K(p)$; more precisely, it holds*

$$K(p) \subset H(p) := \overline{\mathcal{W}^s(\mathcal{O}_f(p)) \pitchfork \mathcal{W}^u(\mathcal{O}_f(p))} \subset \overline{\mathcal{W}^u(\mathcal{O}_f(p))} \subset \Lambda.$$

Proof. The fact that $\mathcal{W}^u(p; f^q) \setminus \{p\}$ and $\mathcal{W}^s(p; f^q) \setminus \{p\}$ intersect transversally guarantees the existence of a horseshoe $K(p)$ which is contained in the homoclinic class $H(p) := \overline{\mathcal{W}^s(\mathcal{O}_f(p)) \pitchfork \mathcal{W}^u(\mathcal{O}_f(p))}$ of p (see [Sma65, Sma67] and [New72, Section 2]). If Λ contains one branch $\mathcal{B}^s \subset \mathcal{W}^s(p; f^q) \setminus \{p\}$, then by Lemma 2.9, $\overline{\mathcal{W}^s(\mathcal{O}_f(p)) \cup \mathcal{W}^u(\mathcal{O}_f(p))} \subset \Lambda$; otherwise, by Proposition 2.8, it holds that $\overline{\mathcal{W}^u(p; f^q)} \subset \Lambda$. In either case, $K(p) \subset H(p) \subset \overline{\mathcal{W}^u(\mathcal{O}_f(p))} \subset \Lambda$. □

3 General facts about dissipative billiards

3.1 Dissipative billiard map

Examples of dissipative maps are given by dissipative billiard maps within convex domains of the plane. Let Ω be a convex domain of the plane $\mathbb{R}^2$ with C^k boundary, $k \geq 2$. Let $\Upsilon\colon \mathbb{T} \to \mathbb{R}^2$ be an arclength parametrization of $\partial\Omega$. We denote by f_1 the conservative billiard map associated to Ω. Now, as in Definition A, we take a C^{k-1} function $\lambda\colon \mathbb{A} \to (0,1)$ satisfying condition (1.2). As in Definition A, the dissipative billiard map within Ω associated to the dissipation function λ is then given by

$$f_\lambda\colon \begin{cases} \mathbb{A} & \to \quad \mathbb{A}\,, \\ (s,r) & \mapsto \quad f_\lambda(s,r) = (s',r')\,, \end{cases}$$

where $\Upsilon(s')$ is the point where the half line, starting at $\Upsilon(s)$ and making an angle $\varphi = \arcsin r$ with the normal, hits the boundary $\partial\Omega$ again, and $r' = \lambda(s', r_1')r_1'$, r_1' being the sine of the outgoing angle of reflection in the case of an elastic collision. Letting $\mathcal{H}_\lambda\colon \mathbb{A} \ni (s,r) \mapsto (s, \lambda(s,r)r) \in \mathbb{A}$, we observe that

$$f_\lambda = \mathcal{H}_\lambda \circ f_1\,. \tag{3.1}$$

Here are some basic properties of the dissipative billiard map.

1. The equality $r' = r_1'$ happens if and only if $r' = r_1' = 0$, i.e., the bounce at x' is perpendicular.
2. We recall that the standard billiard map f_1 preserves the area form $\omega = dr \wedge ds = d\alpha$, where α denotes the 1-form $r\,ds$. Thus, by (3.1), f_λ is a conformally symplectic map; indeed,
$$f_\lambda^*\omega = \big(\partial_r\lambda(s',r_1')r_1' + \lambda(s',r_1')\big)\omega.$$
In particular, if λ is a constant function, then f_λ is a conformally symplectic map with constant factor λ, as $f_\lambda^*\omega = f_1^*(\mathcal{H}_\lambda^*\omega) = \lambda\omega$.
3. The map f_λ is a dissipative map of $\mathbb{A}$, according to Definition 2.1. In particular, f_λ verifies:
   (i) for any $(s,r) \in \mathrm{int}(\mathbb{A})$, it holds
   $$0 < \det Df_\lambda(s,r) = \partial_r\lambda(s',r_1')r_1' + \lambda(s',r_1') < 1\,; \tag{3.2}$$
   in particular, if λ is constant, then for any $(s,r) \in \mathrm{int}(\mathbb{A})$, it holds $0 < \det Df_\lambda(s,r) = \lambda < 1$;
   (ii) it holds $f_\lambda(\mathbb{T} \times [-1,1]) \subset \mathbb{T} \times (-1,1)$; in particular, if λ is constant, then
   $$f_\lambda(\mathbb{T} \times [-1,1]) = \mathbb{T} \times [-\lambda,\lambda] \subset \mathbb{T} \times (-1,1)\,. \tag{3.3}$$
   Moreover, we will see in Section 6 that the map f_λ is a positive twist map.
4. Again from (3.1), for every $(s,r) \in \mathrm{int}(\mathbb{A})$, and $(s',r') := f_\lambda(s,r)$, we have
$$\begin{aligned} Df_\lambda(s,r) &= D\mathcal{H}_\lambda(f_1(s,r))\, Df_1(s,r) \\ &= \begin{bmatrix} 1 & 0 \\ \partial_s\lambda(s',r_1')r_1' & \partial_r\lambda(s',r_1')r_1' + \lambda(s',r_1') \end{bmatrix} \begin{bmatrix} -\frac{\tau\mathcal{K}+\nu}{\nu'} & \frac{\tau}{\nu\nu'} \\ \tau\mathcal{K}\mathcal{K}' + \mathcal{K}\nu' + \mathcal{K}'\nu & -\frac{\tau\mathcal{K}'+\nu'}{\nu} \end{bmatrix}, \end{aligned} \tag{3.4}$$
where $\tau = \ell(s,s') := \|\Upsilon(s) - \Upsilon(s')\|$ is the Euclidean distance between the points $x = \Upsilon(s)$, $x' = \Upsilon(s')$, $\mathcal{K}$, $\mathcal{K}'$ denote the curvatures at $\Upsilon(s), \Upsilon(s')$ respectively and $\nu = \sqrt{1-r^2}$, $\nu' = \sqrt{1-(r_1')^2} = \sqrt{1-\left(\frac{r'}{\lambda}\right)^2}$. Formula (3.4) can be deduced from [CM06, Section 2.11], by applying the change of coordinates $(s,\varphi) \mapsto (s, r = \sin\varphi)$.
If the dissipation is constant, equal to $\lambda \in (0,1)$, then with same notations as above, for $(s,r) \in \mathrm{int}(\mathbb{A})$,
$$Df_\lambda(s,r) = \begin{bmatrix} -\frac{\tau\mathcal{K}+\nu}{\nu'} & \frac{\tau}{\nu\nu'} \\ \lambda\big(\tau\mathcal{K}\mathcal{K}' + \mathcal{K}\nu' + \mathcal{K}'\nu\big) & -\lambda\frac{\tau\mathcal{K}'+\nu'}{\nu} \end{bmatrix}.$$

Notation 3.1. In the sequel, while considering the dissipative billiard map f_λ, we denote its attractor by Λ_λ^0 and its Birkhoff attractor by Λ_λ.

3.2 Properties of the Birkhoff attractor for axially symmetric billiards

This subsection is devoted to proving some properties of the Birkhoff attractor for a dissipative billiard map under some symmetric assumptions on the domain Ω. In particular, they can be applied to the case of elliptic billiards considered in Section 4. In this subsection, we assume that the billiard map f_λ has constant dissipation $\lambda \in (0,1)$. In fact, this would more generally as long as the dissipation function $\lambda\colon \mathbb{A} \to (0,1)$ respects the symmetries of Ω.

Definition 3.2 (Axially symmetric billiard table)**.** *Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain with C^2 boundary. We say that Ω is axially symmetric with respect to some line $\Delta \subset \mathbb{R}^2$ if Ω is invariant under the reflection along the line Δ.*

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain with C^2 boundary, with perimeter $2L > 0$, which is axially symmetric with respect to some line Δ. Let s_0 and $s_0 + L$ be the arclength parameters of the points in $\partial\Omega \cap \Delta$. For $\lambda \in (0,1)$, let f_λ be the associated dissipative billiard map. Then:*

1. *the pair $\{(s_0, 0), (s_0 + L, 0)\}$ corresponds to a 2-periodic orbit;*
2. *let us denote by $\mathcal{I}_\Delta$ the involution $\mathcal{I}_\Delta\colon (s_0 + s, \varphi) \mapsto (s_0 - s, -\varphi)$; then,*

$$\mathcal{I}_\Delta \circ f_\lambda = f_\lambda \circ \mathcal{I}_\Delta. \tag{3.5}$$

Proof. Item (1) follows by noticing that the osculating circles at s_0 and $s_0 + L$ are invariant under the reflection through the axis Δ, hence the line segment connecting the points $s_0, s_0 + L$ (which is collinear to Δ) is perpendicular to the boundary of Ω at the points $s_0, s_0 + L$. The second point is immediate once we have observed that $f_\lambda = \mathcal{H}_\lambda \circ f_1$, as $\mathcal{H}_\lambda$ commutes with $\mathcal{I}_\Delta$ and, under the assumption of axial symmetry, f_1 and $\mathcal{I}_\Delta$ also commute. □

Corollary 3.4. *Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain with C^2 boundary with perimeter $2L > 0$, axially symmetric with respect to some line Δ. Let s_0 and s_0+L be the arclength parameters of the points in $\partial\Omega\cap\Delta$. For $\lambda \in (0,1)$, let f_λ be the associated dissipative billiard map and Λ_λ the corresponding Birkhoff attractor. Then:*

1. *$\mathcal{I}_\Delta(\Lambda_\lambda) = \Lambda_\lambda$;*
2. *$\{(s_0, 0), (s_0 + L, 0)\} \subset \Lambda_\lambda$.*

Proof. Item (1) follows from the fact that Λ_λ is the smallest, with respect to inclusion, compact connected f_λ-invariant subset which separates the annulus. Indeed, $\mathcal{I}_\Delta(\Lambda_\lambda)$ is compact, connected; it is f_λ-invariant, by the fact that Λ_λ is f_λ-invariant, and by (3.5); moreover, $\mathcal{I}_\Delta(\Lambda_\lambda)$ separates the annulus. Thus, $\Lambda_\lambda \subset \mathcal{I}_\Delta(\Lambda_\lambda)$. We conclude because $\mathcal{I}_\Delta^2 = \mathrm{Id}$ and so $\mathcal{I}_\Delta(\Lambda_\lambda) \subset \mathcal{I}_\Delta^2(\Lambda_\lambda) = \Lambda_\lambda$.

To show point (2), let us argue by contradiction, assuming that $(s_0, 0) \notin \Lambda_\lambda$. By compactness of Λ_λ, there exists a connected open neighborhood U of $(s_0, 0)$ that is disjoint from Λ_λ. Since $\mathcal{I}_\Delta(s_0, 0) = (s_0, 0)$, the set $U' := U \cap \mathcal{I}_\Delta(U)$ is a connected open neighborhood of $(s_0, 0)$ that is $\mathcal{I}_\Delta$-invariant and disjoint from Λ_λ. Recall that Λ_λ separates the annulus $\mathbb{A}$, i.e., $\mathbb{A} \setminus \Lambda_\lambda$ is the disjoint union of two open connected components, denoted by U_{Λ_λ} and V_{Λ_λ}. Since $\mathcal{I}_\Delta$ maps the top boundary $\mathbb{T} \times \{1\}$ to the bottom boundary $\mathbb{T} \times \{-1\}$ and viceversa, we have $\mathcal{I}_\Delta(U_{\Lambda_\lambda}) = V_{\Lambda_\lambda}$. As $U' \cap \Lambda_\lambda = \emptyset$, we can assume without loss of generality that $U' \subset U_{\Lambda_\lambda}$. But then, $U' = \mathcal{I}_\Delta(U') \subset \mathcal{I}_\Delta(U_{\Lambda_\lambda}) = V_{\Lambda_\lambda}$, which is a contradiction. Thus, $(s_0, 0) \in \Lambda_\lambda$, and $(s_0 + L, 0) = f_\lambda(s_0, 0) \in \Lambda_\lambda$. □

3.3 Bifurcations of 2-periodic points

As we are going to see, 2-periodic points play a special role for dissipative billiards; this is partly due to the fact that by point (1) in Subsection 3.1, the usual reflection law and the dissipative one have the same effect at an orthogonal collision. In the previous section, we already saw that for convex billiards with symmetries, symmetric 2-periodic orbits have to belong to the Birkhoff attractor. Here we investigate the eigenvalues of 2-periodic points and their bifurcations as the dissipation parameter changes. In fact, as we will see here, although the set of 2-periodic orbits is independent of the value of the dissipation $\lambda\colon \mathbb{A} \to (0,1)$, their type will depend on the strength of the perturbation. Throughout the rest of Subsection 3.3, except in Lemma 3.7 and in the last point of Corollary 3.9, we will assume for simplicity that the dissipation is constant, equal to some $\lambda \in (0,1)$. Yet, the result of Lemma 3.5 could be fully adapted to the case of a non-constant dissipation λ, but the proof would be even more computational; the version we give in Lemma 3.7 is a little less precise but suffices for our purpose. The proofs of the main technical lemma of this subsection, namely Lemma 3.5 and Lemma 3.7 are mainly computational: for this reason, we postpone them to Appendix A.

Let us denote by II the set of 2-periodic points for $\{f_\lambda\}_{\lambda\in[0,1]}$. In the following, for $p = (s,0) \in \mathrm{II}$, let $(s',0) := f_\lambda(p)$ (the point $(s',0)$ is independent of the value of λ as observed above), and denote by $\tau = \ell(s,s') := \|\Upsilon(s) - \Upsilon(s')\|$ the Euclidean distance between the points $\Upsilon(s), \Upsilon(s')$. We also denote by $\mathcal{K}_1$, $\mathcal{K}_2$ the respective curvatures at $\Upsilon(s), \Upsilon(s')$, and let

$$k_{1,2} = k_{1,2}(p) := (\tau\mathcal{K}_1 + 1)(\tau\mathcal{K}_2 + 1). \tag{3.6}$$

Lemma 3.5. *Let $p \in \mathrm{II}$, and let $\tau, \mathcal{K}_1, \mathcal{K}_2, k_{1,2}$ be as above. Fix $\lambda \in (0,1)$, and denote by $\{\mu_1 = \mu_1(\lambda), \mu_2 = \mu_2(\lambda)\}$ the eigenvalues of $Df_\lambda^2(p)$, with $|\mu_1| \le |\mu_2|$.*

- *(a) If $k_{1,2} > 1$, then $0 < \mu_1 < \lambda^2 < 1 < \mu_2$, and the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a* **saddle**.
- *(b) If $k_{1,2} = 1$, then $\mu_1 = \lambda^2$, $\mu_2 = 1$, and the 2-periodic orbit $\{p, f_\lambda(p)\}$ is* **parabolic**.
- *(c) If $k_{1,2} \in (0,1)$, then the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a* **sink**; *moreover, let*

$$\lambda_- = \lambda_-(p) := \frac{1 - \sqrt{1 - k_{1,2}}}{1 + \sqrt{1 - k_{1,2}}} \in (0,1). \tag{3.7}$$

  It holds:

  - *(i) if $\lambda \in (0, \lambda_-)$, then μ_1, μ_2 are real, with $\lambda^2 < \mu_1 < \mu_2 < 1$;*
  - *(ii) if $\lambda = \lambda_-$, then $\mu_1 = \mu_2 = \lambda \in (0,1)$;*
  - *(iii) if $\lambda \in (\lambda_-, 1)$, then μ_1, μ_2 are complex conjugate of modulus λ.*

- *(d) If $k_{1,2} = 0$, then the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a* **sink**, *with $\mu_1 = \mu_2 = -\lambda$, and $Df_\lambda^2(p) = -\lambda\,\mathrm{id}$.*
- *(e) If $k_{1,2} \in (-1,0)$, let*

$$\bar{\lambda} = \bar{\lambda}(p) := \frac{1 - \sqrt{-k_{1,2}}}{1 + \sqrt{-k_{1,2}}} \in (0,1).$$

  It holds:

  - *(i) if $\lambda \in (0, \bar{\lambda})$, then $-1 < \mu_2 < -\lambda < \mu_1 < 0$, and the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a* **sink**;
  - *(ii) if $\lambda = \bar{\lambda}$, then $\mu_1 = -\lambda^2$, $\mu_2 = -1$, and the 2-periodic orbit $\{p, f_\lambda(p)\}$ is* **parabolic**;
  - *(iii) if $\lambda \in (\bar{\lambda}, 1)$, $\mu_2 < -1 < -\lambda^2 < \mu_1 < 0$, and the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a* **saddle**.

- *(f) If $k_{1,2} \le -1$, then $\mu_2 < -1 < -\lambda^2 < \mu_1 < 0$, and the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a* **saddle**.

Moreover, for $\lambda = 0$, the eigenvalues of $Df_0^2(p)$ are $\mu_1 = 0$ and $\mu_2 = k_{1,2}$, and the respective eigenspaces are vertical and horizontal.

Proof. See Appendix A. ∎

Remark 3.6. We use the same notations as above. In the special case of a point $p \in \mathrm{II}$ with $\mathcal{K}_1 = \mathcal{K}_2 =: \mathcal{K} \le 0$, we have $k_{1,2} = (1 + \tau\mathcal{K})^2 \ge 0$, and $\tau\mathcal{K} \le 0$, hence the previous result gives the following outcome.

- (a) If $\tau\mathcal{K} < -2$, then the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a saddle.
- (b) If $\tau\mathcal{K} \in \{-2, 0\}$, then $\mu_1 = \lambda^2$, $\mu_2 = 1$ and the 2-periodic orbit $\{p, f_\lambda(p)\}$ is parabolic.
- (c) If $-2 < \tau\mathcal{K} < 0$, then the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a sink; for $\tau\mathcal{K} = -1$, it holds $Df_\lambda^2(p) = -\lambda\,\mathrm{id}$.

In the case of dissipative billiard maps associated to a non-constant dissipation $\lambda \in (0,1)$, we also have:

Lemma 3.7. *Let $\lambda\colon \mathbb{A} \to (0,1)$ be a C^{k-1} function such that $f_\lambda := \mathcal{H}_\lambda \circ f$ is a dissipative billiard map in the sense of Definition 1.2, where $\mathcal{H}_\lambda\colon (s,r) \mapsto (s, \lambda(s,r)r)$. In particular, f_λ has the same set II of 2-periodic points as f. Fix a 2-periodic orbit $\{p, f_\lambda(p)\}$, and assume that $k_{1,2} \geq 0$, with $k_{1,2} = k_{1,2}(p) := (\tau\mathcal{K}_1 + 1)(\tau\mathcal{K}_2 + 1)$ as in (3.6). Then $\{p, f_\lambda(p)\}$ is parabolic for f_λ if and only if it is for the conservative billiard map $f = f_1$, and this happens if and only if $k_{1,2} = 1$.*

Otherwise, the orbit $\{p, f_\lambda(p)\}$ is either a sink or a saddle for f_λ; more precisely, it is a saddle if and only if $k_{1,2} > 1$, and it is a sink if and only if $k_{1,2} < 1$.

Proof. See Appendix A. □

In the next statement, we summarize some results obtained by Dias Carneiro, Oliffson Kamphorst and Pinto-de-Carvalho, see [DCOKPdC07, Theorem 1] and [DCOKPdC03], and Xia-Zhang [XZ14, Theorem 1.1-Corollary 4.4].

Theorem 3.8. *Fix $k \geq 2$.*

1. *For every $q \geq 2$, there exists an open and dense set $\mathscr{U}^q$ of strongly convex domains with C^k boundary such that the number of q-periodic points (for the usual conservative billiard map) is finite; moreover, all the q-periodic points are either elliptic or hyperbolic.*

2. *There exists a G_δ-dense set $\mathcal{G}^k$ of strongly convex domains with C^k boundary such that, for each $q \geq 2$, the number of q-periodic points is finite. Moreover, all the periodic points are either hyperbolic or elliptic.*

3. *There exists an open and dense set $\mathscr{U}$ of strongly convex domains with C^k boundary, $k \geq 3$, such that for each 2-periodic point $p \in \mathrm{II}$ of saddle type, each branch of $\mathcal{W}^s(p; f^2) \setminus \{p\}$ and $\mathcal{W}^u(p; f^2) \setminus \{p\}$ contains a transverse homoclinic point.*

Corollary 3.9. *Let $k \geq 3$. There exists an open and dense set $\mathscr{U}$ of strongly convex domains with C^k boundary such that, for every $\Omega \in \mathscr{U}$, the following assertions hold.*
If f_λ is a dissipative billiard maps with constant dissipation $\lambda \in (0,1)$, then:

1. *for any $\lambda \in [0,1]$, the set II of 2-periodic points of f_λ is finite;*

2. *for all but at most finitely many $\lambda \in (0,1)$, all the 2-periodic points are non-degenerate, i.e., they are either saddles or sinks;*

3. *there exists $\lambda_*(\Omega) \in (0,1)$ such that for any $\lambda \in [\lambda_*(\Omega), 1)$, and for any point $p \in \mathrm{II}$ of saddle type, each branch of $\mathcal{W}^s(p; f_\lambda^2) \setminus \{p\}$ and $\mathcal{W}^u(p; f_\lambda^2) \setminus \{p\}$ contains a transverse homoclinic point.*

If, moreover, $\Omega \in \mathcal{D}^k \cap \mathscr{U}$, where $\mathcal{D}^k$ is the subset of strongly convex domains with C^k boundary as in Definition D, then for a general dissipative billiard map f_λ in the sense of Definition A (with possibly non-constant dissipation), all the 2-periodic points are either saddles or sinks. It is also true for the (degenerate) map f_0, namely, when the dissipation λ vanishes.

Proof. By the definition of the (dissipative) billiard law given in Definition A, given a convex domain, the set II of 2-periodic points is common to all the maps $\{f_\lambda\}_{\lambda \in [0,1]}$. By Theorem 3.8, we deduce that there exists an open and dense set $\mathscr{U}$ of strongly convex C^k domains, $k \geq 3$, such that, for every $\Omega \in \mathscr{U}$ and for any $\lambda \in [0,1]$, the billiard map f_λ has finitely many 2-periodic points. Fix $\Omega \in \mathscr{U}$, and denote by II its finite set of 2-periodic points. For any $p \in \mathrm{II}$, let $\tau, \mathcal{K}_1, \mathcal{K}_2$ be as in Lemma 3.5, and let $k_{1,2} = k_{1,2}(p) := (\tau\mathcal{K}_1+1)(\tau\mathcal{K}_2+1)$. For any $\lambda \in (0,1)$, the 2-periodic f_λ-orbit $\{p, f_\lambda(p)\}$ is a saddle or a sink, unless $k_{1,2} = 1$ (see Lemma 3.5(b)), or $k_{1,2} \in (-1,0)$ and $\lambda = \bar{\lambda}(p)$ (see Lemma 3.5(e)(ii)). On the one hand, let us examine the case where $k_{1,2} = 1$ for the 2-periodic orbit $\{p, f_1(p)\}$ of the conservative billiard map f_1. By equation (A.2) for $\lambda = 1$, $k_{1,2} = 1$ if and only if $\mathrm{tr} Df_1^2(p) = 2$, i.e., the 2-periodic f_1-orbit $\{p, f_1(p)\}$ is parabolic, which does not occur for the domain Ω, since it is in $\mathscr{U}$. On the other hand, again since $\Omega \in \mathscr{U}$, II is finite, hence so is the set

$$\mathcal{F} := \bigcup_{p \in \mathrm{II}\colon\ k_{1,2}(p) \in (-1,0)} \{\bar{\lambda}(p)\} \subset (0,1)\,.$$

By the above discussion, and by Lemma 3.5, we conclude that for any $p \in \mathrm{II}$, and for any $\lambda \in (0,1) \setminus \mathcal{F}$, the 2-periodic orbit $\{p, f_\lambda(p)\}$ is non-degenerate, i.e. it is either a saddle or a sink.

Point (3) follows immediately from Theorem 3.9; indeed, for the conservative billiard map f_1, there exist transverse homoclinic points on each of the branchs of any 2-periodic point of saddle type; the existence of transverse homoclinic points is stable under C^1-small perturbations of the dynamics, hence the same property holds true for any f_λ with $\lambda \in (0,1)$ close enough to 1.

Finally assume that Ω belongs to the set $\mathcal{D}^k \cap \mathscr{U}$. Let f_λ be a general dissipative billiard map for Ω in the sense of Definition 1.2. It has the same set II of 2-periodic points as f_1. By condition (1.6), for any $p \in \mathrm{II}$, we have $k_{1,2}(p) > 0$. Moreover, $k_{1,2}(p) \neq 1$, since $\Omega \in \mathscr{U}$, hence by Lemma 3.7, the 2-periodic orbit $\{p, f_\lambda(p)\}$ is either a saddle or a sink. Besides, for $\lambda = 0$, Lemma 3.5 says that the eigenvalues of p are $\mu_1 = 0$ and $\mu_2 = k_{1,2}(p)$; as $k_{1,2}(p) > 0$ and $k_{1,2}(p) \neq 1$, we deduce that the 2-periodic $\{p, f_0(p)\}$ of f_0 is either a saddle or a sink. □

Let us remind some classical definitions and results for conservative billiard maps. Let Ω be a strongly convex domain with C^2 boundary. Then, the billiard map expressed in coordinates $(s,\varphi) \in \mathbb{T} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ is a C^1 diffeomorphism, see [Dou82, Proposition I.3.2] and [LC90, Page 11]. Let $F_1 \colon \mathbb{R} \times [-1,1] \to \mathbb{R} \times [-1,1]$ be a lift of the conservative billiard map. Let $\tilde{\pi}_1 \colon \mathbb{R} \times [-1,1] \to \mathbb{R}$ be the projection onto the first coordinate. Then the F_1-orbit of a point (S,r) is completely determined by the bi-infinite sequence $(S_i)_{i\in\mathbb{Z}} := (\tilde{\pi}_1 \circ F_1^i(S,r))_{i\in\mathbb{Z}}$.

Definition 3.10. *Let $(S,r) \in \mathbb{R} \times (-1,1)$. The rotation number of (S,r) is*

$$\rho(S,r) := \lim_{n\to\infty} \frac{\tilde{\pi}_1 \circ F_1^n(S,r)}{n},$$

whenever the limit exists.

Observe that the rotation number depends on the chosen lift F_1. Up to the choice of the lift, i.e., a lift such that F_1 is the identity on the lower boundary $\mathbb{R} \times \{-1\}$, the rotation number of any point belongs to the interval $[0,1]$. Observe that, for such a lift, the rotation number of a periodic (conservative) billiard trajectory corresponds to $\frac{\text{winding number}}{\text{number of reflections}}$.

Let $\ell \colon \mathbb{R}^2 \to \mathbb{R}$ be the generating function of the conservative billiard map. From the geometric point of view, the quantity $\ell(S_i, S_{i+1})$ corresponds to the Euclidean distance on $\mathbb{R}^2$ between $\Upsilon(s_i)$ and $\Upsilon(s_{i+1})$, where $\pi \colon \mathbb{R} \to \mathbb{T}$ is a covering and $s_i = \pi(S_i)$. This is also the quantity previously denoted as $\tau(s_i, s_{i+1})$. In the next proposition, with an abuse of notation, since ℓ is invariant under the action of $\mathbb{Z}$, we also denote by ℓ the function induced on $\mathbb{T}^2$.

By a standard construction due to Birkhoff, see e.g. [Sib04, Theorem 1.2.4], it is well-known that there exist at least two periodic orbits for every rational rotation number. They are obtained by considering the length functional, given by $\sum_{i\in\mathbb{Z}} \ell(S_i, S_{i+1})$. In particular, the first orbit is given by maximizing the functional, while the other one is given by a min-max procedure (sometimes referred to as the *"Mountain Pass Lemma"*). In particular, for the rotation number $\frac{1}{2}$, we obtain two 2-periodic orbits for the conservative billiard map. Then –as remarked at the beginning of the section– the dissipative billiard map f_λ has two 2-periodic orbits for any $\lambda \in [0,1]$. More precisely, for every $\lambda \in (0,1)$, the set of 2-periodic points is non empty and it contains at least 2 different orbits.

Proposition 3.11. *Let f_λ be the dissipative billiard map of a strongly convex domain Ω with C^k boundary, $k \geq 2$, that belongs to the open and dense set $\mathscr{U}$ of Theorem 3.8. Assume that $\{p = (s_1, 0), f_\lambda(p) = (s_2, 0)\}$ is a 2-periodic orbit. We denote by $\mathcal{K}_1, \mathcal{K}_2 < 0$ the respective curvatures at the points $\Upsilon(s_1)$ and $\Upsilon(s_2)$, where $\Upsilon \colon \mathbb{T} \to \mathbb{R}^2$ is an arclength parametrization of the boundary. Let $\tau := \ell(s_1, s_2)$ and $k_{1,2} := (\tau\mathcal{K}_1 + 1)(\tau\mathcal{K}_2 + 1)$. Denote by $\{\mu_1, \mu_2\}$ the eigenvalues of $Df_\lambda^2(p)$, with $|\mu_1| \leq |\mu_2|$. Then:*

- *(a) if (s_1, s_2) corresponds to a local maximum of ℓ (e.g. when $[\Upsilon(s_1), \Upsilon(s_2)]$ is a diameter), then $k_{1,2} > 1$, and the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a saddle;*
- *(b) if (s_1, s_2) corresponds to a critical point of saddle type of ℓ, then $k_{1,2} < 1$, and it holds:*
  - *(i) if $k_{1,2} \geq 0$[4], then the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a sink;*

[4] Note that it is always the case when $\mathcal{K}_1 = \mathcal{K}_2$, or when $\Omega \in \mathcal{D}^k$.

(ii) if $k_{1,2} \in (-1,0)$, let $\bar\lambda = \bar\lambda(p) := \frac{1-\sqrt{-k_{1,2}}}{1+\sqrt{-k_{1,2}}} \in (0,1)$; then, we have:

- *for any $\lambda \in (0,\bar\lambda)$, the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a sink;*
- *for $\lambda = \bar\lambda$, the 2-periodic orbit $\{p, f_\lambda(p)\}$ is parabolic;*
- *for any $\lambda \in (\bar\lambda, 1)$, the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a saddle;*

(iii) if $k_{1,2} \le -1$, then for any $\lambda \in (0,1)$, the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a saddle.

Proof. Fix a 2-periodic orbit $\{p = (s_1, 0), f_\lambda(p) = (s_2, 0)\}$. Since $\partial_1 \ell(s,s') = -\sin\varphi$ and $\partial_2 \ell(s,s') = \sin\varphi'$, the point (s_1,s_2) is a critical point of ℓ. Moreover, we have (see e.g. [CKZ, Lemma 2.1])

$$\ell(s_1+\delta s, s_2+\delta s') - \ell(s_1,s_2) = \frac{1}{2}\begin{bmatrix}\delta s & \delta s'\end{bmatrix} \underbrace{\begin{bmatrix}\mathcal{K}_1 + \frac{1}{\tau} & \frac{1}{\tau} \\ \frac{1}{\tau} & \mathcal{K}_2 + \frac{1}{\tau}\end{bmatrix}}_{=:A} \begin{bmatrix}\delta s \\ \delta s'\end{bmatrix} + o((\delta s)^2 + (\delta s')^2).$$

Let us distinguish between two cases, namely, when the pair (s_1,s_2) corresponds to a local maximum or a critical point of saddle type of the length functional.

(a) In the first case, when ℓ is locally maximal at (s_1,s_2), the Hessian matrix A of ℓ is negative semi-definite, i.e., $\mathrm{tr}A = \mathcal{K}_1 + \mathcal{K}_1 + \frac{2}{\tau} \le 0$, and $\det A = \frac{1}{\tau^2}(k_{1,2}-1) \ge 0$ with $k_{1,2} := (1+\tau\mathcal{K}_1)(1+\tau\mathcal{K}_2)$. Since $\mathcal{K}_1, \mathcal{K}_2 < 0$, from the inequality for $\det A$, we deduce that $k_{1,2} \ge 1$. Therefore, by Lemma 3.5(a)-(b), the real eigenvalues $\mu_1 \le \mu_2$ of $Df_\lambda^2(p)$ satisfy $0 < \mu_1 \le \lambda^2 < 1 \le \mu_2$. In particular, if the local maximum is non-degenerate, then $\det A > 0$, hence $k_{1,2} > 1$, and by Lemma 3.5(a), the 2-periodic point is a saddle, with $0 < \mu_1 < \lambda^2 < 1 < \mu_2$. Note that local maxima of ℓ are always non-degenerate if $\Omega \in \mathscr{U}$; indeed, as in the proof of Corollary 3.9, we see that in that case, $k_{1,2} \neq 1$.

(b) In the second case, the matrix A satisfies $\det A = \frac{1}{\tau^2}(k_{1,2}-1) \le 0$, hence $k_{1,2} \le 1$. When the critical point is non-degenerate (in particular, when $\Omega \in \mathscr{U}$), it holds $\det A < 0$, hence $k_{1,2} < 1$. Then, the conclusion of point (b) in the above statement follows respectively from Lemma 3.5(c)-(d), when $k_{1,2} \in [0,1)$, from Lemma 3.5(e), when $k_{1,2} \in (-1,0)$, and from Lemma 3.5(f), when $k_{1,2} \le -1$. □

4 Birkhoff attractor for circular and elliptic billiards

This section is devoted to the study of the Birkhoff attractor for the dissipative billiard map of (circles and) ellipses. To fix ideas, we can imagine that in what follows, the billiard maps have constant dissipation λ, but in fact, all the results presented in this section hold for a general dissipative billiard map as in Definition A. A useful tool through the whole section is the notion of Lyapunov function.

Definition 4.1. *Let (X,d) be a metric space and let $f\colon X \to X$ be a continuous map. A continuous function $L\colon X \to \mathbb{R}$ is a Lyapunov function for f if $L \circ f(x) \le L(x)$ for every $x \in X$. If L is a Lyapunov function for f, the corresponding neutral set is defined as $\mathcal{N}(L) := \{x \in X :\ L \circ f(x) = L(x)\}$.*

As in the Birkhoff case, the simplest example of dissipative billiard is when the boundary of the billiard table is a circle

$$\mathcal{C} := \left\{x = (x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = R^2\right\}.$$

The proof of the next result is straightforward.

Proposition 4.2. *Let $f_\lambda\colon \mathbb{A} \to \mathbb{A}$ be a dissipative billiard map within a circle $\mathcal{C}$. The corresponding Birkhoff attractor Λ_λ is equal to the attractor Λ_λ^0, and*

$$\Lambda_\lambda = \Lambda_\lambda^0 = \mathbb{T} \times \{0\}.$$

Proof. We notice that, since for any $M \in (0,1]$,

$$f_\lambda(\mathbb{T} \times [-M,M]) \subset \mathbb{T} \times \left[-\max_{\mathbb{A}} \lambda\, M, \max_{\mathbb{A}} \lambda\, M\right] \subset \mathbb{T} \times (-M,M),$$

the attractor (see 2.1) corresponds to $\Lambda_\lambda^0 = \mathbb{T} \times \{0\}$. Since $\mathbb{T} \times \{0\}$ is the minimal element, with respect to the inclusion, in $\mathcal{X}(f_\lambda)$, this concludes the proof. □

Remark 4.3. The following are easy observations about dissipative maps inside a circular billiard. For this remark, we assume that the dissipation λ is constant.

1. Since $\mathcal{D}$ is axially symmetric with respect to every line passing through its center, the fact that $\mathbb{T} \times \{0\} \subset \Lambda_\lambda$ is a direct application of Corollary 3.4.
2. It is worth noting that, in the case of the map f_1 on the disc, the angle $\varphi = \arcsin r$ stays constant along every orbit and it represents an integral of motion; as a consequence, in the dissipative case, $L(s, r) = r$ is a Lyapunov function for f_λ and Λ_λ corresponds to the neutral set $\mathcal{N}(L)$ of L (see Definition 4.1).
3. The foliation $\{\mathbb{T} \times \{r\} : r \in [-1, 1]\}$ is f_λ-invariant, i.e., for every $r \in [-1, 1]$ there exists $r' \in [-1, 1]$ such that $f_\lambda(\mathbb{T} \times \{r\}) = \mathbb{T} \times \{r'\}$. In particular, r and r' have the same sign and $|r'| \leq |r|$.

In the following, we investigate the dynamics of the dissipative billiard map within an ellipse $\mathcal{E}$ of non-zero eccentricity e. As the dynamics is unchanged under rigid motion of the table (affine maps of $\mathbb{R}^2$), without loss of generality, we assume that the major axis is horizontal, and the minor axis is vertical, i.e., for some parameters $a_1 > a_2 > 0$, we have

$$\mathcal{E} := \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} = 1 \right\}, \quad e := \frac{\sqrt{a_1^2 - a_2^2}}{a_1} \in (0, 1) .$$

Denoting by $\cdot$ the Euclidean scalar product, and by B the diagonal matrix

$$B := \begin{bmatrix} \frac{1}{a_1^2} & 0 \\ 0 & \frac{1}{a_2^2} \end{bmatrix},$$

the equation of $\mathcal{E}$ can be abbreviated as $\mathcal{E} = \{Bx \cdot x = 1\}$. In particular, for any $x \in \mathcal{E}$, the vector Bx is collinear with the normal to $\mathcal{E}$ at x; in fact, Bx points outside the convex domain bounded by $\mathcal{E}$. For $\lambda \in (0, 1)$, let $f_\lambda \colon \mathbb{A} \to \mathbb{A}$ be the associated dissipative billiard map where, in such a case, it is convenient to describe the phase-space with the (x, v) coordinates:

$$\left\{ (x, v) \in \mathcal{E} \times T^1\mathcal{E} : Bx \cdot v \leq 0 \right\} .$$

With an abuse of notation, we will refer to this set of coordinates $\{(x, v) \in \mathcal{E} \times T^1\mathcal{E} : Bx \cdot v \leq 0\}$ also as $\mathbb{A}$. In order to lighten notation, f_λ will also denote the dissipative billiard map in (x, v)-coordinates.
Since a point $(s, r) \in \mathbb{A}$ is 2-periodic for f_λ if and only if it is 2-periodic for the standard billiard map f_1, the set of 2-periodic points is reduced to

$$\mathrm{II} := \{E_1, E_2, H_1, H_2\} ,$$

where we denote by

$$\{E_1, E_2 = f_\lambda(E_1)\} \quad \text{and} \quad \{H_1, H_2 = f_\lambda(H_1)\}$$

the 2-periodic orbits of f_λ corresponding to the minor and the major axis respectively.

The next result is a direct outcome of Lemma 3.7.

Lemma 4.4. *Let $f_\lambda \colon \mathbb{A} \to \mathbb{A}$ be a dissipative billiard map within an ellipse $\mathcal{E}$ of non-zero eccentricity. The 2-periodic orbit $\{E_1, E_2 = f_\lambda(E_1)\}$, corresponding to the minor axis, is a sink. The 2-periodic orbit $\{H_1, H_2 = f_\lambda(H_1)\}$, corresponding to the major axis, is a saddle.*

Proof. The statement immediately follows from Lemma 3.7. Indeed, as in Section 3.3, for a 2-periodic orbit $\{p, f_\lambda(p)\} = \{p, f_1(p)\}$, denote by τ and $\mathcal{K}$ respectively the distance between the two bounces and the common curvature at these points, and as in (3.6), let

$$k_{1,2}(p) := (\tau\mathcal{K} + 1)^2 \geq 0 .$$

On the one hand, when $p \in \{E_1, E_2\}$, $\tau\mathcal{K} = 2a_2(-\frac{a_2}{a_1^2}) = -2(\frac{a_2}{a_1})^2 \in (-2, 0)$. Thus, $k_{1,2}(p) \in [0, 1)$, and then, the 2-periodic orbit $\{E_1, E_2 = f_\lambda(E_1)\}$ is a sink. On the other hand, when $p \in \{H_1, H_2\}$, $\tau\mathcal{K} = 2a_1(-\frac{a_1}{a_2^2}) = -2(\frac{a_1}{a_2})^2 < -2$. Thus, $k_{1,2}(p) > 1$, and then, the 2-periodic orbit $\{H_1, H_2 = f_\lambda(H_1)\}$ is a saddle. □

Notation 4.5. For $i = 1, 2$, we denote by $\mathcal{W}^s(H_i; f_\lambda^2)$ (resp. $\mathcal{W}^u(H_i; f_\lambda^2)$) the (1-dimensional) stable (resp. unstable) manifold of H_i for f_λ^2. In order to lighten the notation, for $* = s, u$, $i = 1, 2$, we also denote by $\mathcal{W}^*(\mathcal{O}_\lambda(H_i))$ the union $\mathcal{W}^*(H_1; f_\lambda^2) \cup \mathcal{W}^*(H_2; f_\lambda^2)$. Similarly, let $\mathcal{W}^s(E_i; f_\lambda^2)$ be the (2-dimensional) stable manifold of E_i for f_λ^2. Again, to lighten the notation, for $i = 1, 2$, we denote by $\mathcal{W}^s(\mathcal{O}_\lambda(E_i))$ the union $\mathcal{W}^s(E_1; f_\lambda^2) \cup \mathcal{W}^s(E_2; f_\lambda^2)$.

The main result of the present section is the following characterization of the Birkhoff attractor for dissipative billiard maps within an ellipse.

Theorem 4.6. *Let $f_\lambda \colon \mathbb{A} \to \mathbb{A}$ be a dissipative billiard map within an ellipse $\mathcal{E}$ of non-zero eccentricity. The corresponding Birkhoff attractor Λ_λ is equal to the attractor Λ_λ^0, and we have*

$$\Lambda_\lambda^0 = \Lambda_\lambda = \mathcal{W}^u(\mathcal{O}_\lambda(H_1)) \cup \{E_1, E_2\} = \overline{\mathcal{W}^u(\mathcal{O}_\lambda(H_1))}\,. \tag{4.1}$$

Moreover, for $i = 1, 2$, $\mathcal{W}^u(H_i; f_\lambda^2) \setminus \{H_i\}$ is the disjoint union of two branches $\mathscr{C}_i^1, \mathscr{C}_i^2$, with $\mathscr{C}_i^j \subset \mathcal{W}^s(E_j; f_\lambda^2)$, $j = 1, 2$.

The next two propositions will be used in the proof of Theorem 4.6.

Proposition 4.7. *Let $f_\lambda \colon \mathbb{A} \to \mathbb{A}$ be a dissipative billiard map within an ellipse $\mathcal{E}$ of non-zero eccentricity. The function*

$$L \colon \mathbb{A} \to \mathbb{R}, \quad (x, v) \mapsto Bx \cdot v$$

is a Lyapunov function for f_λ. Moreover, its neutral set $\mathcal{N}(L)$ is equal to $f_\lambda^{-1}(\mathbb{A}_\perp)$, where $\mathbb{A}_\perp$ is the set of points $\{(x, v) \in \mathbb{A} :\ v \text{ is collinear to } x\}$. More precisely, there exists a continuous function $\delta \colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$ such that for any $(x, v) \in \mathbb{A}$,

$$|L(f_\lambda(x, v)) - L(x, v)| < \varepsilon \implies d((x, v), f_\lambda^{-1}(\mathbb{A}_\perp)) < \delta(\varepsilon)\,, \tag{4.2}$$

where d is the usual distance on $\mathbb{A}$.

Proof. For $(x, v) \in \mathbb{A}$, let $(x', v') := f_\lambda(x, v)$. We first observe that

$$B(x' - x) \cdot (x + x') = Bx' \cdot x + Bx' \cdot x' - Bx \cdot x - Bx \cdot x' = 0$$

since $x, x' \in \mathcal{E}$ and the matrix B is symmetric. As $x' - x$ is collinear to the vector v, the previous relation yields $Bv \cdot (x + x') = 0$; by the symmetry of B, we thus obtain

$$-Bx' \cdot v = Bx \cdot v\,. \tag{4.3}$$

Moreover, due to the reflection law, $Bx' \cdot (v + v') < 0$, except when v' is collinear with the normal at x', in which case $Bx' \cdot (v + v') = 0$. By (4.3), we conclude that

$$L(x', v') = Bx' \cdot v' \leq Bx \cdot v = L(x, v)\,,$$

with equality exactly when the bounce at x' is perpendicular to $\mathcal{E}$. This means that $|L(x', v') - L(x, v)| \ll 1$ if and only if $d((x, v), f_\lambda^{-1}(\mathbb{A}_\perp)) \ll 1$. □

Remark 4.8. It is worth noting that $f_\lambda^{-1}(\mathbb{A}_\perp)$ can be alternatively detected as a neutral set in the following way. For $\zeta \in [0, a_2) \cup (a_2, a_1)$, let consider the family of quadrics:

$$\mathcal{E}_\zeta := \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \frac{x_1^2}{a_1^2 - \zeta^2} + \frac{x_2^2}{a_2^2 - \zeta^2} = 1 \right\}.$$

For $\zeta \in [0, a_2)$, $\mathcal{E}_\zeta$ is an ellipse confocal to $\mathcal{E}$ and, for $\zeta \in (a_2, a_1)$, $\mathcal{E}_\zeta$ is a hyperbola confocal to $\mathcal{E}$. Let $F_1 = (-c, 0)$ and $F_2 = (c, 0)$ the two foci of $\mathcal{E}$, where $c := \sqrt{a_1^2 - a_2^2}$. We extend the previous definition by letting $\mathcal{E}_{a_2} := (-\infty, -c] \cup [c, +\infty) \times \{0\}$ and $\mathcal{E}_{a_1} := \{0\} \times \mathbb{R}$. By the theory of usual elliptic billiards, for $(x, v) \in \mathbb{A} \setminus \mathrm{II}$, there exists a unique $\zeta = \zeta(x, v) > 0$ such that any orbit segment of the f_1-trajectory starting at (x, v) is tangent to $\mathcal{E}_\zeta$; moreover, $\mathcal{E}_\zeta$ is an ellipse when the segment $[x, x']$ does not intersect $[F_1, F_2]$, and it is a (possibly degenerate) hyperbola when $[x, x']$ intersects $[F_1, F_2]$. Finally, we set $\zeta(H_1) = \zeta(H_2) := a_2$ and $\zeta(E_1) = \zeta(E_2) := a_1$. Then, comparing the standard reflection law to the dissipative one, it can be proved that the function $-\zeta$ is a Lyapunov function for f_λ, with neutral set $\mathcal{N}(-\zeta) = f_\lambda^{-1}(\mathbb{A}_\perp)$.

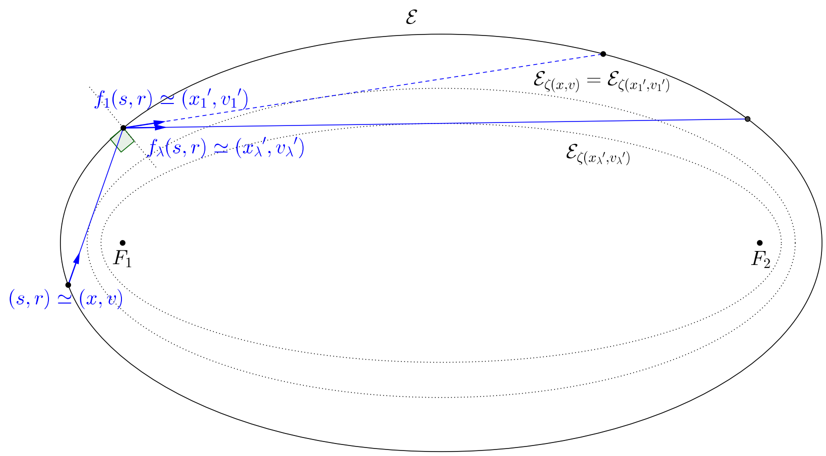

Figure 5: The Lyapunov function $-\zeta$.

Remark 4.9.

(a) Let us recall that $\text{II} := \{E_1, E_2, H_1, H_2\}$ is the set of 2-periodic points. By Proposition 4.7, the function $\mathcal{L}_\lambda := L + L \circ f_\lambda$ is also a Lyapunov function for f_λ, with neutral set

$$\mathcal{N}(\mathcal{L}_\lambda) = f_\lambda^{-1}(\mathbb{A}_\perp) \cap f_\lambda^{-2}(\mathbb{A}_\perp) = f_\lambda^{-1}\big(f_\lambda^{-1}(\mathbb{A}_\perp) \cap \mathbb{A}_\perp\big) = f_\lambda^{-1}(\text{II}) = \text{II}\,.$$

Indeed, an orbit with two consecutive perpendicular bounces is necessarily 2-periodic. Moreover, by (4.2), there exists a continuous function $\hat{\delta}\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\varepsilon\to 0}\hat{\delta}(\varepsilon) = 0$ such that for any $(x,v) \in \mathbb{A}$,

$$0 \le \mathcal{L}_\lambda(x,v) - \mathcal{L}_\lambda(f_\lambda(x,v)) < \varepsilon \implies d((x,v), \text{II}) < \hat{\delta}(\varepsilon)\,, \tag{4.4}$$

where d is the usual distance on $\mathbb{A}$.

(b) For any $(x,v) \in \mathbb{A}_\perp$, Bx and v are collinear, with opposite orientations, hence

$$L(x,v) = -\|Bx\| = -\sqrt{\frac{x_1^2}{a_1^4} + \frac{x_2^2}{a_2^4}} = -\sqrt{\frac{x_2^2}{a_2^2}\Big(\frac{1}{a_2^2} - \frac{1}{a_1^2}\Big) + \frac{1}{a_1^2}}\,.$$

Therefore, $L|_{\mathbb{A}_\perp}$ is maximal when $x_2 = 0$ (and takes the value $-\frac{1}{a_1}$), i.e., for $(x,v) \in \{H_1, H_2\}$, and $L|_{\mathbb{A}_\perp}$ is minimal when $x_2^2 = a_2^2$ (and takes the value $-\frac{1}{a_2}$), i.e., for $(x,v) \in \{E_1, E_2\}$.

(c) Let $X \subset \text{int}(\mathbb{A})$ be a f_λ-invariant set. Then, both $-L|_X$ and $-\mathcal{L}_\lambda|_X$ are Lyapunov functions for f_λ^{-1}.

Proposition 4.10. *Let $f_\lambda\colon \mathbb{A} \to \mathbb{A}$ be a dissipative billiard map within an ellipse $\mathcal{E}$ of non-zero eccentricity. All the orbits are attracted by a 2-periodic orbit, i.e., for any $(s,r) \in \mathbb{A}$, there exists $p_+ = p_+(s,r) \in \text{II} = \{E_1, E_2, H_1, H_2\}$ such that*

$$\lim_{n\to+\infty} f_\lambda^{2n}(s,r) = p_+, \quad \lim_{n\to+\infty} f_\lambda^{2n+1}(s,r) = f_\lambda(p_+)\,.$$

In particular, the set of periodic points for f_λ is reduced to the set II *of 2-periodic points. Moreover,* $\text{II} \subset \Lambda_\lambda \subset \Lambda_\lambda^0$*, and for any $(s,r) \in \Lambda_\lambda^0 \setminus \text{II}$, there exist $i_-, i_+ \in \{1,2\}$ such that*

$$\lim_{n\to-\infty} f_\lambda^{2n}(x,v) = H_{i_-}, \quad \lim_{n\to-\infty} f_\lambda^{2n-1}(x,v) = f_\lambda(H_{i_-})\,,$$
$$\lim_{n\to+\infty} f_\lambda^{2n}(x,v) = E_{i_+}, \quad \lim_{n\to+\infty} f_\lambda^{2n-1}(x,v) = f_\lambda(E_{i_+})\,.$$

Proof. Fix $(s,r) \in \mathbb{A}$. By Remark 4.9(a), the function $\mathcal{L}_\lambda$ is a Lyapunov function for f_λ whose neutral set is II; consequently the omega-limit set $\omega_{f_\lambda}(s,r)$ satisfies $\omega_{f_\lambda}(s,r) \subset \mathcal{N}(\mathcal{L}_\lambda) = \mathrm{II}$. For each $n \geq 0$, we set $u_n := \mathcal{L}_\lambda(f_\lambda^n(s,r))$. The sequence $(u_n)_{\geq 0}$ is decreasing and bounded from below by $\min_{\mathbb{A}} \mathcal{L}_\lambda > -\infty$, hence is convergent. In particular, $\lim_{n\to+\infty}(u_n - u_{n+1}) = 0$; by (4.4), we deduce that

$$\lim_{n\to+\infty} d(f_\lambda^n(s,r), \mathrm{II}) = 0\,,$$

where d is the usual distance inherited from $\mathbb{A}$. Recall that II is formed of only four different points and let $\epsilon := \frac{1}{3}\min_{p\neq q\in \mathrm{II}} d(p,q) > 0$. From previous limit, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, we have $d(f_\lambda^n(s,r), \mathrm{II}) < \epsilon$. Actually, for $n \geq n_0$, there exists a unique point $p(s,r,n) \in \mathrm{II}$ such that $d(f_\lambda^n(s,r), p(s,r,n)) < \epsilon$. In particular, since the bounce at $f_\lambda^{n+1}(s,r)$ gets closer and closer to being perpendicular as $n \to +\infty$, we have that

$$\lim_{n\to+\infty} d(f_\lambda^{n+2}(s,r), f_\lambda^n(s,r)) = 0\,.$$

Let us then fix $n_1 \geq n_0$ such that for any $n \geq n_1$, it holds

$$d(f_\lambda^{n+2}(s,r), f_\lambda^n(s,r)) < \epsilon\,.$$

Then, for any $n \geq n_1$, we have

$$\begin{aligned}
&d(p(s,r,n), p(s,r,n+2))\\
&\leq d(p(s,r,2n), f_\lambda^n(s,r)) + d(f_\lambda^n(s,r), f_\lambda^{n+2}(s,r)) + d(f_\lambda^{n+2}(s,r), p(s,r,n+2)) < 3\epsilon\,.
\end{aligned}$$

By the choice of $\epsilon > 0$, it follows that $p(s,r,n+2) = p(s,r,n)$, for any $n \geq n_1$. Let us then set $p_+ = p_+(s,r) := p(s,r,2n)$, for any $2n \geq n_1$ (it is well-defined by the previous discussion). Then we conclude that

$$\lim_{n\to+\infty} f_\lambda^{2n}(s,r) = p_+, \quad \lim_{n\to+\infty} f_\lambda^{2n+1}(s,r) = f_\lambda(p_+)\,.$$

In particular, we deduce also that $p_+(f_\lambda(s,r)) = f_\lambda(p_+(s,r))$ and that $\omega_{f_\lambda}(s,r) = \{p_+, f_\lambda(p_+)\}$.
The sets $\Lambda_\lambda, \Lambda_\lambda^0$ are f_λ-invariant; moreover, $f_\lambda|_{\Lambda_\lambda}$, resp. $f_\lambda|_{\Lambda_\lambda^0}$ is invertible, and $\tilde{\mathcal{L}}_\lambda := -\mathcal{L}_\lambda|_{\Lambda_\lambda}$, resp. $\tilde{\mathcal{L}}_\lambda^0 := -\mathcal{L}_\lambda|_{\Lambda_\lambda^0}$ is a Lyapunov function for $(f_\lambda|_{\Lambda_\lambda})^{-1}$, resp. $(f_\lambda|_{\Lambda_\lambda^0})^{-1}$, see Remark 4.9(c). Since $\Lambda_\lambda, \Lambda_\lambda^0$ are compact, for any $(s,r) \in \Lambda_\lambda$, resp. $(s,r) \in \Lambda_\lambda^0$, we have that the alpha-limit set $\alpha_{f_\lambda}(s,r)$ satisfies $\emptyset \neq \alpha_{f_\lambda}(s,r) \subset \mathcal{N}(\tilde{\mathcal{L}}_\lambda) \cap \Lambda_\lambda \subset \mathrm{II}$, resp. $\emptyset \neq \alpha_{f_\lambda}(s,r) \subset \mathcal{N}(\tilde{\mathcal{L}}_\lambda^0) \cap \Lambda_\lambda^0 \subset \mathrm{II}$; in particular, we deduce that $\emptyset \neq \Lambda_\lambda \cap \mathrm{II} \subset \Lambda_\lambda^0 \cap \mathrm{II}$. Fix $(s,r) \in \Lambda_\lambda$, resp. $(s,r) \in \Lambda_\lambda^0$. Arguing as above, we see that there exists $p_- = p_-(s,r) \in \mathrm{II} \cap \Lambda_\lambda$, resp. $p_- = p_-(s,r) \in \mathrm{II} \cap \Lambda_\lambda^0$, such that

$$\lim_{n\to-\infty} f_\lambda^{2n}(s,r) = p_-, \quad \lim_{n\to-\infty} f_\lambda^{2n-1}(s,r) = f_\lambda(p_-)\,.$$

Then, it holds that $\alpha_{f_\lambda}(s,r) = \{p_-, f_\lambda(p_-)\}$. There are two cases for the point $(s,r) \in \Lambda_\lambda$, resp. Λ_λ^0:

- either $\alpha_{f_\lambda}(s,r) \cap \omega_{f_\lambda}(s,r) \neq \emptyset$, and then, the whole orbit $(f_\lambda^k(s,r))_{k\in\mathbb{Z}}$ is in the neutral set $\mathcal{N}(\mathcal{L}_\lambda)$, i.e., $(s,r) = p_-(s,r) = p_+(s,r)$ is 2-periodic;

- otherwise, $(s,r) \notin \mathrm{II}$,[5] and the 2-periodic orbits $\alpha_{f_\lambda}(s,r) = \{p_-, f_\lambda(p_-)\}$ and $\omega_{f_\lambda}(s,r) = \{p_+, f_\lambda(p_+)\}$ are distinct. By Remark 4.9(b) and since the orbits of p_+ and p_- are different, we have that $\mathcal{L}_\lambda(p_-) > \mathcal{L}_\lambda(p_+)$ and actually, $\alpha_{f_\lambda}(s,r) = \{H_1, H_2\}$ and $\omega_{f_\lambda}(s,r) = \{E_1, E_2\}$. As $\Lambda_\lambda, \Lambda_\lambda^0$ are closed, we also deduce that $\mathrm{II} \subset \Lambda_\lambda \subset \Lambda_\lambda^0$. □

Remark 4.11. When the dissipation λ is constant, the fact that $\mathrm{II} \subset \Lambda_\lambda \subset \Lambda_\lambda^0$ proven in Proposition 4.10 also follows from Lemma 3.3, due to the symmetries of the ellipse $\mathcal{E}$.

Let us also note that, by Proposition 4.10, for any $(s,r) \in \mathbb{A} \setminus \mathcal{W}^s(H_1; f_\lambda)$, the forward orbit of (s,r) converges to the 2-periodic orbit $\{E_1, f_\lambda(E_1) = E_2\}$.

We are now ready to give the proof of Theorem 4.6.

[5] This case clearly occurs, as Λ_λ separates $\mathbb{A}$, while II is a finite set.

Notation 4.12. Given some small $\delta > 0$, we denote by $\mathcal{W}^u_\delta(H_i; f^2_\lambda)$ the δ-local unstable manifold of H_i with respect to f^2_λ, i.e., the set

$$\mathcal{W}^u_\delta(H_i; f^2_\lambda) := \{(s,r) \in \mathbb{A} : \ d(f^{2n}_\lambda(s,r), H_i) \le \delta, \ \forall\, n \le 0\}.$$

Similarly, the δ-local stable manifold of H_i with respect to f^2_λ is

$$\mathcal{W}^s_\delta(H_i; f^2_\lambda) := \{(s,r) \in \mathbb{A} : \ d(f^{2n}_\lambda(s;r), H_i) \le \delta, \ \forall\, n \ge 0\}.$$

For $* = s, u$, $i = 1,2$, the notation $\mathcal{W}^*_\delta(\mathcal{O}_\lambda(H_i))$ refers to $\mathcal{W}^*_\delta(H_1; f^2_\lambda) \cup \mathcal{W}^*_\delta(H_2; f^2_\lambda)$. We denote by

$$\mathcal{W}^s_\delta(E_i; f^2_\lambda) := \{(s,r) \in \mathbb{A} : \ d(f^{2n}_\lambda(s,r), E_i) \le \delta, \ \forall\, n \ge 0\}$$

the δ-local stable manifold of E_i with respect to f^2_λ. Similarly, for $i = 1,2$, the notation $\mathcal{W}^s_\delta(\mathcal{O}_\lambda(E_i))$ refers to $\mathcal{W}^s_\delta(E_1; f^2_\lambda) \cup \mathcal{W}^s_\delta(E_2; f^2_\lambda)$.

Proof of Theorem 4.6. By Proposition 4.10, for any $(s,r) \in \Lambda_\lambda \setminus \{E_1, E_2\}$, resp. $(s,r) \in \Lambda^0_\lambda \setminus \{E_1, E_2\}$, there exists $i_- \in \{1,2\}$ such that $\lim_{n\to-\infty} f^{2n}_\lambda(s,r) = H_{i_-}$, hence $(s,r) \in \mathcal{W}^u(\mathcal{O}_\lambda(H_{i_-}))$. We deduce that

$$\Lambda_\lambda \subset \Lambda^0_\lambda \subset \mathcal{W}^u(\mathcal{O}_\lambda(H_1)) \cup \{E_1, E_2\} = \overline{\mathcal{W}^u(\mathcal{O}_\lambda(H_1))}, \tag{4.5}$$

where the last equality follows again from Proposition 4.10; indeed, the points E_1, E_2 are accumulated by the forward orbit of any point $(s,r) \in \Lambda_\lambda \setminus \{E_1, E_2\} \subset \mathcal{W}^u(\mathcal{O}_\lambda(H_1))$. From (4.5), and applying Lemma 2.7 to $\overline{\mathcal{W}^u(\mathcal{O}_\lambda(H_1))}$, we deduce that $\overline{\mathcal{W}^u(\mathcal{O}_\lambda(H_1))}$ separates $\mathbb{A}$. Since it is also compact, connected and f_λ-invariant, it holds that $\overline{\mathcal{W}^u(\mathcal{O}_\lambda(H_1))} \in \mathcal{X}(f_\lambda)$.

Claim 4.13. *There is $\delta > 0$ such that, for any $x \in \mathcal{W}^u_\delta(\mathcal{O}_\lambda(H_1))$, $\overline{\mathcal{W}^u(\mathcal{O}_\lambda(H_1))} \setminus \{x\}$ does not separate the annulus.*

Proof of the claim. Let $\eta > 0$ be small enough such that the balls of radius η centered at points in $\{H_1, H_2, E_1, E_2\}$ are pairwise disjoint. From Proposition 4.10, for $i = 1,2$, we have that $\mathcal{W}^u(H_i; f^2_\lambda) \setminus \{H_i\}$ is contained in the stable manifold of $\{E_1, E_2\}$; in particular, there exists $N \in \mathbb{N}$ such that for every $n > N$, it holds

$$f^n_\lambda(\mathcal{W}^u_\eta(\mathcal{O}_\lambda(H_1)) \setminus f^N_\lambda(\mathcal{W}^u_\eta(\mathcal{O}_\lambda(H_1)) \subset B(E_1, \eta) \cup B(E_2, \eta). \tag{4.6}$$

We can then choose $\delta > 0$ small enough such that, for $i = 1,2$, the δ-local unstable manifold of H_i is a C^1 graph over the first coordinate projection of $B(H_i, \delta)$ and such that $B(H_i, \delta) \cap \mathcal{W}^u(H_i; f^2_\lambda) = \mathcal{W}^u_\delta(H_i; f^2_\lambda)$, i.e., the unstable manifold meets the ball only at the local unstable manifold. This last property is possible thanks to (4.6). The δ-local unstable manifold $\mathcal{W}^u_\delta(H_i; f^2_\lambda)$ separates the ball $B(H_i, \delta)$, i.e., we have $B(H_i, \delta) \setminus \mathcal{W}^u_\delta(H_i; f^2_\lambda) = \mathcal{U} \cup \mathcal{V}$, for two disjoint connected open sets $\mathcal{U}$ and $\mathcal{V}$. For any $x \in \mathcal{W}^u_\delta(H_i; f^2_\lambda)$, the set $B(H_i, \delta) \setminus (\mathcal{W}^u_\delta(H_i; f^2_\lambda) \setminus \{x\})$ is path-connected. We conclude that $\overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1))} \setminus \{x\}$ does not separate the annulus if

$$\mathcal{U} \subset U_{\overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1))}} \quad \text{and} \quad \mathcal{V} \subset V_{\overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1))}},$$

where $\mathbb{A} \setminus \overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1))} = U_{\overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1))}} \cup V_{\overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1))}}$. This follows from the fact that $H_i \in \Lambda_\lambda$ and that $\overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1))} \in \mathcal{X}(f_\lambda)$: indeed, by Proposition 2.5(2), $H_i \in \mathrm{Fr}(U_{\overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1))}} \cap \mathrm{Fr}(V_{\overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1))}})$ and in particular the two connected open sets $\mathcal{U}$ and $\mathcal{V}$ cannot be contained in the same connected component of $\mathbb{A} \setminus \overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1))}$. □

Thus, by Lemma 2.6, we have that

$$\mathcal{W}^u_\delta(\mathcal{O}_\lambda(H_1)) \subset \Lambda_\lambda. \tag{4.7}$$

Let us also recall that for $i = 1,2$, we have $\mathcal{W}^u(\mathcal{O}_\lambda(H_1)) = \bigcup_{j\ge0} f^j_\lambda(\mathcal{W}^u_\delta(\mathcal{O}_\lambda(H_1)))$. By (4.7), and as Λ_λ is f_λ-invariant and closed, we obtain

$$\overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1))} \subset \Lambda_\lambda. \tag{4.8}$$

Comparing (4.5) and (4.8), we deduce that all the inclusions are actually equalities, which concludes the proof of (4.1).
By the previous discussion, $\Lambda_\lambda \backslash \text{II}$ is the disjoint union of four connected components $\mathscr{C}_1, \mathscr{C}_1', \mathscr{C}_2 = f_\lambda(\mathscr{C}_1), \mathscr{C}_2' = f_\lambda(\mathscr{C}_1')$, where $\mathscr{C}_i$ and $\mathscr{C}_i'$ correspond to the two branches of $\mathcal{W}^u(H_i; f_\lambda^2) \setminus \{H_i\}$, for $i = 1, 2$. Moreover, by Corollary 4.10, $\mathscr{C}_1 \subset \mathcal{W}^s(E_j; f_\lambda^2)$ and $\mathscr{C}_1' \subset \mathcal{W}^s(E_k; f_\lambda^2)$ for some $j, k \in \{1, 2\}$. We claim that $j \neq k$. Assume by contradiction that $j = k$ and set $\widehat{\mathscr{C}_1} := \mathscr{C}_1 \cup \mathscr{C}_1' \cup \{H_1, E_j\} = \mathcal{W}^u(H_1; f_\lambda^2) \cup \{E_j\}$. Since $\mathcal{W}^u(H_1; f_\lambda^2)$ has no self-intersection, we have that $\widehat{\mathscr{C}_1}$ is a f_λ^2-invariant simple closed curve. We distinguish between two cases:

1. either $\widehat{\mathscr{C}_1}$ separates the annulus $\mathbb{A}$; then, we would have $\Lambda_\lambda = \widehat{\mathscr{C}_1} \sqcup f_\lambda(\widehat{\mathscr{C}_1})$, where $\widehat{\mathscr{C}_1}$ and $f_\lambda(\widehat{\mathscr{C}_1})$ are compact, connected, and both separate $\mathbb{A}$. This would imply that $\mathbb{A} \setminus \Lambda_\lambda$ is the disjoint union of 3 connected open sets, one of whose is a f_λ-invariant bounded open set. This would contradict the dissipative character of the map;
2. or the curve $\widehat{\mathscr{C}_1}$ is homotopic to a point; in particular it bounds a f_λ^2-invariant open set. Again, this would contradict the dissipative character of f_λ^2.

Thus, $j \neq k$. setting $\mathscr{C}_1^j := \mathscr{C}_1$, $\mathscr{C}_1^k := \mathscr{C}_1'$, $\mathscr{C}_2^k := \mathscr{C}_2 = f_\lambda(\mathscr{C}_1^j)$, $\mathscr{C}_2^j := \mathscr{C}_2' = f_\lambda(\mathscr{C}_1^k)$, this concludes the proof. ☐

In the next corollary, we prove that the conclusion of Theorem 4.6 remains true for strictly convex domains whose boundary is sufficiently C^2-close to an ellipse. For simplicity, in the rest of this section, we will assume that the dissipation λ is constant.

Corollary 4.14. *Let $\mathcal{E}$ be an ellipse and fix $\lambda \in (0,1)$. Then, there exists $\epsilon = \epsilon(\mathcal{E}, \lambda) > 0$ such that for any domain $\Omega \subset \mathbb{R}^2$ whose boundary $\partial\Omega$ is C^k, $k \geq 2$, and satisfies $d_{C^2}(\partial\Omega, \mathcal{E}) < \epsilon$, the following holds. Let $f_\lambda \colon \mathbb{A} \to \mathbb{A}$ be the dissipative billiard map within Ω. There exist 2-periodic orbits $\{H_1(\Omega), H_2(\Omega)\}$ and $\{E_1(\Omega), E_2(\Omega)\}$ of saddle and sink type respectively, and the Birkhoff attractor is equal to*

$$\Lambda_\lambda = \mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1(\Omega)) \cup \{E_1(\Omega), E_2(\Omega)\} = \overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1(\Omega))}\,,$$

where $\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1(\Omega)) := \mathcal{W}^u(H_1(\Omega); f_\lambda^2) \cup \mathcal{W}^u(H_2(\Omega); f_\lambda^2)$. Moreover, the function $(\mathcal{E}, \lambda) \mapsto \epsilon(\mathcal{E}, \lambda)$ can be chosen to be continuous.

Proof. Denote by g_λ the dissipative billiard map of $\mathcal{E}$ with dissipation parameter $\lambda \in (0,1)$. Let $\{H_1, H_2\}$ and $\{E_1, E_2\}$ be the 2-periodic orbits for g_λ of saddle and sink type respectively, see Lemma 4.4. For $\partial\Omega$ sufficiently C^2-close to $\mathcal{E}$, the associated domain Ω is still strongly convex, as the curvature function depends continuously on the domain in the C^2-topology, and $\mathcal{E}$ is strongly convex. Without loss of generality, we can assume that the perimeter of $\partial\Omega$ is still one (as the dynamics is invariant under rescaling), so that the dissipative billiard map f_λ is defined on the same phase space $\mathbb{A}$ as g_λ. Moreover, for any $\eta > 0$, there exists $\epsilon_0(\eta) > 0$ such that if $d_{C^2}(\partial\Omega, \mathcal{E}) < \epsilon_0(\eta)$, then $d_{C^1}(f_\lambda, g_\lambda) < \eta$.[6] Fix $\eta > 0$ small enough such that for any Ω with $d_{C^2}(\partial\Omega, \mathcal{E}) < \epsilon_0 := \epsilon_0(\eta)$, the 2-periodic orbits $\{H_1, H_2\}$ and $\{E_1, E_2\}$ have continuations $\{H_1(\Omega), H_2(\Omega)\}$ (of saddle type) and $\{E_1(\Omega), E_2(\Omega)\}$ (of sink type) for f_λ.
Let $\delta_0 > 0$ be such that for every $0 < \delta < \delta_0$, the balls $B(E_1, \delta), B(E_2, \delta)$ are both contractible, i.e., any closed path contained in $B(E_1, \delta)$, resp. $B(E_2, \delta)$, is homotopic to a point. As already noticed with inclusion (4.6) in Proposition 4.10, we can fix $0 < \delta < \delta_0$ small enough such that there exists $N \in \mathbb{N}$ with the property that for any $n > N$, the set

$$g_\lambda^n(\mathcal{W}_{\delta/2}^u(\mathcal{O}_{g_\lambda}(H_1))) \setminus g_\lambda^N(\mathcal{W}_{\delta/2}^u(\mathcal{O}_{g_\lambda}(H_1))) \subset B(E_1, \delta/2) \cup B(E_2, \delta/2)\,.$$

where $\mathcal{W}_{\delta/2}^u(\mathcal{O}_{g_\lambda}(H_1)) := \mathcal{W}_{\delta/2}^u(H_1; g_\lambda^2) \cup \mathcal{W}_{\delta/2}^u(H_2; g_\lambda^2)$. By Hadamard-Perron Theorem –see e.g. [BS15, Proposition 5.6.1]– local invariant manifolds of hyperbolic fixed points depend continuously on the dynamics in the C^1-topology (hence depend continuously on $\partial\Omega$ in the C^2-topology). Consequently, there exists $0 < \epsilon_1 < \epsilon_0$ such that for every domain Ω with $d_{C^2}(\partial\Omega, \mathcal{E}) < \epsilon_1$, for any $n > N$, the set

$$f_\lambda^n(\mathcal{W}_\delta^u(\mathcal{O}_{f_\lambda}(H_1(\Omega)))) \setminus f_\lambda^N(\mathcal{W}_\delta^u(\mathcal{O}_{f_\lambda}(H_1(\Omega))))$$

[6] Indeed, by (3.4), the differentials of f_λ, g_λ depend continuously on the curvature function.

is contained in $B(E_1,\delta)\cup B(E_2,\delta)$, where $\mathcal{W}^u_\delta(\mathcal{O}_{f_\lambda}(H_1(\Omega))) := \mathcal{W}^u_\delta(H_1(\Omega); f_\lambda^2)\cup \mathcal{W}^u_\delta(H_2(\Omega); f_\lambda^2)$. Observe that $E_i(\Omega)\in B(E_i,\delta)$, for $i=1,2$. In particular, the 2-periodic orbit $\{E_1(\Omega),E_2(\Omega)\}$ belongs to

$$\overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1(\Omega))} := \overline{\mathcal{W}^u(H_1(\Omega); f_\lambda^2)\cup \mathcal{W}^u(H_2(\Omega); f_\lambda^2)}$$

and the latter is an f_λ-invariant, compact and connected set.

Claim 4.15. *The set $\overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1(\Omega))}$ separates $\mathbb{A}$.*

Proof of the claim. As for $\overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1(\Omega))}$, we use the notation

$$\overline{\mathcal{W}^u(\mathcal{O}_{g_\lambda}(H_1))} := \overline{\mathcal{W}^u(H_1(\Omega); g_\lambda^2)\cup \mathcal{W}^u(H_2(\Omega); g_\lambda^2)}\,.$$

Observe that the second one separates the annulus because, by Theorem 4.6, it is the Birkhoff attractor of g_λ. The claim then follows if we show that they are homotopic. From the previous choice of $\epsilon_1>0$ we can decompose

$$\overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1(\Omega)))} = \gamma_1\cup\gamma_2\cup\hat\gamma_1\cup\hat\gamma_2\,,$$

where for $i=1,2$, $\gamma_i\subset B(E_i,\delta)$ is a continuous path containing $E_i(\Omega)$ whose endpoints q_i^1,q_i^2 lie on the circle $\mathcal{C}(E_i,\delta)$ (centered at E_i of radius δ), while $\hat\gamma_i\subset f_\lambda^N(\mathcal{W}^u_\delta(H_i(\Omega); f_\lambda^2))$ is an unstable arc with endpoints q_1^i and q_2^i. Similarly,

$$\overline{\mathcal{W}^u(\mathcal{O}_{g_\lambda}(H_1))} = \Gamma_1\cup\Gamma_2\cup\hat\Gamma_1\cup\hat\Gamma_2\,,$$

where for $i=1,2$, $\Gamma_i\subset B(E_i,\delta)$ is a continuous path containing E_i whose endpoints Q_i^1,Q_i^2 lie on the circle $\mathcal{C}(E_i,\delta)$, while $\hat\Gamma_i\subset g_\lambda^N(\mathcal{W}^u_\delta(H_i;g_\lambda^2))$ is an unstable arc with endpoints Q_1^i and Q_2^i.
Fix $0<\delta'\ll\delta$. We can retract a δ'-neighborhood of the circles $\mathcal{C}(E_1,\delta)$ and $\mathcal{C}(E_2,\delta)$ in such a way that the respective images $\{\gamma_i',\hat\gamma_i',\Gamma_i',\hat\Gamma_i'\}_{i=1,2}$ of $\{\gamma_i,\hat\gamma_i,\Gamma_i,\hat\Gamma_i\}_{i=1,2}$ after retraction satisfy that for $i=1,2$, γ_i',Γ_i' have the same endpoints, and $\hat\gamma_i',\hat\Gamma_i'$ have the same endpoints. In particular, it holds that $\gamma_1\cup\gamma_2\cup\hat\gamma_1\cup\hat\gamma_2$ is homotopic to $\Gamma_1\cup\Gamma_2\cup\hat\Gamma_1\cup\hat\Gamma_2$ if and only if $\gamma_1'\cup\gamma_2'\cup\hat\gamma_1'\cup\hat\gamma_2'$ is homotopic to $\Gamma_1'\cup\Gamma_2'\cup\hat\Gamma_1'\cup\hat\Gamma_2'$. We denote by B_1',B_2' the respective images of $B(E_1,\delta)$ and $B(E_2,\delta)$ after retraction.
Since for $i=1,2$, the set B_i' is contractible, and $\gamma_i'\cup\Gamma_i'$ is a closed loop, we deduce that γ_i' and Γ_i' are homotopic. Besides, by the continuous dependence of the local unstable manifolds on the dynamics, the unstable arcs $\hat\gamma_i$ and $\hat\Gamma_i$ are C^0-close to each other, and then, the paths $\hat\gamma_i'$ and $\hat\Gamma_i'$ are homotopic. We conclude that $\gamma_1'\cup\gamma_2'\cup\hat\gamma_1'\cup\hat\gamma_2'$ is homotopic to $\Gamma_1'\cup\Gamma_2'\cup\hat\Gamma_1'\cup\hat\Gamma_2'$; by construction, it follows that $\overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1(\Omega)))} = \gamma_1\cup\gamma_2\cup\hat\gamma_1\cup\hat\gamma_2$ is also homotopic to $\overline{\mathcal{W}^u(\mathcal{O}_{g_\lambda}(H_1))} = \Gamma_1\cup\Gamma_2\cup\hat\Gamma_1\cup\hat\Gamma_2$. ☐

As a consequence of the previous claim, we have $\overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1(\Omega))}\in\mathcal{X}(f_\lambda)$, hence –by Proposition 2.5– the Birkhoff attractor of f_λ is contained in it. Since the Birkhoff attractor cannot be reduced to $\{E_1(\Omega),E_2(\Omega)\}$, it must contain points in the unstable manifold of the saddle periodic orbit. Since the Birkhoff attractor is invariant and closed, it holds that $H_1(\Omega),H_2(\Omega)\in\Lambda_\lambda$. Repeating the proof of Claim 4.13, we can show that for any point x in the local unstable manifold of $\{H_1(\Omega),H_2(\Omega)\}$, the set $\overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1(\Omega)))}\setminus\{x\}$ does not separate the annulus. By Lemma 2.6, we deduce that the local unstable manifold is contained in the Birkhoff attractor. Again, since Λ_λ is invariant and closed, we deduce that $\Lambda_\lambda = \overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(H_1(\Omega)))}$.
Finally, the fact that the function $(\mathcal{E},\lambda)\mapsto\epsilon(\mathcal{E},\lambda)$ can be chosen to be continuous follows from the fact that the objects for which certain conditions have to be satisfied while choosing ϵ_0,ϵ_1 above depend continuously on the dynamics f_λ (in the C^1-topology), which itself depends continuously on the eccentricity of the ellipse $\mathcal{E}$ and on the dissipation parameter $\lambda\in(0,1)$. ☐

5 Birkhoff attractors for strong dissipation

In the previous section, as a first example, we have seen that, in the case of a circular table, the Birkhoff attractor is the simplest possible, i.e. the graph of the zero function. We are thus naturally led to investigate when Λ_λ is topologically simple, that is, it is a graph. The main result of the present section is proving that the geometric condition contained in Definition D together with the hypothesis that the dissipation is strong, i.e., with λ close to 0, are sufficient for the Birkhoff attractor to be a graph. The corresponding result (Theorem 5.7) contains also details on dynamics' and graphs' regularity and utilizes the notions of dominated splitting for an invariant set and of normally contracted (called also hyperbolic) manifold. These definitions are recalled at the beginning of the next section.

Throughout this section, for simplicity, we will assume that the dissipative billiard maps f_λ have constant dissipation $\lambda \in (0,1)$. Yet, we will argue in Remark 5.17 that the same results can be obtained for a general dissipative billiard map as in Definition A.

5.1 Normally contracted Birkhoff attractors for strong dissipation

Let $f_\lambda \colon \mathbb{A} \to \mathbb{A}$, $(s,r) \mapsto (s',r')$ be the dissipative billiard map within a convex domain $\Omega \subset \mathbb{R}^2$. We use notations of Section 3.1.

Proposition 5.1. *Assume that $\Omega \in \mathcal{D}^k$, where $\mathcal{D}^k$ has been introduced in Definition D. Then, there exists $\lambda(\Omega) \in (0,1)$ and a cone-field $\mathcal{C} = (\mathcal{C}(s,r))_{(s,r)\in\mathbb{A}}$ containing the horizontal direction:*

$$\mathcal{C}(s,r) := \{v \in T_{(s,r)}\mathbb{A} : \ v = (v_s, v_r), |v_r| \leq \eta(r)|v_s|\},$$

where $\eta : [-1,1] \to \mathbb{R}_+$ is a continuous function, such that for each $\lambda \in (0, \lambda(\Omega))$, and for each $(s,r) \in \mathbb{T} \times [-\lambda(\Omega), \lambda(\Omega)]$

$$Df_\lambda(s,r)\mathcal{C}(s,r) \subset \operatorname{int}\mathcal{C}(f_\lambda(s,r)) \cup \{0\}.$$

Proof. We recall that for any $(s,r) \in \mathbb{A}$, and $(s',r') := f_\lambda(s,r)$, $\tau(s,r) := \ell(s,s') = \|\Upsilon(s) - \Upsilon(s')\|$ is the Euclidean distance between the consecutive bounces $\Upsilon(s)$, $\Upsilon(s')$, so that the quantity $\tau(s)$ in Definition D is merely $\tau(s) = \tau(s,0)$. Moreover, since $\Omega \in \mathcal{D}^k$, there exists a constant $c_0 > 0$ such that

$$\max_{s\in\mathbb{T}} \tau(s,0)\mathcal{K}(s) < -1 - c_0.$$

By compactness and continuity of the involved functions, we can fix $\delta_0 > 0$ and $K_0 > 0$ such that

$$\max_{(s,r)\in\mathbb{A}} \tau(s,r) \leq \operatorname{diam}\Omega < \delta_0, \qquad \max_{s\in\mathbb{T}} |\mathcal{K}(s)| < \mathcal{K}_0.$$

We can find $\lambda_1 \in (0,1)$ small enough such that

$$\max_{(s,r)\in\mathbb{T}\times[-\lambda_1,\lambda_1]} \tau(s,r)\mathcal{K}(s) + \nu(r) < -c_0, \quad \max_{(s,r)\in\mathbb{T}\times[-\lambda_1,\lambda_1]} \frac{\tau(s,r)}{\nu(r)} < \delta_0, \tag{5.1}$$

where $\nu(r) := \sqrt{1-r^2}$. By (3.4), for each $(s,r) \in \operatorname{int}(\mathbb{A})$, we have

$$Df_\lambda(s,r)e_1 = \begin{bmatrix} -\frac{\tau\mathcal{K}+\nu}{\nu'} \\ \lambda\big(\tau\mathcal{K}\mathcal{K}' + \mathcal{K}\nu' + \mathcal{K}'\nu\big) \end{bmatrix}, \qquad Df_\lambda(s,r)e_2 = \begin{bmatrix} \frac{\tau}{\nu\nu'} \\ -\lambda\frac{\tau\mathcal{K}'+\nu'}{\nu} \end{bmatrix}, \tag{5.2}$$

where $e_1 = (1,0)^T$, $e_2(0,1)^T$ are the vectors of the canonical basis. Moreover, $\mathcal{K}$, $\mathcal{K}'$ denote the curvatures at the points corresponding to s, s', while $\nu := \nu(r) = \sqrt{1-r^2}$, $\nu' := \nu'(r') = \sqrt{1 - \left(\frac{r'}{\lambda}\right)^2}$.
Let $\alpha_0 := \frac{c_0}{2\delta_0} > 0$. At each $(s,r) \in \mathbb{A}$, we identify $T_{(s,r)}\mathbb{A}$ with $\mathbb{R}^2$, and define the cone

$$\mathcal{C}^{\alpha_0}(s,r) := \{u = ae_1 + be_2 \in \mathbb{R}^2 : |b| \leq \alpha_0\nu(r)|a|\}.$$

We note that this cone always contains the horizontal direction $\mathbb{R} \times \{0\}$. Fixed now $\lambda \in (0, \lambda_1)$, let $(s,r) \in f_\lambda(\mathbb{A}) = \mathbb{T} \times [-\lambda, \lambda]$, $(s',r') := f_\lambda(s,r)$. By (5.1) and (5.2), for any $u = ae_1 + be_2 \in \mathcal{C}^{\alpha_0}(s,r)$, its image u' by $Df_\lambda(s,r)$ is equal to

$$u' = \left[-a\frac{\tau\mathcal{K}+\nu}{\nu'} + b\frac{\tau}{\nu\nu'}, a\lambda\big(\tau\mathcal{K}\mathcal{K}' + \mathcal{K}\nu' + \mathcal{K}'\nu\big) - b\lambda\frac{\tau\mathcal{K}'+\nu'}{\nu}\right]^T =: a'e_1 + b'e_2,$$

where $\nu = \nu(r), \nu' = \nu'(r')$. Thus, we have

$$\nu(r')|a'| \geq \nu'(r')|a'| \geq |a|c_0 - |b|\delta_0 \geq \frac{c_0}{2}|a|,$$
$$|b'| \leq |a|\lambda\left((\delta_0\mathcal{K}_0^2 + 2\mathcal{K}_0) + \alpha_0(\delta_0\mathcal{K}_0 + 1)\right).$$

Given now $\mu_0 \in (0,1)$, it holds
$$Df_\lambda(s,r)\mathcal{C}^{\alpha_0}(s,r) \subset \mathcal{C}^{\mu_0\alpha_0}(s',r'),$$
provided that $\lambda \in (0, \lambda(\Omega))$, with
$$\lambda(\Omega) = \lambda(\delta_0, \mathcal{K}_0, c_0, \mu_0) := \min\left(\lambda_1, \frac{\mu_0\alpha_0 c_0}{2(\delta_0\mathcal{K}_0^2 + 2\mathcal{K}_0) + 2\alpha_0(\delta_0\mathcal{K}_0 + 1)}\right) \in (0,1).$$

Setting $\mathcal{C}(s,r) = \mathcal{C}^{\alpha_0}(s,r)$ for each $(s,r) \in \mathbb{A}$, we conclude the proof. □

In order to continue on this section, we need to recall some notions and results: the definition of dominated splitting for an invariant set, the definition of normally contracted (hyperbolic) manifold (see e.g. [Sam16, Definition 2.2] and [BB13] respectively) and a Theorem by Hirsch-Pugh-Shub on the regularity of such normally contracted manifolds (see [HPS77]).

Definition 5.2 (Dominated splitting). *Let M be a compact Riemannian manifold without boundary. Let $f\colon M \to M$ be a C^ℓ diffeomorphism onto its image, $\ell \geq 1$. Let K be an invariant set for f. Then, K has a* **dominated splitting** *if the tangent bundle over K splits into two subbundles $T_K M = E \oplus F$ such that*

1. *E and F are invariant by Df;*
2. *the subbundles E and F vary continuously with respect to the point $x \in K$;*
3. *there exist $C > 0$ and $0 < \nu < 1$ such that for any $x \in K$,*
$$\|Df^n(x)|_E\| \cdot \|Df^{-n}(f^n(x))|_F\| \leq C\,\nu^n, \quad \forall\, n \geq 0.$$

Roughly speaking, the previous definition says that any direction not contained in the subbundle E converges exponentially fast to the direction F under iteration of Df. For the following definitions of (ℓ-)normal contraction, we refer to [HPS77] and [CP15].

Definition 5.3 (Normally contracted manifold). *Let M be a compact Riemannian manifold without boundary. Let $f\colon M \to M$ be a C^ℓ diffeomorphism onto its image, $\ell \geq 1$. Let N be a closed C^1 manifold, invariant under f. Then, we say that N is* **normally contracted** *if N has a dominated splitting $T_N M = E^s \oplus TN$ such that E^s is uniformly contracted, i.e., there exists $n_0 \in \mathbb{N}$ and $\mu \in (0,1)$ such that for any $n \geq n_0$ it holds*
$$\|Df^n(x)|_{E^s}\| \leq \mu^n, \quad \forall\, x \in N.$$
Moreover, we say that N is **ℓ-normally contracted** *if the above splitting satisfies the following stronger condition: there exist $C > 0$ and $0 < \nu < 1$ such that for any $x \in N$, and for any $1 \leq j \leq \ell$,*
$$\|Df^n(x)|_{E^s}\| \cdot \|Df^{-n}(f^n(x))|_{TN}\|^j \leq C\,\nu^n, \quad \forall\, n \geq 0.$$

Once we have a ℓ-normally contracted manifold, then the following theorem by Hirsch-Pugh-Shub assures that the manifold is as regular as the dynamics.

Theorem 5.4. *[HPS77] Let M be a compact Riemannian manifold without boundary. Let $f\colon M \to M$ be a C^ℓ diffeomorphism onto its image, $\ell \geq 1$. Let N be a closed C^1 manifold, invariant under f. If N is ℓ-normally contracted, then N is actually a C^ℓ manifold.*

We can now state an interesting outcome of Proposition 5.1.

Proposition 5.5. *Let $\Omega \in \mathcal{D}^k$, $k \geq 2$. Let $\lambda(\Omega) \in (0,1)$ be given by Proposition 5.1. Then, for any $\lambda \in (0, \lambda(\Omega))$, the attractor Λ_λ^0 has a dominated splitting $E^s \oplus E^c$ where the bundle E^s is uniformly contracted, and each point $(s,r) \in \Lambda_\lambda^0$ has a stable manifold $\mathcal{W}^s(s,r)$, which is transverse to the horizontal. Moreover, there exists $0 < \lambda'(\Omega) < \lambda(\Omega)$ such that for some $C > 0$ and $0 < \nu < 1$, we have that for any $\lambda \in (0, \lambda'(\Omega))$, for any $x \in \Lambda_\lambda^0$, and for any $1 \leq j \leq k-1$,*

$$\|Df_\lambda^n(x)|_{E^s}\| \cdot \|Df_\lambda^{-n}(f_\lambda^n(x))|_{E^c}\|^j \leq C\,\nu^n, \quad \forall\, n \geq 0. \tag{5.3}$$

Proof. Let $\lambda(\Omega) \in (0,1)$ be as in Proposition 5.1. Take $\lambda \in (0, \lambda(\Omega))$. By the cone-field criterion (see e.g. [CP15, Theorem 2.6] and [Sam16, Proposition 2.2]) for the cone-field $\mathcal{C} = (\mathcal{C}(s,r))_{(s,r)\in\mathbb{A}}$ constructed in Proposition 5.1, we deduce that the attractor $\Lambda^0_\lambda \subset f_\lambda(\mathbb{A})$ has a dominated splitting $E^s_\lambda \oplus E^c_\lambda = E^s \oplus E^c$, where $E^c(s,r)$ is contained in the horizontal cone $\mathcal{C}(s,r)$, for each $(s,r) \in \Lambda^0_\lambda$. Moreover, the fiber bundle E^s is uniformly contracted. Indeed, by the domination, there exist $\mu \in (0,1)$ and $n_0 \in \mathbb{N}$ such that for each $(s,r) \in \Lambda^0_\lambda$, and for each $n \geq n_0$,

$$\|Df^n_\lambda(s,r)|_{E^s}\| \leq \mu^n \|Df^n_\lambda(s,r)|_{E^c}\| .$$

Observe that, by (3.2), we have $\det Df^n_\lambda(s,r) = \lambda^n$. As the angle between E^s and E^c is bounded away from zero at each point of Λ^0_λ, thus uniformly as Λ^0_λ is compact, and by a change of basis, we conclude that there exists $n_1 \in \mathbb{N}$ such that for each $(s,r) \in \Lambda^0_\lambda$, and for each $n \geq n_1$,

$$\|Df^n_\lambda(s,r)|_{E^s}\| \leq \lambda^{\frac{n}{2}},$$

i.e., the bundle E^s is uniformly contracted. In particular, by the Stable Manifold Theorem (see e.g. [HPS77] or [CP15]), each point $(s,r) \in \Lambda^0_\lambda$ has a stable manifold $\mathcal{W}^s(s,r)$, which is uniformly transverse to the cone-field $\mathcal{C}$ which contains the horizontal direction.

The center bundle E^c is contained in a cone around the horizontal direction and independent of λ. By (5.2), the modulus of the projection over the first coordinate of $Df_\lambda(s,r)(1,0)^T$ does not depend on λ. Since the central direction E^c is contained in a cone around the horizontal direction, we deduce that there exist constants $0 < C_1 < C_2$ such that for any $\lambda \in (0, \lambda(\Omega))$, it holds

$$C_1 \leq \|Df_\lambda(s,r)|_{E^c}\| \leq C_2, \quad \forall\, (s,r) \in \Lambda^0_\lambda . \tag{5.4}$$

Since $\det Df_\lambda(s,r) = \lambda$, reasoning as above, we deduce that there exists a constant $C_3 > 0$ such that for any $\lambda \in (0, \lambda(\Omega))$, it holds

$$\|Df_\lambda(s,r)|_{E^s}\| \leq C_3\lambda, \quad \forall\, (s,r) \in \Lambda^0_\lambda . \tag{5.5}$$

By (5.4) and (5.5), we conclude that for $\lambda'(\Omega) \in (0, \lambda(\Omega))$ sufficiently small, (5.3) holds for any $\lambda \in (0, \lambda'(\Omega))$, for any $x \in \Lambda^0_\lambda$, and for any $1 \leq j \leq k-1$. ☐

Remark 5.6. Observe that, if we could say *a priori* that Λ^0_λ is a C^1 manifold, then Proposition 5.5 would be saying that Λ^0_λ is ℓ-normally contracted.

The following proposition guarantees that the center space E^c of the dominated splitting of Λ^0_λ integrates uniquely to the Birkhoff attractor (see Bonatti-Crovisier [BC16] for related results in this direction).

Theorem 5.7. *Let $\Omega \in \mathcal{D}^k$, $k \geq 2$, and let $\lambda(\Omega) \in (0,1)$ be given by Proposition 5.1. Then, for $\lambda \in (0, \lambda(\Omega))$, the Birkhoff attractor Λ_λ of f_λ coincides with the attractor Λ^0_λ and is a normally contracted C^1 graph over $\mathbb{T} \times \{0\}$. Let $\lambda'(\Omega) < \lambda(\Omega)$ be given by Proposition 5.5. Then, for $\lambda \in (0, \lambda'(\Omega))$, $\Lambda_\lambda = \Lambda^0_\lambda$ is actually a C^{k-1} graph and Λ_λ converges in the C^1 topology to the zero section $\mathbb{T} \times \{0\}$ as $\lambda \to 0$.*

Proof. Fix $\lambda \in (0, \lambda(\Omega))$, and let $\mathcal{C} = (\mathcal{C}(s,r))_{(s,r)\in\mathbb{A}}$ be the cone-field in $\mathbb{T}\times[-\lambda,\lambda]$ constructed in Proposition 5.1; let us recall that it contains the horizontal direction, as $\mathcal{C}(s,r) = \{v \in T_{(s,r)}\mathbb{A} : v = (v_s, v_r), |v_r| \leq \alpha_0\, \nu(r)\, |v_s|\}$. Let

$$\mathscr{F} := \Big\{\gamma\colon \mathbb{T} \to [-\lambda,\lambda] \text{ such that } \gamma \in C^1(\mathbb{T}) \text{ and } (1,\gamma'(s)) \in \mathcal{C}(s,\gamma(s)),\ \forall\, s \in \mathbb{T}\Big\}.$$

The map f_λ acts on $\mathscr{F}$ by the graph transform

$$\mathscr{G}_{f_\lambda}\colon \mathscr{F} \to \mathscr{F}, \quad \gamma \mapsto \big(s \mapsto \pi_2 \circ f_\lambda\big(g_\lambda^{-1}(s), \gamma(g_\lambda^{-1}(s))\big)\big)\,,$$

where π_1, π_2 denote the projection on the first and second coordinate, respectively, and $g_\lambda\colon \mathbb{T} \to \mathbb{T}$ is the map $s \mapsto \pi_1 \circ f_\lambda(s,\gamma(s))$. Indeed, the cone-field $\mathcal{C}$ around the horizontal direction is contracted by the dynamics, i.e., $Df_\lambda(s,r)\mathcal{C}(s,r) \subset \mathrm{int}(\mathcal{C}(f_\lambda(s,r))) \cup \{0\}$, hence for any $\gamma \in \mathscr{G}_{f_\lambda}$, the image by f_λ of the graph of γ is still the graph of a C^1 function, such that the vector tangent to $f_\lambda(\mathrm{graph}(\gamma))$ is in $\mathcal{C}$, and $\pi_1 \circ f_\lambda|_{\mathrm{graph}(\gamma)}$ is a homeomorphism between graph(γ) and $\mathbb{T}$. In particular, $f_\lambda\big(\text{graph}(\gamma)\big) = \text{graph}(\mathscr{G}_{f_\lambda}(\gamma))$ for a well-defined function $\mathscr{G}_{f_\lambda}(\gamma) \in \mathscr{F}$.

For any $k \in \mathbb{N}$, let denote $\mathbb{A}_k := f_\lambda^k(\mathbb{A})$, and let $\mathscr{F}_k$ be the subset of functions $\gamma \in \mathscr{F}$ whose graph is contained in $\mathbb{A}_k$. Note that $\mathscr{F}_{k+1} \subset \mathscr{F}_k$, and by construction, it holds $\mathscr{F} = \mathscr{F}_1$. Moreover, if $\gamma \in \mathscr{F}_k$, $k \geq 1$, then it holds $\mathscr{G}_{f_\lambda}(\gamma) \in \mathscr{F}_{k+1}$.

In the sequel, let $\|\cdot\|_\infty$ be the sup-norm on the space $C^0(\mathbb{T}, [-1, 1])$. That is, for $\gamma_1, \gamma_2 \in C^0(\mathbb{T}, [-1, 1])$, we let $\|\gamma_2 - \gamma_1\|_\infty := \max_{s \in \mathbb{T}} |\gamma_2(s) - \gamma_1(s)|$. The graph transform acts as a contraction on $\mathscr{F}$, for $\|\cdot\|_\infty$.

Claim 5.8. *There exists a contant $c > 0$ such that for any $\gamma_1, \gamma_2 \in \mathscr{F}$, it holds*

$$\|\mathscr{G}_{f_\lambda}^n(\gamma_2) - \mathscr{G}_{f_\lambda}^n(\gamma_1)\|_\infty \leq c\lambda^{\frac{n}{2}} \|\gamma_2 - \gamma_1\|_\infty, \quad \forall\, n \geq 0.$$

Proof of the claim. For $k_0 \geq 1$ sufficiently large, $\mathbb{A}_{k_0}$ is foliated by stable leaves $\{\mathcal{W}^s(x) \cap \mathbb{A}_{k_0} : x \in \Lambda_\lambda^0\}$, and by the transversality between E^s and $\mathcal{C}$ on Λ_λ^0, there exists $\theta_0 > 0$ such that

$$\angle(T_x\mathcal{W}^s(x), T_x\Gamma_\gamma) \geq \theta_0, \quad \forall\, x \in \text{graph}(\gamma),\ \gamma \in \mathscr{F}_{k_0}, \tag{5.6}$$

where $\angle$ denotes the (non-oriented) angle between the considered vector subspaces.

Let $\gamma_1, \gamma_2 \in \mathscr{F}_{k_0}$. For each $s \in \mathbb{T}$, we denote by $H_{\gamma_1,\gamma_2}^s(s) \in \text{graph}(\gamma_2)$ the image of $(s, \gamma_1(s)) \in \text{graph}(\gamma_1)$ by the holonomy map from graph(γ_1) to graph(γ_2) along the leaves of $\mathcal{W}^s$. That is, follow the stable leaf passing through $(s, \gamma_1(s))$ until it intersects the graph of γ_2: such intersection point is $H_{\gamma_1,\gamma_2}^s(s)$.

For $n \geq 1$, denote by γ_i^n the image $\mathscr{G}_{f_\lambda}^n(\gamma_i)$, $i = 1, 2$. Then, by (5.6), there exists a constant $c > 1$ such that for each $\gamma_1, \gamma_2 \in \mathscr{F}_{k_0}$,

$$c^{-1}|\gamma_2(s) - \gamma_1(s)| \leq d_{\mathcal{W}^s}((s, \gamma_1(s)), H_{\gamma_1,\gamma_2}^s(s)) \leq c|\gamma_2(s) - \gamma_1(s)|, \quad \forall\, s \in \mathbb{T}, \tag{5.7}$$

where $d_{\mathcal{W}^s}$ denotes the distance along a stable leaf[7]. Moreover, since $(s, \gamma_1(s))$ and $H_{\gamma_1,\gamma_2}^s(s))$ belong to the same stable leaf, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$d_{\mathcal{W}^s}(f_\lambda^n(s, \gamma_1(s)), f_\lambda^n \circ H_{\gamma_1,\gamma_2}^s(s)) \leq \lambda^{\frac{n}{2}} d_{\mathcal{W}^s}((s, \gamma_1(s)), H_{\gamma_1,\gamma_2}^s(s)), \quad \forall\, s \in \mathbb{T}. \tag{5.8}$$

For each $s \in \mathbb{T}$ and $n \geq 0$, let us set $s_{-n} := g_\lambda^{-n}(s)$, with $g_\lambda \colon s \mapsto \pi_1 \circ f_\lambda(s, \gamma_1(s))$ as above, so that $f_\lambda^n(s_{-n}, \gamma_1(s_{-n})) = (s, \gamma_1^n(s))$, and $f_\lambda^n \circ H_{\gamma_1,\gamma_2}^s(s_{-n}) = H_{\gamma_1^n,\gamma_2^n}^s(s)$. Indeed, for $i = 1, 2$, f_λ^n sends the graph of γ_i to the graph of γ_i^n, and it sends stable leaves to stable leaves. Then, by (5.7)-(5.8), for $n \geq n_0$, it holds

$$\begin{aligned}
|\gamma_2^n(s) - \gamma_1^n(s)| &\leq c d_{\mathcal{W}^s}((s, \gamma_1^n(s)), H_{\gamma_1^n,\gamma_2^n}^s(s)) \\
&= c d_{\mathcal{W}^s}(f_\lambda^n(s_{-n}, \gamma_1(s_{-n})), f_\lambda^n \circ H_{\gamma_1,\gamma_2}^s(s)) \\
&\leq c\lambda^{\frac{n}{2}} d_{\mathcal{W}^s}((s_{-n}, \gamma_1(s_{-n})), H_{\gamma_1,\gamma_2}^s(s_{-n})) \\
&\leq c^2\lambda^{\frac{n}{2}} |\gamma_2(s_{-n}) - \gamma_1(s_{-n})| \leq c^2\lambda^{\frac{n}{2}} \|\gamma_2 - \gamma_1\|_\infty.
\end{aligned}$$

Now, for any $\bar{\gamma}_1, \bar{\gamma}_2 \in \mathscr{F} = \mathscr{F}_1$, their images under $\mathscr{G}_{f_\lambda}^{k_0-1}$ are in $\mathscr{F}_{k_0}$, hence, up to enlarging the constant c, we conclude that for any $n \geq 0$,

$$\|\mathscr{G}_{f_\lambda}^n(\bar{\gamma}_2) - \mathscr{G}_{f_\lambda}^n(\bar{\gamma}_1)\|_\infty \leq c^2\lambda^{\frac{n}{2}} \|\bar{\gamma}_2 - \bar{\gamma}_1\|_\infty. \quad \square$$

Let $\gamma_0^\pm \colon s \in \mathbb{T} \mapsto \pm\lambda \in (-1, 1)$; clearly, $\gamma_0^\pm \in \mathscr{F}$. For $n \geq 0$, let $\gamma_n^\pm := \mathscr{G}_{f_\lambda}^n(\gamma_0^\pm)$. Observe that for any $n \geq 0$, $\gamma_n^\pm$ is in $\mathscr{F}_{n+1}$. By Claim 5.8, the sequences $(\gamma_n^\pm)_{n\geq 0}$ are Cauchy sequences, hence converge to a continuous function $\gamma_\infty^\pm$.

Claim 5.9. *The equality $\gamma_\infty^+ = \gamma_\infty^- =: \gamma_\infty$ holds and $\Lambda_\lambda^0 = \Lambda_\lambda = \text{graph}(\gamma_\infty)$.*

Proof. Let us denote by $\Gamma_\infty^\pm$ the graph of $\gamma_\infty^\pm$. By construction, $\Gamma_\infty^\pm$ is invariant under f_λ, it is compact, connected, and separates the annulus, i.e., $\Gamma_\infty^\pm \in \mathcal{X}(f_\lambda)$. On one hand, by Proposition 2.5, we have $\Lambda_\lambda \subset \Gamma_\infty^\pm$. On the other hand, by the graph property, for any $x \in \Gamma_\infty^\pm$, the set $\Gamma_\infty^\pm \setminus \{x\}$ does not separate the annulus. By Lemma 2.6, we conclude that $\Gamma_\infty^+ = \Gamma_\infty^- = \Lambda_\lambda$.

[7]More precisely, the distance $d_{\mathcal{W}^s}$ comes from the restriction of the Riemannian metric to the stable leaves.

We can now show that the Birkhoff attractor coincides with the attractor. By definition of the attractor Λ^0_λ, we have $\Lambda^0_\lambda \subset \cap_{n\geq 0}\mathbb{A}_n$. Observe that, for each $n \geq 0$, $\mathbb{A}_n$ is a cylinder bounded by the graphs of $\gamma_n^\pm$. Since the sequences $(\mathrm{graph}(\gamma_n^\pm))_{n\geq 0}$ converge to the same limit graph $\Gamma^+_\infty = \Gamma^-_\infty = \Lambda_\lambda$, it follows that $\Lambda^0_\lambda \subseteq \Gamma^+_\infty = \Gamma^-_\infty = \Lambda_\lambda \subseteq \Lambda^0_\lambda$. □

Claim 5.10. *The function γ_∞ is C^1, and $(1, \gamma'_\infty(s)) \in E^c(s, \gamma_\infty(s))$ for every $s \in \mathbb{T}$.*

Proof. The function $\gamma_\infty = \gamma^+_\infty$ is the C^0 limit of the sequence of C^1 functions $(\gamma_n^+)_{n\geq 0}$. To show that γ_∞ is C^1, it suffices to show that the derivatives $((\gamma_n^+)')_{n\geq 0}$ also converge uniformly. For each $n \geq 0$, let us denote by Γ_n^+ the graph of γ_n^+. With the same notations as in Claim 5.8, for each $s \in \mathbb{T}$, it holds

$$T_{(s,\gamma_n^+(s))}\Gamma_n^+ = Df^n_\lambda(s_{-n}, \gamma_0^+(s_{-n}))T_{(s_{-n},\gamma_0^+(s_{-n}))}\Gamma_0^+ \subset Df^n_\lambda(s_{-n}, \gamma_0^+(s_{-n}))\mathcal{C}(s_{-n}, \gamma_0^+(s_{-n})),$$

and by the cone-field criterion, the cone $Df^n_\lambda(s_{-n}, \gamma_0^+(s_{-n}))\mathcal{C}(s_{-n}, \gamma_0^+(s_{-n}))$ is exponentially small with respect to n, uniformly in $s \in \mathbb{T}$, that is, the amplitude of each cone $Df^n_\lambda\mathcal{C}$ is, up to a uniform constant, equal to μ^n times the amplitude of the cone $\mathcal{C}$, for some uniform constant $\mu \in (0, 1)$. We conclude that the sequence $((\gamma_n^+)')_{n\geq 0}$ converges in the C^1-topology. Moreover, it also implies that at any point $(s, \gamma_\infty(s)) = \lim_{n\to+\infty} (s, \gamma_n^+(s))$, the tangent space of the limit graph Γ_∞ is equal to $E^c(s, \gamma_\infty(s))$. □

So far, we have shown that the (Birkhoff) attractor $\Lambda^0_\lambda = \Lambda_\lambda$ is the graph of a C^1 function γ_∞. Moreover, the graph of γ_∞ is tangent to E^c, and so Λ_λ is a normally contracted C^1 graph, since

$$T\mathbb{A}|_{\Lambda_\lambda} = E^s \oplus E^c = E^s \oplus T\Lambda_\lambda.$$

Now, let $\lambda'(\Omega) < \lambda(\Omega)$ be given by Proposition 5.5. The domain $\partial\Omega$ is of class C^k, and thus, the dissipative billiard map f_λ is of class C^{k-1} for any $\lambda \in (0, 1)$. Fix $\lambda \in (0, \lambda'(\Omega))$. Since $T\Lambda_\lambda = E^c$, and by (5.3), we deduce that Λ_λ is $(k-1)$-normally contracted in the sense of Definition 5.3, hence by Theorem 5.4, the function γ_∞ is actually C^{k-1}.

By construction, $\Lambda_\lambda = \Lambda^0_\lambda \subset f_\lambda(\mathbb{A}) = \mathbb{T}\times[-\lambda, \lambda]$, hence Λ_λ converges to the zero section $\mathbb{T}\times\{0\}$ in the C^0-topology. To show the convergence in the C^1-topology, it suffices to show that $T\Lambda_\lambda = E^c$ converges uniformly to the horizontal space. By construction, at any $s \in \mathbb{T}$, $E^c(s, \gamma^\infty(s)) \subset Df_\lambda(s_{-1}, \gamma_0^+(s_{-1}))\mathcal{C}(s_{-1}, \gamma_0^+(s_{-1}))$, and by (5.2), the vertical component of vectors in the latter cone is less than $\tilde{c}\lambda$, for some constant $\tilde{c} > 0$. □

5.2 Examples and further consequences

As a first consequence of Theorem 5.7, we prove that –when the dissipation is strong, i.e., λ is close to zero– the Birkhoff attractor of ellipses is a normally contracted C^1 graph. Given an ellipse $\mathcal{E}$ of non-zero eccentricity, we let $\{E_1, E_2\}$ be the 2-periodic orbit corresponding to the minor axis; it is a sink, by Lemma 4.4. Then, we define

$$\lambda_-(\mathcal{E}) := \frac{1 - \sqrt{1 - (-2(\frac{a_2}{a_1})^2 + 1)^2}}{1 + \sqrt{1 - (-2(\frac{a_2}{a_1})^2 + 1)^2}} \in (0, 1),$$

i.e., $\lambda_-(\mathcal{E}) = \lambda_-(p)$ for $p = E_1$ or E_2 as in (3.7).

By Theorem 4.6, we have $\Lambda_\lambda = \overline{\mathcal{W}^u(H_1; f^2_\lambda) \cup \mathcal{W}^u(H_2; f^2_\lambda)}$, and for $i = 1, 2$, $\mathcal{W}^u(H_i; f^2_\lambda) \setminus \{H_i\}$ is the disjoint union of two branches $\mathscr{C}^1_i, \mathscr{C}^2_i$, with $\overline{\mathscr{C}^j_i} = \mathscr{C}^j_i \cup \{H_i, E_j\}$, $j = 1, 2$. Thus, Λ_λ is a manifold if and only if for $i, j \in \{1, 2\}$, the tangent space $T_x\mathscr{C}^j_i = T_x\Lambda_\lambda$ has a limit V^j_i as $\mathscr{C}^j_i \ni x \to E_j$. Indeed, by Corollary 3.4, if so, we necessarily have $V^j_1 = V^j_2$. Clearly, a necessary condition for this to hold is that the eigenvalues of $Df^2_\lambda(E_i)$ are real, for $i = 1, 2$, i.e., $\lambda \in (0, \lambda_-(\mathcal{E}))$. Actually, the following holds.

Corollary 5.11. *Let $f_\lambda \colon \mathbb{A} \to \mathbb{A}$ be the dissipative billiard map inside an ellipse $\mathcal{E}$ with eccentricity $e \in (0, \frac{\sqrt{2}}{2})$. Then, there exists $\lambda(\mathcal{E}) < \lambda_-(\mathcal{E})$ such that, for $\lambda \in (0, \lambda(\mathcal{E}))$, the Birkhoff attractor $\Lambda_\lambda = \overline{\mathcal{W}^u(H_1; f^2_\lambda) \cup \mathcal{W}^u(H_2; f^2_\lambda)}$ is a normally contracted C^1 graph, which is actually C^∞ except possibly at E_i, $i = 1, 2$, where Λ_λ is tangent to the weak stable space of $Df^2_\lambda(E_i)$.*

Proof. Let $\Upsilon\colon \mathbb{T} \to \mathbb{R}^2$ be a parametrization of the boundary by arclength such that $\Upsilon(0), \Upsilon(\frac{1}{2})$ correspond to the trace on $\mathcal{E}$ of the 2-periodic orbit of maximal length. For each $s \in \mathbb{T}$, let $\tau(s) = \tau(s,0) := \|\Upsilon(s) - \Upsilon(s')\|$, where $\Upsilon(s), \Upsilon(s')$ are the two points of intersection of $\mathcal{E}$ and the normal to $\mathcal{E}$ at $\Upsilon(s)$, and let $\mathcal{K}(s) \le 0$ be the curvature at $\Upsilon(s)$. The function $[0, \frac{1}{4}] \ni s \mapsto \tau(s)\mathcal{K}(s)$ is increasing, with $\tau(\frac{1}{4})\mathcal{K}(\frac{1}{4}) = -2\big(\frac{a_2}{a_1}\big)^2 = 2(e^2-1)$, where $e > 0$ is the eccentricity of $\mathcal{E}$. Thus, if $e \in (0, \frac{\sqrt{2}}{2})$, the domain bounded by $\mathcal{E}$ is in $\mathcal{D}^\infty$ (recall Definition D), hence by Proposition 5.5 and Theorem 5.7, there exists $\lambda(\mathcal{E}) < \lambda_-(\mathcal{E}))$ such that for $\lambda \in (0, \lambda(\mathcal{E}))$, the Birkhoff attractor Λ_λ is a normally contracted C^1 graph. In fact, it is C^∞ everywhere except possibly at E_1, E_2; indeed, near any other point, it coincides with some piece of the unstable manifold of H_1 or H_2, which is C^∞. By Lemma 3.5, the eigenvalues μ_1, μ_2 of $Df_\lambda^2(E_i)$ satisfy $\lambda^2 < \mu_1 < \mu_2 < 1$. As Λ_λ is C^1 and f_λ-invariant, for $i = 1,2$, any tangent vector $v \in T_{E_i}\Lambda_\lambda$ is an eigenvector of $Df_\lambda^2(E_i)$; since Λ_λ is normally contracted, any such v has to be in the eigenspace associated to the weak eigenvalue μ_2. □

Remark 5.12. If the eccentricity is larger than $\frac{\sqrt{2}}{2}$, then we loose the graph property, even for small dissipation parameters $\lambda \in (0,1)$, see Proposition 5.16 below.

Remark 5.13. It was asked to us by Viktor Ginzburg whether the phase space $\mathbb{A}$ of dissipative billiards admits an invariant foliation by curves homotopic to the zero-section $\mathbb{T} \times \{0\}$. Indeed, in the case of a dissipative billiard within a circle considered at the beginning of Section 4, it is clear that the horizontal foliation $\{\mathbb{T} \times \{r\}\}_{r \in [-1,1]}$ is preserved by any dissipative map f_λ, $\lambda \in (0,1)$. More generally, the existence of such foliations seems much less rigid than in the conservative case, where it is related to the famous Birkhoff conjecture (see e.g. [ADSK16, KS18, BM22] for recent progress in this direction).

Indeed, fix a domain $\Omega \in \mathcal{D}^k$, $k \ge 2$, and a dissipation parameter $\lambda \in (0, \lambda(\Omega))$. With the notations of Theorem 5.7, for any $k \ge 0$, let $\mathbb{A}_k := f_\lambda^k(\mathbb{A})$, and let $\mathcal{F}_1$ be a foliation of $\mathbb{A}_1 \setminus \mathbb{A}_2$ defined as follows. Note that $\mathbb{A}_1 \setminus \mathbb{A}_2$ has two connected components $\mathbb{A}_1^+$ and $\mathbb{A}_1^-$, where $\mathbb{A}_1^\pm$ is bounded by the leaves $\mathbb{T} \times \{\pm\lambda\}$ (in $\mathbb{A}_1^\pm$) and $f_\lambda(\mathbb{T} \times \{\pm\lambda\})$ (in the complement of $\mathbb{A}_1^\pm$). Let then $\mathcal{F}_1$ be the disjoint union of two foliations $\mathcal{F}_1^+$ and $\mathcal{F}_1^-$, where $\mathcal{F}_1^\pm$ is a foliation of $\mathbb{A}_1^\pm$ by C^1 graphs over $\mathbb{T} \times \{0\}$ whose tangent space remains in the cone-field $\mathcal{C}$ constructed in Proposition 5.1, and whose boundary leaves are $\mathbb{T} \times \{\pm\lambda\}$ and $f_\lambda(\mathbb{T} \times \{\pm\lambda\})$. For $k \ge 0$, let $\mathcal{F}_k$ be the foliation of $\mathbb{A}_k \setminus \mathbb{A}_{k+1}$ whose leaves are images by f_λ^{k-1} of the leaves of $\mathcal{F}_1$. Since the cone-field $\mathcal{C}|_{\mathbb{T} \times [-\lambda,\lambda]}$ is contracted under forward iteration, for each $k \ge 1$, the leaves of $\mathcal{F}_k$ are C^1 graphs over $\mathbb{T} \times \{0\}$ whose tangent space is contained in the cone-field $\mathcal{C}$. Moreover, the same argument as in the proof of Theorem 5.7 says that the collection of leaves of $\mathcal{F}_k$ converges uniformly to the Birkhoff attractor Λ_λ in the C^1-topology as $k \to +\infty$. Let $\mathcal{F}_0 := f_\lambda^{-1}(\mathcal{F}_1)$, and let $\mathcal{F}$ be the foliation $\mathcal{F} := \sqcup_{k \ge 0} \mathcal{F}_k \sqcup \Lambda_\lambda$. By construction, it is a foliation of $\mathbb{A}$ by C^1 curves, and it is (forward-)invariant under f_λ. Moreover, the leaves of $\mathcal{F} \cap \mathbb{A}_1$ are C^1 graphs over $\mathbb{T} \times \{0\}$.

As λ gets smaller and smaller, the Birkhoff attractor of f_λ is contained in a smaller and smaller strip around the zero section; actually, we can use the C^1 convergence of the Birkhoff attractor to $\mathbb{T} \times \{0\}$ to deduce interesting information on the dynamics of $f_\lambda|_{\Lambda_\lambda}$, when $\lambda \in (0,1)$ is small, from the degenerate 1-dimensional dynamics of f_0, namely when $\lambda = 0$.

Theorem 5.14. *Let $k \ge 2$. For a C^k-generic billiard $\Omega \in \mathcal{D}^k$ there exists $\lambda''(\Omega) \in (0,1)$ such that for any $\lambda \in (0, \lambda''(\Omega))$, the Birkhoff attractor Λ_λ is a C^{k-1} normally contracted graph of rotation number $\frac{1}{2}$, and, moreover,*

$$\Lambda_\lambda = \bigcup_{i=1}^{\ell} \overline{\mathcal{W}^u(H_i; f_\lambda^2) \cup \mathcal{W}^u(f_\lambda(H_i); f_\lambda^2)},$$

for some finite collection $\{H_i, f_\lambda(H_i)\}_{i=1,\cdots,\ell}$ of 2-periodic orbits of saddle type.

Proof. Let $\Omega \in \mathcal{D}^k$ be a C^k-generic billiard as in Corollary 3.9, and let $\lambda'(\Omega) \in (0,1)$ be given by Theorem 5.7. For any $\lambda \in (0, \lambda'(\Omega))$, the Birkhoff attractor Λ_λ of f_λ is normally contracted, and is equal to the graph Γ_{γ_λ} of some C^{k-1} function $\gamma_\lambda\colon \mathbb{T} \to [-1,1]$. We let $g_\lambda\colon \mathbb{T} \to \mathbb{T}$ be the circle map $\mathbb{T} \ni s \mapsto \pi_1 \circ f_\lambda(s, \gamma_\lambda(s))$ induced by $f_\lambda|_{\Lambda_\lambda}$, where $\pi_1\colon \mathbb{A} = \mathbb{T} \times [-1,1] \to \mathbb{T}$ is the projection over the first coordinate. For any $s \in \mathbb{T}$, let $(s_1, \gamma_\lambda(s_1)) := f_\lambda(s, \gamma_\lambda(s))$. By (**??**), it holds

$$g_\lambda'(s) = -\frac{\tau(s,\gamma_\lambda(s))\mathcal{K}(s) + \nu(s)}{\nu'(s)} + \frac{\tau(s,\gamma_\lambda(s))}{\nu(s)\nu'(s)}\gamma_\lambda'(s), \tag{5.9}$$

where $\tau(s,\gamma_\lambda(s))$ is the length of the orbit segment for f_λ (also of f_1) connecting the points $\Upsilon(s)$ and $\Upsilon(s_1)$, $\mathcal{K}(s)$ is the curvature at $\Upsilon(s)$, and $\nu(s)=\sqrt{1-\gamma_\lambda^2(s)}$, $\nu'(s):=\sqrt{1-\left(\frac{\gamma_\lambda(s_1)}{\lambda}\right)^2}$.

Let us note that the function $\mathbb{T}\ni s\mapsto \pi_1\circ f_\lambda(s,0)$ is independent of the value of $\lambda\in[0,1]$. In particular, for any $\lambda\in(0,1)$, it holds $\pi_1\circ f_\lambda|_{\mathbb{T}\times\{0\}}=\pi_1\circ f_1|_{\mathbb{T}\times\{0\}}=\pi_1\circ f_0|_{\mathbb{T}\times\{0\}}$. We denote such a function by g_0. Note that the function $g_\lambda=\pi_1\circ f_\lambda|_{\Lambda_\lambda}$ is C^0-converging to g_0 as $\lambda\to 0$, since Λ_λ converges to the zero section, by Theorem 5.7. The extended family $(g_\lambda)_{\lambda\in[0,\lambda'(\Omega))}$ satisfies:

Claim 5.15. *The family of maps $(g_\lambda)_{\lambda\in[0,\lambda'(\Omega))}$ depends continuously on λ in the C^1-topology.*

Proof. By the theory of normally contracted invariant manifolds and their persistence (see e.g. [BB13, Theorem 2.1 and Corollary 2.2]), γ_λ depends continuously on $\lambda\in(0,\lambda'(\Omega))$ in C^{k-1}-topology, hence g_λ also depends continuously on λ in C^{k-1}-topology. Moreover, by Theorem 5.7, γ_λ converges to the zero function 0 in the C^1-topology as $\lambda\to 0$, hence g_λ converges to the map g_0 in the C^1-topology. □

In particular, g_λ converges to the map g_0 in the C^1-topology, with $g_0'\colon s\mapsto -\tau(s,0)\mathcal{K}(s)-1$, where $\tau(s,0)$ is the length of the first orbit segment for f_0 (also for f_1) starting at $(s,0)$, and $\mathcal{K}(s)$ is the curvature at $\Upsilon(s)$. Note that $-\tau(s,0)\mathcal{K}(s)-1\neq 0$, since Ω is in $\mathcal{D}^k$. In particular, g_0 is a circle diffeomorphism. Let us denote by Π the set of 2-periodic points of the family $\{f_\lambda\}_{\lambda\in[0,1]}$. As already observed, the set Π is common to every f_λ. Since the set Π is contained in the zero section $\mathbb{T}\times\{0\}$, the circle diffeomorphism g_0 has rotation number $\frac{1}{2}$. Moreover, by Corollary 3.9, for a C^k-generic domain Ω, for any $\lambda\in[0,1)$, all the 2-periodic points of the billiard map f_λ are either saddles or sinks. In particular, the latter persist under perturbation, even when we consider the 1-dimensional dynamics on the corresponding Birkhoff attractor, as we are going to show. In fact, for any 2-periodic point $p=(s,0)\in\Pi$, denoting by $\mathcal{K}_1,\mathcal{K}_2$ the curvatures at the two bounces, and by τ the Euclidean distance between the two bounces, according to (5.9), the multiplier $(g_0^2)'(s)=\frac{d}{ds}g_0(g_0(s))$ is equal to $k_{1,2}:=(\tau\mathcal{K}_1+1)(\tau\mathcal{K}_2+1)$. In particular, for the circle diffeomorphism g_0, the 2-periodic point s is repelling when $|k_{1,2}|>1$, and attracting when $|k_{1,2}|<1$. By Claim 5.15, for $\lambda>0$ small, the circle diffeomorphism g_λ is C^1-close to g_0. Thus, for any $p=(s,0)\in\Pi$, the 2-periodic point s for g_0 admits a continuation for g_λ. Since the set Π is common to all functions f_λ and since generically 2-periodic points are isolated, we deduce that s is also 2-periodic for g_λ. Therefore, there exists $\lambda''(\Omega)\in(0,\lambda'(\Omega))$ such that for any $\lambda\in(0,\lambda''(\Omega))$, the restriction $f_\lambda|_{\Lambda_\lambda}$ still has rotation number $\frac{1}{2}$. Observe that, on the one hand, if $p=(s,0)$ is a sink for f_λ, then s is an attracting 2-periodic point for g_λ; on the other hand, if $p=(s,0)$ is of saddle type for f_λ, then s is a 2-periodic repulsive point, because the Birkhoff attractor is normally contracted. By standard facts of the theory of circle homeomorphisms with rational rotation number, the α-limit set $\alpha_{f_\lambda^2}(s,r)$ of any point $(s,r)\in\Lambda_\lambda\setminus\Pi$ is a 2-periodic point $H=H(s,r)$, which has to be of saddle type (as sinks are repulsive for the past dynamics). Arguing as in the proof of Proposition 4.10, we deduce that $(s,r)\in\mathcal{W}^u(H;f_\lambda^2)$. Similarly, $\omega_{f_\lambda^2}(s,r)=E\in\Pi$, with E a sink periodic point in $\overline{\mathcal{W}^u(H;f_\lambda^2)}$. We conclude that

$$\Lambda_\lambda=\bigcup_{i=1}^{\ell}\overline{\mathcal{W}^u(H_i;f_\lambda^2)\cup\mathcal{W}^u(f_\lambda(H_i);f_\lambda^2)}\,,$$

for some finite collection $\{H_i,f_\lambda(H_i)\}_{i=1,\cdots,\ell}$ of 2-periodic orbits of saddle type, which concludes the proof. □

We will now show that for any C^k convex domain in the interior of the complement of $\mathcal{D}^k$, $k\geq 2$, we loose the graph property of Λ_λ for small dissipation parameters $\lambda\in(0,1)$. This is the case in particular for any ellipse $\mathcal{E}$ of eccentricity e larger than $\frac{\sqrt{2}}{2}$.[8] As previously, given a strongly convex billiard Ω, for $(s,r)\in\mathbb{A}$, we denote by $\tau(s,r)$ the length of the first orbit segment for the (conservative) billiard map starting at (s,r), and by $\mathcal{K}(s)<0$ the curvature at the point of $\partial\Omega$ associated to s.

Proposition 5.16. *Let $k\geq 2$, and let Ω be a strongly convex domain with C^k boundary in the complement of $\mathcal{D}^k$, such that there exists $s_0\in\mathbb{T}$ with $\tau(s_0,0)\mathcal{K}(s_0)>-1$. Then, for $\lambda\in(0,1)$ sufficiently small, the Birkhoff attractor Λ_λ is not a graph over $\mathbb{T}\times\{0\}$.*

[8] Indeed, if $\{(s_0,0),f_1(s_0,0)\}$ is the 2-periodic orbit along the minor axis of $\mathcal{E}$, then with the notations of Proposition 5.16, an easy computation shows that $\tau(s_0,0)\mathcal{K}(s_0)=2(e^2-1)>-1$, hence the assumption of Proposition 5.16 is satisfied.

Proof. By contradiction, let us assume that there exists a sequence $(\lambda_n)_{n\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ converging to 0 such that Λ_{λ_n} is the graph of some function $\gamma_n\colon \mathbb{T} \to [-1,1]$. As Λ_{λ_n} separates $\mathbb{A}$, the function γ_n is necessarily continuous. We can then define the map

$$g_{\lambda_n}\colon \mathbb{T} \to \mathbb{T}, \quad s \mapsto \pi_1 \circ f_{\lambda_n}(s, \gamma_n(s))\,,$$

where $\pi_1\colon \mathbb{A} \to \mathbb{T}$ denotes the projection on the first coordinate. Note that by the graph hypothesis, g_{λ_n} is invertible. Let us also define $g_0\colon s \mapsto \pi_1 \circ f_0(s,0)$; note that $g_0 = \pi_1 \circ f_\lambda(s,0)$, for any $\lambda \in [0,1]$, and that g_0 is C^1. By construction, $\Lambda_\lambda \subset \mathbb{T} \times [-\lambda, \lambda]$, and $f_\lambda\colon (s,r) \mapsto (s', \lambda r_1')$. Hence for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for any $n \geq n_\varepsilon$, $d_{C^0}(g_0, g_{\lambda_n}) < \varepsilon$. Now, $g_0'(s) = -(\tau(s)\mathcal{K}(s)+1)$. On the one hand, if $\{(s_1,0),(s_2,0)\}$ is a 2-periodic orbit of maximal perimeter, then, as in the proof of Proposition 3.11,

$$(\tau(s_1)\mathcal{K}(s_1)+1)(\tau(s_2)\mathcal{K}(s_2)+1) \geq 1\,.$$

Since $\mathcal{K} < 0$, we conclude that there exists $i \in \{1,2\}$ such that $\tau(s_i)\mathcal{K}(s_i) + 1 \leq -1$, i.e., $g_0'(s_i) \geq 1$. On the other hand, by assumption, $g_0'(s_0) < 0$. We conclude that there exist $s_* \in \mathbb{T}$ and $\eta_1, \eta_2 > 0$ such that $g_0(s_* - \eta_1) = g_0(s_* + \eta_2)$ but $g_0(s_*) \neq g_0(s_* - \eta_1)$. Let $\varepsilon := \frac{1}{3}|g_0(s_*) - g_0(s_* - \eta_1)|$. We deduce that for any $n \geq n_\varepsilon$,

$$\begin{aligned} &\text{either } g_{\lambda_n}(s_*) > g_{\lambda_n}(s_* - \eta_1) \text{ and } g_{\lambda_n}(s_*) > g_{\lambda_n}(s_* + \eta_2)\,, \\ &\text{or } g_{\lambda_n}(s_*) < g_{\lambda_n}(s_* - \eta_1) \text{ and } g_{\lambda_n}(s_*) < g_{\lambda_n}(s_* + \eta_2)\,. \end{aligned}$$

By the continuity of g_{λ_n}, we deduce that g_{λ_n} is not injective, a contradiction. □

We conclude this section by discussing the case where the dissipative billiard map f_λ has non-constant dissipation.

Remark 5.17. Let us consider a general dissipative billiard map f_λ as in Definition A, for some C^{k-1} dissipation function $\lambda\colon \mathbb{A} \to (0,1)$. The results presented in this section can be obtained for such a map f_λ, as long as $\|\lambda\|_{C^1} \ll 1$.

6 Topologically complex Birkhoff attractors for mild dissipation

Birkhoff attractors for dissipative billiards described in Sections 4 and 5 do not make the idea of their possible topological complexity. In fact, following a celebrated result by M. Charpentier [Cha34, Section 20] –here Theorem 6.8– a Birkhoff attractor for a dissipative diffeomorphism may be an "indecomposable continuum" (see Fig. 6), and a sufficient condition for this occurrence is that two rotation numbers associated to the Birkhoff attractor itself are different. The aim of this section is proving that such a phenomenon occurs also for Birkhoff attractors of dissipative billiard maps. Moreover, we discuss some topological and dynamical consequences of this phenomenon.

The section is organized as follows. After recalling the main definition and properties of a twist map, we present the construction of the upper and the lower rotation numbers associated to the Birkhoff attractor, as well as the statement of Charpentier's Theorem. Finally, in the case of dissipative billiards, we give a sufficient condition assuring that the corresponding Birkhoff attractor has different upper and lower rotation numbers and we discuss the dynamical consequences of this fact.

6.1 Twist diffeomorphisms

Fix the standard metric and trivialisation of the tangent space of $\mathbb{A} := \mathbb{T} \times [-1,1]$, as well as the counter-clockwise orientation of the plane. Let $\beta \in (0, \frac{\pi}{2})$ and denote by v the unitary vertical vector $(0,1)$. For any $x \in \mathbb{A}$, the cone $C_+(x,\beta)$ is the set of vectors $w \in T_x\mathbb{A}$ such that the angle $\theta(v,w)$ (with respect to the fixed metric and trivialisation) admits a lift in $(-\pi+\beta, -\beta)$; similarly, the cone $C_-(x,\beta)$ is the set of vectors $w \in T_x\mathbb{A}$ such that the angle $\theta(v,w)$ admits a lift in $(\pi-\beta, \beta)$, see Fig. 7. For the next definition, we refer to [Her83, Section 1.2].

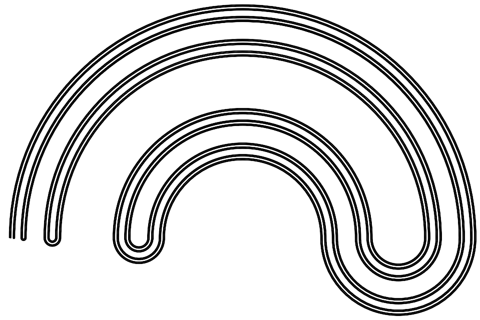

Figure 6: An example of indecomposable continuum (L Rempe-Gillen, CC BY-SA 3.0 `https://creativecommons.org/licenses/by-sa/3.0`, via Wikimedia Commons).

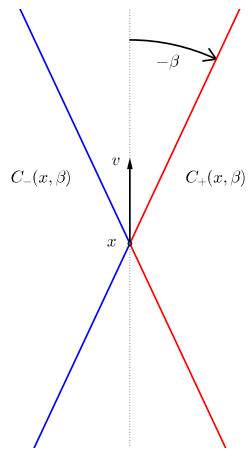

Figure 7: The cones $C_+(x,\beta)$ and $C_-(x,\beta)$.

Definition 6.1. *Let U be an open subset of $\mathbb{A}$. A C^1 orientation-preserving diffeomorphism $f\colon U \subset \mathbb{A} \to f(U) \subset \mathbb{A}$ is a positive, resp. negative twist map on U if there exists $\beta \in (0, \frac{\pi}{2})$ such that:*

$$Df(x)v \in C_+(f(x),\beta), \quad \textit{resp. } Df(x)v \in C_-(f(x),\beta), \qquad \forall\, x \in U.$$

We are mostly interested in dissipative twist maps. Nevertheless, if we restrict to constant conformally symplectic twist maps, a variational setting can be described, following [Ban88]. Let f be a constant conformally symplectic twist diffeomorphism of $\mathrm{int}(\mathbb{A})$ into its image of conformality ratio $a > 0$ with respect to the area form $\omega = dr \wedge ds = d\alpha$, where $\alpha = rds$ is the Liouville 1-form. Denote by $F\colon (S,r) \in \mathbb{R}\times[-1,1] \mapsto (S',r') \in \mathbb{R}\times[-1,1]$ a lift of f to the universal cover. The map f/a is an exact symplectic twist diffeomorphism of $\mathrm{int}(\mathbb{A})$; this means that there exists a generating function $H \in C^2(\mathbb{R}^2;\mathbb{R})$ for F/a such that $F^*\alpha - a\alpha = adH$, that is

$$r'dS' - ardS = adH(S,S')\,. \tag{6.1}$$

We define a (formal) action functional as

$$\mathcal{H}\colon \{S_i\}_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}} \longmapsto \sum_{i\in\mathbb{Z}} \frac{H(S_i,S_{i+1})}{a^i}\,.$$

Definition 6.2. *A bi-infinite sequence $\{S_i\}_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ is stationary for $\mathcal{H}$ if*

$$\partial_1 H(S_i,S_{i+1}) + a\partial_2 H(S_{i-1},S_i) = 0 \qquad \forall i \in \mathbb{Z}\,.$$

We can then characterize the orbits of F in terms of stationary sequences. Indeed, equality (6.1) means that, for every $S, S' \in \mathbb{R}$,

$$\begin{cases} r &= -\partial_1 H(S,S'), \\ r' &= a\partial_2 H(S,S'). \end{cases}$$

As a consequence, $\{(S_i, r_i)\}_{i\in\mathbb{Z}}$ is an orbit of F if and only if for every $i \in \mathbb{Z}$, it holds

$$-\partial_1 H(S_i, S_{i+1}) = r_i = a\partial_2 H(S_{i-1}, S_i)\,. \tag{6.2}$$

This implies the following.

Proposition 6.3. *A bi-infinite sequence $\{(S_i, r_i)\}_{i\in\mathbb{Z}}$ is an orbit of F if and only if the bi-infinite sequence $\{S_i\}_{i\in\mathbb{Z}}$ is stationary.*

An important example of twist map is given by the billiard map within a convex domain, as recalled in the following proposition.

Proposition 6.4. *Let $\Omega \subset \mathbb{R}^2$ be a convex domain, with C^k boundary, $k \geq 2$. Then, the associated billiard map $f = f_1 \colon \mathbb{A} \to \mathbb{A}$ given by (1.1) is a positive twist map when restricted to* $\mathrm{int}(\mathbb{A})$.

Proof. Let $(s,r) \in \mathrm{int}(\mathbb{A})$. We consider the image of the vertical direction by the differential of f:

$$Df(s,r)\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\frac{\tau}{\nu\nu'}\\ -\frac{(\tau\mathcal{K}'+\nu')}{\nu}\end{bmatrix}.$$

In order to conclude that f is a positive twist map, it is sufficient to show that for some $M > 0$, independent of the point $(s,r) \in \mathrm{int}(\mathbb{A})$, it holds

$$\frac{|\tau\mathcal{K}' + \nu'|/\nu}{\tau/\nu\nu'} = |\tau\mathcal{K}' + \nu'|\frac{\nu'}{\tau} \leq M\,.$$

Observe that for any point we have $|\tau\mathcal{K}' + \nu'| \leq \mathrm{diam}(\Omega)\mathcal{K}_0 + 1$, where $\mathcal{K}_0$ denotes the maximum in absolute value of the curvature of $\partial\Omega$. Thus, it suffices to get a uniform upper bound on $\frac{\nu'}{\tau}$ to conclude. Whenever τ –which is the Euclidean distance between two consecutive points– is bounded away from zero, the quantity we are interested in is then clearly bounded. The points for which τ is approaching zero are points closer and closer to the boundary. Let then $(s_n, r_n)_n \in (\mathrm{int}(\mathbb{A}))^{\mathbb{N}}$ be a sequence of points converging to a point $(s_\infty, \pm 1)$. Without loss of generality, assume that we converge to $(s_\infty, 1)$. Let $\mathcal{K}_\infty \leq 0$ be the curvature at the point on $\partial\Omega$ corresponding to s_∞. We distinguish between two cases: either $\mathcal{K}_\infty < 0$, or $\mathcal{K}_\infty = 0$.

In the first case, we let $R_\infty := |\mathcal{K}_\infty^{-1}| > 0$ be the radius of curvature at s_∞. By approximating our convex domain with the osculating circle at the point s_∞, we obtain

$$\tau_n \sim 2\nu_n' R_\infty\,,$$

denoting by τ_n the Euclidean distance between the points corresponding to s_n and s_n', where $(s_n', r_n') := f(s_n, r_n)$, and with $\nu_n' := \sqrt{1-(r_n')^2}$ (see e.g. [Dou82, Chapter 4, I.3.4.]). Thus

$$\lim_{n\to+\infty}\frac{\nu_n'}{\tau_n} = \frac{1}{2R_\infty} = -\frac{\mathcal{K}_\infty}{2} \leq \frac{\mathcal{K}_0}{2}\,,$$

which provides the required uniform bound.

In the second case, namely when $\mathcal{K}_\infty = 0$, the boundary $\partial\Omega$ is approximated up to order 2 by the tangent space at s_∞. Let $(\tilde{s}_n, \tilde{r}_n)$ and $(\tilde{s}_n', \tilde{r}_n')$ be the respective approximations of (s_n, r_n) and $(s_n', r_n') := f(s_n, r_n)$; then $\tilde{r}_n = \tilde{r}_n' = 1$ (and the corresponding $\tilde{\nu}_n, \tilde{\nu}_n'$ satisfy $\tilde{\nu}_n = \tilde{\nu}_n' = 0$). Besides, in our approximation, τ_n is approximated by $|\tilde{s}_n - \tilde{s}_n'|$. This yields

$$\lim_{n\to+\infty}\frac{\nu_n'}{\tau_n} = 0\,.$$

In either case, we go get the required uniform bound. ☐

6.2 Upper and lower rotation numbers and Charpentier's result

We follow the presentation contained in [LC88, Sections 4 and 5]. Recall from Definition 2.1 that

$$C = \{(s,r) \in \mathbb{A} : \ \phi^-(s) \le r \le \phi^+(s)\} \subset \mathbb{A},$$

where $\phi^-, \phi^+ \colon \mathbb{T} \to \mathbb{R}$ are continuous maps. For $\lambda \in (0,1)$, let f_λ be a dissipative (see Definition 2.1) positive twist map of C into its image and Λ_λ be its corresponding Birkhoff attractor (see Definition 2.5). Denote by C_λ^+ (resp. C_λ^-) the connected component of $C \setminus \Lambda_\lambda$ containing $\{(s, \phi^+(s)) \in \mathbb{A} : \ s \in \mathbb{T}\}$ (resp. $\{(s, \phi^-(s)) \in \mathbb{A} : \ s \in \mathbb{T}\}$). For any $(s,r) \in \mathbb{A}$ the upper (resp. lower) vertical line is

$$V^+(s,r) := \{(s,y) \in \mathbb{A} : \ y \ge r\}$$

(resp. $V^-(s,r) := \{(s,y) \in \mathbb{A} : \ y \le r\}$). Let us now define (see Fig. 8)

$$\Lambda_\lambda^+ := \{x \in \Lambda_\lambda : \ V^+(x) \setminus \{x\} \subset C_\lambda^+\} \qquad \text{and} \qquad \Lambda_\lambda^- := \{x \in \Lambda_\lambda : \ V^-(x) \setminus \{x\} \subset C_\lambda^-\}\,.$$

Therefore, we can define two functions $\mu_\lambda^\pm \colon \mathbb{T} \to [-1,1]$ whose graphs $\Gamma_{\mu_\lambda^\pm}$ satisfy $\Gamma_{\mu_\lambda^\pm} = \Lambda_\lambda^\pm$.

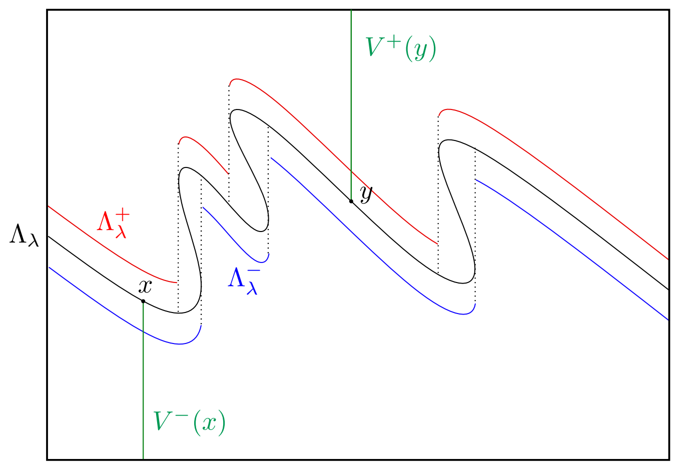

Figure 8: The lower and upper verticals.

In the sequel, we fix a covering $\pi \colon \mathbb{R} \times [-1,1] \to \mathbb{T} \times [-1,1]$ of $\mathbb{A}$, and let $\tilde{\Lambda}_\lambda := \pi^{-1}(\Lambda_\lambda)$, $\tilde{\Lambda}_\lambda^\pm := \pi^{-1}(\Lambda_\lambda^\pm)$; we also denote by $\tilde{\mu}_\lambda^\pm \colon \mathbb{R} \to [-1,1]$ the lifts of $\mu_\lambda^\pm \colon \mathbb{T} \to [-1,1]$. Moreover, we let $\pi_1 \colon \mathbb{T} \times [-1,1] \to \mathbb{T}$ and $\tilde{\pi}_1 \colon \mathbb{R} \times [-1,1] \to \mathbb{R}$ be the first coordinate projections. The next propositions are Corollary 4.8 and Corollary 4.7-Corollary 4.5 in [LC88] respectively.

Proposition 6.5. *The map $\tilde{\mu}_\lambda^+ \colon \mathbb{R} \to [-1,1]$ (resp. $\tilde{\mu}_\lambda^- \colon \mathbb{R} \to [-1,1]$) is upper (resp. lower) semi-continuous. Moreover,*

$$\tilde{\mu}_\lambda^\pm(\tilde{\theta}') - \tilde{\mu}_\lambda^\pm(\tilde{\theta}) \le (\tilde{\theta}' - \tilde{\theta}) \operatorname{cotan} \beta, \qquad \forall\, \tilde{\theta} < \tilde{\theta}',$$

where $\beta \in (0, \frac{\pi}{2})$ is the constant in Definition 6.1.

Proposition 6.6. *The following properties hold:*

(a) $f_\lambda^{-1}(\Lambda_\lambda^\pm) \subset \Lambda_\lambda^\pm$, and the order defined by the first coordinate projection is preserved by f_λ^{-1}.

(b) Let $\hat{U}_\lambda^\pm := \{x \in C : \ V^\pm(x) \subset C_\lambda^\pm\}$ be the set of points radially accessible from below/above. If $x \in f_\lambda(C) \cap \hat{U}_\lambda^\pm$, then $f_\lambda^{-1}(x) \in \hat{U}_\lambda^\pm$ and $f_\lambda(V^\pm(f_\lambda^{-1}(x))) \subset \hat{U}_\lambda^\pm$.

Let $F_\lambda \colon \mathbb{R} \times [-1,1] \to F_\lambda(\mathbb{R} \times [-1,1])$ be a continuous lift of f_λ. The next result, due to G.D. Birkhoff, is [LC88, Proposition 4.11].

Proposition 6.7. *The sequence* $\left(\frac{\tilde{\pi}_1\circ F_\lambda^n-\tilde{\pi}_1}{n}\right)_{n\in\mathbb{N}}$ *converges uniformly on* $\tilde{\Lambda}_\lambda^+$ *(resp.* $\tilde{\Lambda}_\lambda^-$*) to a constant* ρ_λ^+ *(resp.* ρ_λ^-*). The constants* ρ_λ^+ *and* ρ_λ^- *–called upper and lower rotation numbers– do depend on the chosen lift, but not their difference.*

From the previous result, we immediately conclude that if $\Lambda_\lambda^+\cap\Lambda_\lambda^-\neq\emptyset$ (equivalently, if there is at least a point where Λ_λ is a graph) then $\rho_\lambda^+=\rho_\lambda^-$. We finally recall that, when the upper and lower rotation numbers are different, then the corresponding Birkhoff attractor is topologically complicated, in the sense made precise by the following result of M. Charpentier (see [Cha34, Section 20]).

Theorem 6.8 ([Cha34])**.** *If* $\rho_\lambda^+-\rho_\lambda^->0$*, then* Λ_λ *is an indecomposable continuum, i.e., it cannot be written as union of two compact connected non-trivial sets.*

6.3 The dissipative billiard case

In this section –for a dissipative billiard map– we give a sufficient condition assuring that the corresponding Birkhoff attractor has different upper and lower rotation numbers (see Proposition 6.10). Clearly, this is not the case of the billiard tables studied respectively in Sections 4 and 5. Indeed, for an ellipse, the corresponding Birkhoff attractor –independently from the dissipative parameter– is not an indecomposable continuum and, in particular, it holds that $\rho_\lambda^+=\rho_\lambda^-=\frac{1}{2}\mod\mathbb{Z}$. This can be proved even more directly. Indeed, since the rotation number is invariant under the dynamics and, for the dissipative billiard map on an ellipse, every point of the Birkhoff attractor is in the omega-limit set of a two periodic point, we can deduce that every point of both $\tilde{\Lambda}_\lambda^+$ and $\tilde{\Lambda}_\lambda^-$ has rotation number equal to that of the 2-periodic point, i.e., equal to $\frac{1}{2}$. In Section 5, we study billiards whose Birkhoff attractor is a graph: in this case, we clearly have that $\rho_\lambda^+=\rho_\lambda^-$.

Let $\Omega\subset\mathbb{R}^2$ be a strongly convex domain (i.e. whose curvature never vanishes) with C^k, $k\geq 2$, boundary $\partial\Omega$. Then the associated (conservative) billiard map $f=f_1$ is a C^{k-1} positive twist map (with respect to some $\beta\in(0,\frac{\pi}{2})$) of $\mathbb{A}:=\mathbb{T}\times[-1,1]$ into itself. Consequently, for every $\lambda\in(0,1)$, the dissipative billiard map f_λ defined in Section 3.1 is a C^{k-1} positive twist map (with respect to some $\beta'\geq\beta\in(0,\frac{\pi}{2})$) of $\mathbb{A}:=\mathbb{T}\times[-1,1]$ into its image.

We recall that an essential curve in $\mathbb{A}$ is a topological embedding of $\mathbb{T}$ that is not homotopic to a point. The next proposition is an adaptation of [LC88]: mainly, the only difference concerns the type of maps considered, but the proof follows the main lines of [LC88, Section 8]. Some computations in the proof are simpler because of the kind of maps studied. More precisely, given a C^1 function $\lambda\colon\mathbb{A}\to(0,1)\subset\mathbb{R}$, we consider compositions of twist maps with some homothety $\mathcal{H}_\lambda$ of factor $\lambda(s,r)$ in the second variable, but in the inverse order with respect to [LC88]. This class of maps contains in particular the dissipative billiard maps considered in the present work. Let us recall the notion of instability region (see e.g. [Arn16, Definition 2.18]).

Definition 6.9. *Let* $C=\{(s,r)\in\mathbb{A}:\ \phi^-(s)\leq r\leq\phi^+(s)\}\subset\mathbb{A}$*, where* $\phi^-,\phi^+\colon\mathbb{T}\to\mathbb{R}$ *are continuous maps. Let* $f\colon C\to f(C)$ *be a twist map on* $\operatorname{int}(C)$*. Let* $\mathscr{V}(f)$ *be the union of all* f*-invariant essential curves in* C*. An instability region* $\mathscr{I}$ *is an open bounded connected component of* $C\setminus\mathscr{V}(f)$ *that contains in its interior an essential curve.*

Proposition 6.10. *Let* $C=\{(s,r)\in\mathbb{A}:\ \phi^-(s)\leq r\leq\phi^+(s)\}\subset\mathbb{A}$*, where* $\phi^-,\phi^+\colon\mathbb{T}\to\mathbb{R}$ *are continuous maps. Let* $f\colon C\to C$ *be an orientation-preserving homeomorphism, homotopic to the identity, such that*

1. f *preserves the standard 2-form* $\omega=dr\wedge ds$*;*
2. $f\colon\operatorname{int}(C)\to\operatorname{int}(C)$ *is a positive twist map on* $\operatorname{int}(C)$ *with respect to* $\beta\in(0,\frac{\pi}{2})$*;*
3. $\mathscr{I}:=\operatorname{int}(C)$ *is an instability region for* f *that contains the zero section* $\mathbb{T}\times\{0\}$*.*

Then, there exists $\bar{\epsilon}>0$ *such that for any* $\epsilon\leq\bar{\epsilon}$*, for any* C^1 *function* $\lambda\colon C\to(0,1)$ *such that* $\frac{\epsilon}{2}<d_{C^0}(\lambda,1)$[9] $<\epsilon$ *and* $\|D\lambda\|<\epsilon^2$*, the Birkhoff attractor*[10] *of* $f_\lambda:=\mathcal{H}_\lambda\circ f$ *has* $\rho_\lambda^+-\rho_\lambda^->0$*, where* $\mathcal{H}_\lambda\colon(s,r)\mapsto(s,\lambda(s,r)r)$*.*

[9] Here, the notation 1 stands for the constant function.

[10] Let us observe that by the assumptions on C, f, and by the definition of f_λ, the Birkhoff attractor of f_λ is contained in $\operatorname{int}(C)$.

Remark 6.11. Observe that, in Proposition 6.10, if we restrict to the class of constant functions λ, we are stating that there exists $\lambda_0 \in (0,1)$ such that, for any $\lambda \in [\lambda_0, 1)$, the Birkhoff attractor of the dissipative map $f_\lambda := \mathcal{H}_\lambda \circ f$ has $\rho_\lambda^+ - \rho_\lambda^- > 0$, where $\mathcal{H}_\lambda\colon (s,r) \mapsto (s, \lambda r)$.

As a corollary of Proposition 6.10, we obtain a sufficient condition to assure that a dissipative billiard map has different rotation numbers.

Corollary 6.12. *Let $\Omega \subset \mathbb{R}^2$ be a strongly convex domain with C^k, boundary, $k \geq 2$. Let $f = f_1$ be the associated (conservative) billiard map. If f admits an instability region $\mathscr{I}$ that contains the zero section $\mathbb{T} \times \{0\}$, then there exists $\lambda_0 \in (0,1)$ such that, for any $\lambda \in [\lambda_0, 1)$, the Birkhoff attractor of the corresponding dissipative billiard map f_λ has $\rho_\lambda^+ - \rho_\lambda^- > 0$, with $\frac{1}{2} \in (\rho_\lambda^-, \rho_\lambda^+) \mod \mathbb{Z}$.*

Remark 6.13. Both in Proposition 6.10 and in Corollary 6.12, the boundary of the instability region is made up of the graphs of two continuous functions $\phi^- < 0 < \phi^+$. These functions are actually Lipschitz by Birkhoff's Theorem, see [Bir22]. In Corollary 6.12, by the time-reversal symmetry of the conservative billiard map, it even holds $\phi^- = -\phi^+$.

Proof of Proposition 6.10. Since λ is, in particular, continuous on the compact set C, it takes values in $(0,1)$ and since $d_{C^0}(\lambda, 1) < \epsilon$, there exist $\lambda_{\min}, \lambda_{\max} \in (1-\epsilon, 1)$ such that, for any $(s,r) \in C$ it holds

$$1 - \epsilon < \lambda_{\min} \leq \lambda(s,r) \leq \lambda_{\max} < 1\,.$$

Since $\|D\lambda\| < \epsilon^2$, we also have that $\lambda_{\max} - \lambda_{\min} < \epsilon^2$. Since also $d_{C^0}(\lambda, 1) > \frac{\epsilon}{2}$, we have that for every $(s,r) \in C$ it holds $\lambda(s,r) \leq \lambda_{\max} < 1 - \frac{\epsilon}{2}(1-2\epsilon)$.

The map $f_\lambda\colon C \to f_\lambda(C) \subset \mathrm{int}(C)$, defined by $f_\lambda := \mathcal{H}_\lambda \circ f$, is a dissipative map, according to Definition 2.1. Indeed, for every $(s,r) \in \mathrm{int}(C)$, it holds

$$\det(Df_\lambda(s,r)) = \det(D\mathcal{H}_\lambda(s',r')) = r'\partial_2\lambda(s',r') + \lambda(s',r') < 1 - \frac{\epsilon}{2} + 2\epsilon^2\,,$$

where $f(s,r) = (s',r')$; there exists ϵ_0 small enough such that for every $\epsilon < \epsilon_0$ the value $\det(Df_\lambda(s,r))$ is uniformly smaller than 1. Let F be a lift of f. We denote by Λ_λ the Birkhoff attractor of f_λ and by $\rho_\lambda^\pm$ its lower an upper rotation numbers with respect to the lift $\mathcal{H}_\lambda \circ F$. Recall that f is a positive twist map with respect to $\beta \in (0, \frac{\pi}{2})$. Observe that, up to consider a smaller ϵ_0, for any $\epsilon \leq \epsilon_0$, for any function λ that is ϵ-C^1-close to 1, the map f_λ is still a positive twist map on $\mathrm{int}(C)$ with respect to $\frac{\beta}{2} \in (0, \frac{\pi}{4})$.
Consider now the annulus $\mathcal{A}$ bounded by $\Gamma_{\phi^-} := \{(s, \phi^-(s)) \in \mathbb{A} :\ s \in \mathbb{T}\}$ and its image $f_\lambda(\Gamma_{\phi^-})$. Since $f(\Gamma_{\phi^-}) = \Gamma_{\phi^-}$, we have

$$f_\lambda(\Gamma_{\phi^-}) = \mathcal{H}_\lambda \circ f(\Gamma_{\phi^-}) = \mathcal{H}_\lambda(\Gamma_{\phi^-}) = \Gamma_{\lambda\phi^-}\,,$$

and consequently $m(\mathcal{A}) = -\int_{\mathbb{T}}(1 - \lambda(s, \phi^-(s)))\phi^-(s)\,ds$; in particular, we have

$$-(1-\lambda_{\max})\int_{\mathbb{T}} \phi^-(s)\,ds \leq m(\mathcal{A}) \leq -(1-\lambda_{\min})\int_{\mathbb{T}} \phi^-(s)\,ds\,.$$

We denote by C_λ^- the connected component of $C \setminus \Lambda_\lambda$ containing $\Gamma_{\phi^-} = \{(s, \phi^-(s)) \in \mathbb{A} :\ s \in \mathbb{T}\}$. Observe that, for every $n \in \mathbb{N}^*$, it holds

$$m(f_\lambda^n(\mathcal{A})) = \int_{f_\lambda^{n-1}(\mathcal{A})} |r\partial_2\lambda(s,r) + \lambda(s,r)|\,dr \wedge ds \leq (\lambda_{\max} + \epsilon^2)m(f_\lambda^{n-1}(\mathcal{A}))\,.$$

Then, we have

$$m(C_\lambda^-) = \sum_{n=0}^{+\infty} m(f_\lambda^n(\mathcal{A})) \leq \sum_{n=0}^{+\infty}(\lambda_{\max} + \epsilon^2)^n\, m(\mathcal{A}) \leq \tag{6.3}$$

$$-(1-\lambda_{\min})\sum_{n=0}^{+\infty}(\lambda_{\max} + \epsilon^2)^n \int_{\mathbb{T}} \phi^-(s)\,ds = \frac{1-\lambda_{\min}}{1-\lambda_{\max}-\epsilon^2}\, m(C^-)$$

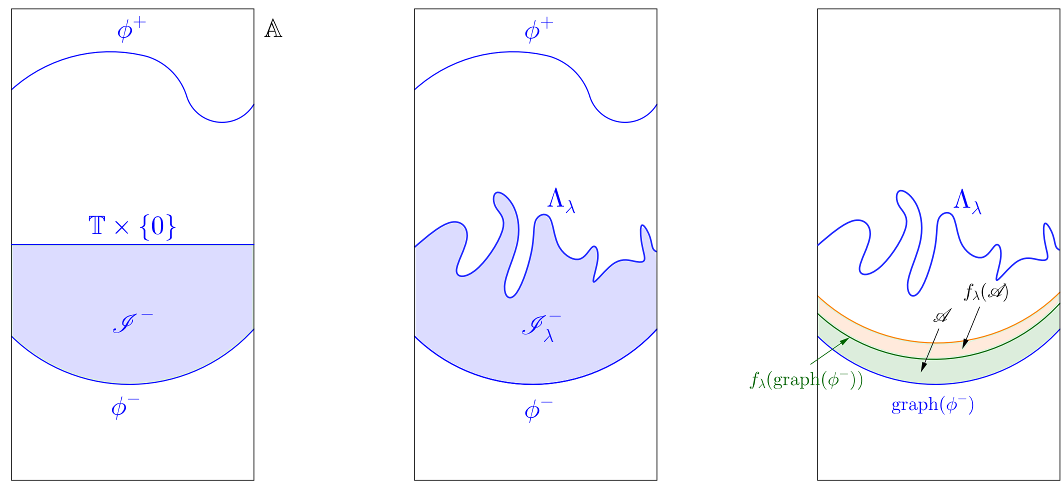

Figure 9: Here, $\mathscr{I}^- := \mathrm{int}(C^-)$ is the part of the instability region $\mathscr{I} = \mathrm{int}(C)$ that lies below the zero section $\mathbb{T}\times\{0\}$, while $\mathscr{I}^-_\lambda := \mathrm{int}(C^-_\lambda)$ is the connected component of $\mathscr{I}\setminus\Lambda_\lambda$ bounded by Γ_{ϕ^-}.

where $C^\pm := \{(s,r)\in C,\, \pm r\geq 0\}$. A consequence of (6.3) is that, up to choose a smaller ϵ_0, for every $\epsilon\leq\epsilon_0$

(6.4)
$$\forall\,\lambda \text{ such that } \frac{\epsilon}{2}<d_{C^0}(\lambda,1)<\epsilon,\, \|D\lambda\|<\epsilon^2,\, \exists \text{ at least one point } (s_\lambda,r_\lambda)\in\Lambda^-_\lambda \text{ with } r\leq\min_{s\in\mathbb{T}}\phi^+(s)/2\,.$$

Indeed, if for some function λ every point of Λ^-_λ is contained in the interior of $C^+ = \{(s,r)\in C : r\geq \min_{s\in\mathbb{T}}\phi^+(s)/2\}$, then $\int_{\mathbb{T}}\mu^-_\lambda(s)\,ds > \min_{s\in\mathbb{T}}\phi^+(s)/2>0$. We would then obtain $m(C^-_\lambda) = m(C^-) + \int_{\mathbb{T}}\mu^-_\lambda(s)\,ds > m(C^-)+\min_{s\in\mathbb{T}}\phi^+(s)/2$. Nevertheless, by (6.3), it holds $m(C^-_\lambda)\leq \frac{1-\lambda_{\min}}{1-\lambda_{\max}-\epsilon^2}m(C^-)$ and we get

$$\min_{s\in\mathbb{T}}\phi^+(s)/2 < m(C^-)\left(\frac{\lambda_{\max}-\lambda_{\min}+\epsilon^2}{1-\lambda_{\max}-\epsilon^2}\right)\leq m(C^-)\frac{2\epsilon}{\frac{1}{2}-2\epsilon}\,;$$

if ϵ is small enough, this provides the required contradiction.
Denote by $\mathscr{C}_\epsilon$ the set of C^1 functions $\lambda\colon C\to(0,1)$ such that $\frac{\epsilon}{2}<d_{C^0}(\lambda,1)<\epsilon$ and $\|D\lambda\|<\epsilon^2$. For every $\epsilon\in(0,\epsilon_0)$, let λ_ϵ be a function in $\mathscr{C}_\epsilon$.

Claim 6.14. *[LC88, Proposition 8.3] It holds*

(6.5)
$$\lim_{\epsilon\to 0}\inf_{s\in\mathbb{T}}\mu^\pm_{\lambda_\epsilon}(s)-\phi^\pm(s)=0\,.$$

Proof of the claim: Let us show the claim when $\pm=-$. By contradiction, assume that this does not hold. In particular, there exists $M>0$ and a sequence $\epsilon_n\to 0$ as $n\to+\infty$ such that, for every $n\in\mathbb{N}$, the function $\lambda_{\epsilon_n}\in\mathscr{C}_{\epsilon_n}$ and it holds

$$\Lambda_{\lambda_{\epsilon_n}}\cap H_M=\emptyset\,,$$

where $H_M := \{(s,r) : \phi^-(s)\leq r\leq\phi^-(s)+M\}\subset C$.
By hypothesis, f preserves the standard 2-form and it is a positive twist map on $\mathrm{int}(C)$. Thus, by [Bir32, Section 6] (see also [Her83, Proposition 5.9.2]), we have

(6.6)
$$\Gamma_{\phi^+}=\{(s,\phi^+(s))\in\mathbb{A}:\ s\in\mathbb{T}\}\subset\overline{\bigcup_{k\in\mathbb{Z}}f^k(H_M)}=\overline{\bigcup_{n\in\mathbb{N}}f^{-n}(H_M)}\,.$$

Denote by L the Lipschitz constant of ϕ^+ and let $E:=\min_{s\in\mathbb{T}}\phi^+(s)>0$. Moreover, recall that every $f_{\lambda_{\epsilon_n}}$ is a positive twist map with respect to the constant $\frac{\beta}{2}\in(0,\frac{\pi}{4})$, where β is the twist constant of f.

Choose $j \in \mathbb{N}$ such that

$$\frac{1}{j\frac{E}{2} - 2 - L} \leq \frac{\tan(\beta/2)}{2}\,. \tag{6.7}$$

For $i = 0, \ldots, j-1$, denote $s_i = i/j \mod 1$ and

$$B_i := \left\{ (s,r) \in C : |s - s_i| < \frac{1}{2j},\ |\phi^+(s) - r| < \frac{2}{j} \right\}\,.$$

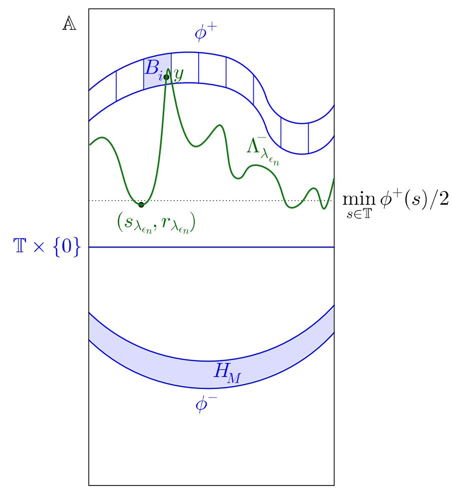

Figure 10: Controlling the shape of $\Lambda^-_{\lambda_{\epsilon_n}}$.

By using inclusion (6.6), we deduce the existence of an index $m_0 \in \mathbb{N}$ such that

$$\bigcup_{0 \leq m \leq m_0} f^{-m}(H_M) \cap B_i \neq \emptyset\,,$$

for every $i = 0, \ldots, j-1$. Therefore, since $f_{\lambda_{\epsilon_n}}$ converges uniformly to f when $\epsilon_n \to 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $i = 0, \ldots, j-1$,

$$\bigcup_{0 \leq m \leq m_0} f^{-m}_{\lambda_{\epsilon_n}}(H_M) \cap B_i \neq \emptyset\,. \tag{6.8}$$

Recall that $V^-(s,r) := \{(s,y) \in \mathbb{A} :\ y \leq r\}$ and that, for any λ, the set C^-_λ is the connected component of $C \setminus \Lambda_\lambda$ containing $\{(s, \phi^-(s)) \in \mathbb{A} :\ s \in \mathbb{T}\}$. Let us denote by $\hat{U}^-_\lambda$ the set of points which are radially accessible from below, i.e., the points $(s,r) \in C$ such that $V^-(s,r) \cap C \subset \overline{C^-_\lambda}$. Clearly, $H_M \subset \hat{U}^-_\lambda$ for any λ. For any $n \geq n_0$ and any $i \in \{0, \ldots, j-1\}$, by (6.8), there exist $m = m(n,i) \in \{0, \ldots, m_0\}$ and a point $y = y(n,i,m) \in f^{-m}_{\lambda_{\epsilon_n}}(H_M)$ such that

$$y \in f^{-m}_{\lambda_{\epsilon_n}}(H_M) \cap B_i\,.$$

This means that

$$x := f^m_{\lambda_{\epsilon_n}}(y) \in H_M \subset \hat{U}^-_{\lambda_{\epsilon_n}} \implies x \in f^m_{\lambda_{\epsilon_n}}(C) \cap \hat{U}^-_{\lambda_{\epsilon_n}}\,,$$

and –by using Proposition 6.6(b)– we get $y = f^{-m}_{\lambda_{\epsilon_n}}(x) \in \hat{U}^-_{\lambda_{\epsilon_n}}$. Since $y \in B_i$, it holds

$$U^-_{\lambda_{\epsilon_n}} \cap B_i \neq \emptyset \implies \Lambda^-_{\lambda_{\epsilon_n}} \cap B_i \neq \emptyset\,. \tag{6.9}$$

In order to conclude the proof, fix $n \geq n_0$. By (6.4), there exists a point $(s_{\lambda_{\epsilon_n}}, r_{\lambda_{\epsilon_n}}) \in \Lambda^-_{\lambda_{\epsilon_n}}$ such that $r_{\lambda_{\epsilon_n}} \leq \min_{s\in\mathbb{T}} \phi^+(s)/2$. Let us denote by $(\tilde{s}_{\lambda_{\epsilon_n}}, r_{\lambda_{\epsilon_n}}) \in \tilde{\Lambda}^-_{\lambda_{\epsilon_n}}$ a lift of $(s_{\lambda_{\epsilon_n}}, r_{\lambda_{\epsilon_n}})$. By (6.9), we can also find a point $(\tilde{s}'_{\lambda_{\epsilon_n}}, r'_{\lambda_{\epsilon_n}}) \in \tilde{\Lambda}^-_{\lambda_{\epsilon_n}}$ such that

$$r'_{\lambda_{\epsilon_n}} \geq \phi^+(s'_{\lambda_{\epsilon_n}}) - \frac{2}{j} \qquad \text{and} \qquad \tilde{s}_{\lambda_{\epsilon_n}} < \tilde{s}'_{\lambda_{\epsilon_n}} < \tilde{s}_{\lambda_{\epsilon_n}} + \frac{1}{j}\,.$$

Thus

$$r'_{\lambda_{\epsilon_n}} - r_{\lambda_{\epsilon_n}} \geq \frac{E}{2} - \frac{2}{j} - \frac{L}{j} \qquad \text{and} \qquad \frac{\tilde{s}'_{\lambda_{\epsilon_n}} - \tilde{s}_{\lambda_{\epsilon_n}}}{r'_{\lambda_{\epsilon_n}} - r_{\lambda_{\epsilon_n}}} \leq \frac{1}{j\frac{E}{2} - 2 - L} \leq \frac{\tan(\beta/2)}{2}.$$

Since $r_{\lambda_{\epsilon_n}} = \tilde{\mu}^-_{\lambda_{\epsilon_n}}(\tilde{s}_{\lambda_{\epsilon_n}})$ and $r'_{\lambda_{\epsilon_n}} = \tilde{\mu}^-_{\lambda_{\epsilon_n}}(\tilde{s}'_{\lambda_{\epsilon_n}})$, this contradicts Proposition 6.5 and completes the proof. □

From the twist condition on the conservative map f, it holds that, once we fix a lift F of the map, the rotation numbers of the graphs of ϕ^+ and ϕ^- are well-defined. Denoting them by ρ^+ and ρ^- respectively, we also have that $\rho^- < \rho^+$. It is then sufficient to show that Claim 6.14 implies

$$\lim_{\epsilon \to 0} \rho^{\pm}_{\lambda_\epsilon} = \rho^{\pm}. \tag{6.10}$$

We can follow *verbatim* the proof of [LC88, Corollary 4.8] to deduce (6.10) and then conclude the proof. □

Now, Corollary 6.12 is a straightforward consequence of Proposition 6.4, Proposition 6.10, Remark 6.11, Remark 6.13, and the following observation: with the notation of the above proof, $\frac{1}{2} \in (\rho^-, \rho^+) \mod \mathbb{Z}$, hence for any C^1 function λ whose C^0-distance from the constant function 1 is in $(\frac{\epsilon}{2}, \epsilon)$ and such that $\|D\lambda\| < \epsilon^2$, (6.10) implies that $\frac{1}{2} \in (\rho^-_\lambda, \rho^+_\lambda) \mod \mathbb{Z}$.

In the sequel we show that the Birkhoff attractor of a dissipative billiard map may have different upper and lower rotation numbers, provided that the dissipation is mild enough. Moreover, we emphasize the main dynamical consequences of this fact. We start by recalling Corollary G, stated in the Introduction.

Corollary G. Fix $k \geq 3$. There exists an open and dense subset $\mathscr{U}$ of C^k strongly convex domains such that the following holds. For any $\Omega \in \mathscr{U}$, there exists $\lambda_0(\Omega) \in (0,1)$ such that, for any $\lambda \in [\lambda_0(\Omega), 1)$, the Birkhoff attractor Λ_λ of the corresponding dissipative billiard map f_λ has $\rho^+_\lambda - \rho^-_\lambda > 0$, with $\frac{1}{2} \in (\rho^-_\lambda, \rho^+_\lambda) \mod \mathbb{Z}$. Moreover, there exists $\lambda_1(\Omega) \in [\lambda_0(\Omega), 1)$ such that for any $\lambda \in [\lambda_1(\Omega), 1)$, and for any 2-periodic point p of saddle type (e.g., when the 2-periodic orbit $\{p, f_\lambda(p)\}$ corresponds to a diameter of the table), there exists a horseshoe $K_\lambda(p) \subset \Lambda_\lambda$ in the homoclinic class $H_\lambda(p) := \overline{\mathcal{W}^s(\mathcal{O}_{f_\lambda}(p)) \pitchfork \mathcal{W}^u(\mathcal{O}_{f_\lambda}(p))}$ of p; more precisely, it holds

$$K_\lambda(p) \subset H_\lambda(p) := \overline{\mathcal{W}^s(\mathcal{O}_{f_\lambda}(p)) \pitchfork \mathcal{W}^u(\mathcal{O}_{f_\lambda}(p))} \subset \overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(p))} \subset \Lambda_\lambda.$$

The proof of Corollary G relies on the following proposition.

Proposition 6.15. *For each $k \geq 2$, there exists an open and dense subset $\mathscr{U}$ of C^k strongly convex domains such that for any $\Omega \in \mathscr{U}$, the corresponding billiard map has an instability region $\mathscr{I} \subset \mathbb{A} := \mathbb{T} \times [-1,1]$ that contains (a neighborhood of) the zero-section $\mathbb{T} \times \{0\}$.*

Proof. The argument follows the work [DCOKPdC07] of Dias Carneiro, Oliffson Kamphorst and Pinto-de-Carvalho. For a convex domain Ω, let $f = f_1 : \mathbb{A} \to \mathbb{A}$ be the associated (conservative) billiard map. We denote by $I : (s,r) \mapsto (s,-r)$ the time-reversal involution; recall that $f \circ I = I \circ f^{-1}$. Let $\Gamma \subset \mathbb{A}$ be a f-invariant essential curve. In particular, by Birkhoff's Theorem (see [Bir22]), there exists a Lipschitz function $\phi : \mathbb{T} \to \mathbb{R}$ such that $\Gamma = \{(s, \phi(s)) \in \mathbb{A} : \ s \in \mathbb{T}\}$. The symmetric graph $I(\Gamma) = \{(s, -\phi(s)) \in \mathbb{A} : \ s \in \mathbb{T}\}$ is also f-invariant, as $f(I(\Gamma)) = I(f^{-1}(\Gamma)) = I(\Gamma)$. Moreover, we observe that $\Gamma \cap I(\Gamma) \subset \mathbb{T} \times \{0\}$. In particular, any point in $\Gamma \cap I(\Gamma)$ is a 2-periodic point. Indeed, the intersection is also f-invariant: given any $x_0 = (s_0, 0) \in \Gamma \cap I(\Gamma)$, then also $f(x_0) = (s_1, 0) \in \Gamma \cap I(\Gamma)$. Thus, the bounces at x_0 and $f(x_0)$ are perpendicular, hence $\{x_0, f(x_0)\}$ is a 2-periodic orbit. We conclude that the rotation number of Γ (and so of $I(\Gamma)$) is equal to $\frac{1}{2}$.

Now, by [DCOKPdC07, Section 3], given a rational number $\frac{p}{q} \in \mathbb{Q}$, there exists an open and dense subset $\mathscr{U}_{\frac{p}{q}}$ of C^k strongly convex billiards which have no rotational invariant curve with rotation number $\frac{p}{q}$. Let us briefly recall the argument. If γ is such a curve, then the restriction $f|_\gamma$ of the billiard map to γ is a homeomorphism of the circle, and since the rotation number is equal to $\frac{p}{q}$, there are periodic points on γ of type (p,q). But these cannot be linearly elliptic, since the curve is a Lipschitz graph over $\mathbb{T}$, by Birkhoff's Theorem. Then, these periodic points are either degenerate, as in the circular billiard, or hyperbolic, in which case, γ will be a union of periodic points and saddle connections. By [DCOKPdC07, Theorem 1 and

Theorem 2] (see also [DCOKPdC03, XZ14]), both cases are not allowed for a strongly convex billiard Ω in an open and dense subset of domains.

It follows from the previous discussion that there exists an open and dense subset $\mathscr{U} = \mathscr{U}_{\frac{1}{2}}$ of C^k strongly convex domains such that any $\Omega \in \mathscr{U}$ has no invariant essential curve crossing the zero-section $\mathbb{T} \times \{0\}$, and thus, has an instability region containing a neighborhood of the zero-section $\mathbb{T} \times \{0\}$. □

Proof of Corollary G. Fix $k \geq 3$. As an immediate outcome of Corollary 6.12 and Proposition 6.15, there exists an open and dense subset $\mathscr{U}$ of C^k strongly convex domains such that for any $\Omega \in \mathscr{U}$, there exists $\lambda_0(\Omega) \in (0,1)$ such that, for any $\lambda \in [\lambda_0(\Omega), 1)$, the Birkhoff attractor Λ_λ of the corresponding dissipative billiard map f_λ has $\rho_\lambda^+ - \rho_\lambda^- > 0$, with $\frac{1}{2} \in (\rho_\lambda^-, \rho_\lambda^+) \mod \mathbb{Z}$. Let us denote by $\mathrm{II}_{\max}$ the set of 2-periodic points for f_1 with locally maximal perimeter. As noted at the beginning of Section 3.3, for any $p \in \mathrm{II}_{\max}$, $\{p, f_\lambda(p)\}$ is still a 2-periodic orbit for each dissipative map f_λ, $\lambda \in (0,1)$, and by Proposition 3.11(a), it is actually of saddle type, for any Ω in an open and dense subset of C^k domains. Moreover, for $\lambda \in [\lambda_0(\Omega), 1)$, $\frac{1}{2} \in (\rho_\lambda^-, \rho_\lambda^+) \mod \mathbb{Z}$, hence $\mathrm{II}_{\max} \subset \Lambda_\lambda$, by [LC88, Proposition 14.2]. By Corollary 3.9(3), there exists $\lambda_1(\Omega) \in [\lambda_0(\Omega), 1)$ such that, for any $p \in \mathrm{II}_{\max}$, and for any $\lambda \in [\lambda_1(\Omega), 1)$, each branch of $\mathcal{W}^s(p; f_\lambda^2) \setminus \{p\}$ and $\mathcal{W}^u(p; f_\lambda^2) \setminus \{p\}$ contains a transverse homoclinic point. Now, Corollary 2.10 implies that for any $\lambda \in [\lambda_1(\Omega), 1)$, the Birkhoff attractor Λ_λ of f_λ contains a horseshoe $K_\lambda(p)$, with

$$K_\lambda(p) \subset H_\lambda(p) := \overline{\mathcal{W}^s(\mathcal{O}_{f_\lambda}(p)) \pitchfork \mathcal{W}^u(\mathcal{O}_{f_\lambda}(p))} \subset \overline{\mathcal{W}^u(\mathcal{O}_{f_\lambda}(p))} \subset \Lambda_\lambda. \quad \square$$

We can guarantee that the upper and lower rotation numbers of the Birkhoff attractor are different also in the case of or every C^2-convex domain with a point with vanishing curvature, as explained in the sequel.

Corollary 6.16. *Let Ω be a convex domain with C^2 boundary such there is a point at which the curvature vanishes. Then for any $\epsilon > 0$, there exists $\lambda_0 = \lambda_0(\Omega, \epsilon) \in (0,1)$ such that for any $\lambda \in [\lambda_0, 1)$, the Birkhoff attractor of f_λ has $\rho_\lambda^+ - \rho_\lambda^- \in (1-\epsilon, 1)$.*

Corollary 6.16 is a consequence of Proposition 6.4, Proposition 6.10, Remark 6.11 and the next well-known result by Mather (see [Mat82] and also [Tab05, Corollary 5.29]-[GK95, Theorem 1.1]).

Theorem 6.17. *If the curvature of a C^2-convex billiard curve vanishes at some point, then the associated conservative billiard map has no invariant essential curves.*

Proof of Corollary 6.16. Let Ω be a convex domain with C^2 boundary. Then, the associated billiard map $f : \mathbb{A} \to \mathbb{A}$ is an orientation-preserving homeomorphism, homotopic to the identity, it preserves the standard 2-form $\omega = dr \wedge ds$, and the restriction of f to $\mathrm{int}(\mathbb{A})$ is a C^1 diffeomorphism, see [LC90] and also [Dou82] for all details. By Proposition 6.4, it is a positive twist map on $\mathrm{int}(\mathbb{A})$. Since there exists a point of zero curvature, by Theorem 6.17, the whole $\mathrm{int}(\mathbb{A})$ is an instability region. We conclude the proof by applying Proposition 6.10. □

Let us conclude this section by the following remark, which provides a different proof of Proposition 6.10. We are grateful to Patrice Le Calvez for suggesting this argument. It is possible to show that, as λ tends to 1 in the C^0 topology, the Birkhoff attractor comes closer and closer to both Γ_{ϕ^+} and Γ_{ϕ^-}. From this, again following the argument of Corollary 4.8 in [LC88], it can be deduced that $\lim_{\lambda \to 1} \rho_\lambda^\pm = \rho^\pm$, and thus, for λ close enough to 1, it holds $\rho_\lambda^+ - \rho_\lambda^- > 0$, since $\rho^+ - \rho^- > 0$.

A Proof of Lemma 3.5 and Lemma 3.7: bifurcation of eigenvalues at 2-periodic points for dissipative billiard maps

Let us recall that II denotes the set of 2-periodic points for the conservative billiard map $f = f_1$. For $p = (s, 0) \in \mathrm{II}$, we denote by $\tau = \ell(s, s') := \|\Upsilon(s) - \Upsilon(s')\|$ the Euclidean distance between the points $\Upsilon(s), \Upsilon(s')$, where $(s', 0) := f(p)$. We also denote by $\mathcal{K}_1, \mathcal{K}_2$ the respective curvatures at $\Upsilon(s), \Upsilon(s')$.

Let us fix a C^{k-1} function $\lambda : \mathbb{A} \to (0,1)$ such that $f_\lambda := \mathcal{H}_\lambda \circ f$ is a dissipative billiard map in the sense of Definition A, where $\mathcal{H}_\lambda : (s, r) \mapsto (s, \lambda(s, r) r)$. In particular, f_λ has the same set II of 2-periodic points as f, and for any 2-periodic orbit $\{p = (s, 0), f_1(p) = f_\lambda(p) = (s', 0)\}$, we have

$$D\mathcal{H}_\lambda(p) = \begin{bmatrix} 1 & 0 \\ 0 & \lambda_1 \end{bmatrix}, \quad D\mathcal{H}_\lambda(f_\lambda(p)) = \begin{bmatrix} 1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

with $\lambda_1 := \lambda(p)$, and $\lambda_2 := \lambda(f_\lambda(p))$. By formula (3.4), we have

$$Df_\lambda(p) = D\mathcal{H}_\lambda(f_\lambda(p))Df(p) = \begin{bmatrix} -(\tau\mathcal{K}_1+1) & \tau \\ \frac{\lambda_2}{\tau}(k_{1,2}-1) & -\lambda_2(\tau\mathcal{K}_2+1) \end{bmatrix},$$
$$Df_\lambda(f_\lambda(p)) = D\mathcal{H}_\lambda(p)Df(f_\lambda(p)) = \begin{bmatrix} -(\tau\mathcal{K}_2+1) & \tau \\ \frac{\lambda_1}{\tau}(k_{1,2}-1) & -\lambda_1(\tau\mathcal{K}_1+1) \end{bmatrix},$$

where we have set

$$k_{1,2} := (\tau\mathcal{K}_1+1)(\tau\mathcal{K}_2+1)\,.$$

Observe that $\det Df_\lambda(p) = D\mathcal{H}_\lambda(f_\lambda(p)) = \lambda_2$, as f is conservative; similarly, $\det Df_\lambda(f_\lambda(p)) = \lambda_1$. We thus obtain

$$Df_\lambda^2(p) = \begin{bmatrix} k_{1,2}(1+\lambda_2)-\lambda_2 & * \\ * & k_{1,2}\lambda_1(1+\lambda_2)-\lambda_1 \end{bmatrix}.$$

In particular,

$$\text{(A.1)} \qquad \det Df_\lambda^2(p) = \lambda_1\lambda_2, \quad \mathrm{tr} Df_\lambda^2(p) = (1+\lambda_1)(1+\lambda_2)k_{1,2} - (\lambda_1+\lambda_2)\,.$$

Proof of Lemma 3.5. We consider the case where the dissipation is constant, equal to some $\lambda \in (0,1)$. Let us denote by $\mu_1 = \mu_1(\lambda), \mu_2 = \mu_2(\lambda)$ the eigenvalues of $Df_\lambda^2(p)$, with $|\mu_1| \le |\mu_2|$. In particular, with the above notations, we have $\lambda_1 = \lambda_2 = \lambda$, and

$$\text{(A.2)} \qquad \mu_1\mu_2 = \det Df_\lambda^2(p) = \lambda^2, \quad \mathrm{tr} Df_\lambda^2(p) = (1+\lambda)^2 k_{1,2} - 2\lambda\,.$$

By (A.2), the characteristic polynomial $\chi_{p,\lambda}(x) = \det(Df_\lambda^2(p) - x\,\mathrm{id})$ is equal to

$$\chi_{p,\lambda}(x) = x^2 - \big((1+\lambda)^2 k_{1,2} - 2\lambda\big)x + \lambda^2\,,$$

with $k_{1,2} = (1+\tau\mathcal{K}_1)(1+\tau\mathcal{K}_2)$. Then, $\chi_{p,\lambda}$ has discriminant $\Delta = \big((1+\lambda)^2k_{1,2}-2\lambda\big)^2 - 4\lambda^2 = k_{1,2}(1+\lambda)^2\big((1+\lambda)^2k_{1,2}-4\lambda\big)$, which has the same sign as

$$\tilde\Delta := k_{1,2}\left((1+\lambda)^2k_{1,2}-4\lambda\right) = \lambda^2k_{1,2}^2 + 2\lambda k_{1,2}(k_{1,2}-2) + k_{1,2}^2\,.$$

The quantity $\tilde\Delta$ is a quadratic polynomial in λ, whose discriminant is equal to

$$\delta = 4k_{1,2}^2\big((k_{1,2}-2)^2 - k_{1,2}^2\big) = -16k_{1,2}^2(k_{1,2}-1)\,.$$

(*a*) If $k_{1,2} > 1$, then $\delta < 0$, hence $\tilde\Delta > 0$, $\Delta > 0$, and the eigenvalues μ_1, μ_2 of $Df_\lambda^2(p)$ are real, with $|\mu_1| \le |\mu_2|$. Their product $\mu_1\mu_2 = \det Df_\lambda^2(p) = \lambda^2$ is positive; their sum $\mu_1+\mu_2$ is also positive because

$$\text{(A.3)} \qquad \mu_1+\mu_2 = \mathrm{tr} Df_\lambda^2(p) = (1+\lambda)^2k_{1,2} - 2\lambda > (1+\lambda)^2 - 2\lambda = 1+\lambda^2 > 0\,,$$

where the first inequality comes from the hypothesis $k_{1,2} > 1$. Therefore, both eigenvalues are positive, and $0 < \mu_1 \le \mu_2$. In particular, by (A.2), $0 < \mu_1^2 \le \mu_1\mu_2 = \lambda^2$ hence $\mu_1 \in (0,1)$. Let us show that in fact, $0 < \mu_1 \le \lambda^2 < 1 < \mu_2$. Indeed, for $i = 1,2$, by (A.2)-(A.3), we have $\mu_i + \frac{\lambda^2}{\mu_i} > 1+\lambda^2$, hence $P(\mu_i) > 0$, where $P(X) = X^2 - (1+\lambda^2)X + \lambda^2$. Since the roots of P are $\{\lambda^2, 1\}$, and as $\mu_1 \in (0,1)$, we deduce that $\mu_1 \in (0,\lambda^2)$, and then $\mu_2 = \frac{\lambda^2}{\mu_1} > 1$.

(*b*) If $k_{1,2} = 1$, then $\chi_{p,\lambda}(x) = x^2 - (1+\lambda^2)x + \lambda^2 = (x-\lambda^2)(x-1)$, thus $\mu_1 = \lambda^2$, $\mu_2 = 1$.

If $k_{1,2} < 1$, $k_{1,2} \neq 0$, then $\delta > 0$. Let $\lambda_\pm := -1 + \frac{2}{k_{1,2}} \pm 2\sqrt{\frac{1-k_{1,2}}{k_{1,2}^2}}$.

(*c*) Assume now that $k_{1,2} \in (0,1)$. Note that $\lambda_+ = -1 + \frac{2}{k_{1,2}}(1+\sqrt{1-k_{1,2}}) \ge -1 + \frac{2}{k_{1,2}} > 1$, and $\lambda_- = -1 + \frac{2}{k_{1,2}}(1-\sqrt{1-k_{1,2}}) = \frac{\xi^2-2\xi+1}{1-\xi^2} = \frac{1-\xi}{1+\xi}$, with $\xi := \sqrt{1-k_{1,2}} \in (0,1)$, so that $\lambda_- \in (0,1)$. Then,

1. for $\lambda \in (\lambda_-,1)$, $\tilde\Delta < 0$, $\Delta < 0$, hence the eigenvalues of $Df_\lambda^2(p)$ are complex conjugate;

2. for $\lambda \in (0, \lambda_-]$, $\tilde{\Delta} \geq 0$, $\Delta \geq 0$, hence the eigenvalues of $Df_\lambda^2(p)$ are real. Moreover, $\mu_1 = \mu_2$ if and only if $\lambda = \lambda_-$.

In case (1), by (A.2), it holds $|\mu_1| = |\mu_2| = \sqrt{\mu_1\mu_2} = \sqrt{\lambda^2} = \lambda \in (0,1)$, hence the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a sink.

Let us consider case (2). By (A.2), μ_1, μ_2 have the same sign, and $0 < \mu_1^2 \leq \mu_1\mu_2 = \lambda^2$ hence $|\mu_1| \in (0,1)$. Moreover, as $k_{1,2} \in (0,1)$, and by (A.2), for $i = 1,2$, we have

$$\mu_i + \frac{\lambda^2}{\mu_i} = \mathrm{tr} Df_\lambda^2(p) = (1+\lambda)^2 k_{1,2} - 2\lambda \in (-2\lambda, 1+\lambda^2)\,. \tag{A.4}$$

Assume that μ_1, μ_2 are negative. By (A.4), for $i = 1,2$, it holds $\mu_i + \frac{\lambda^2}{\mu_i} > -2\lambda$, hence $0 > \mu_i^2 + 2\lambda\mu_i + \lambda^2 = (\mu_i + \lambda)^2$, a contradiction. Thus, μ_1, μ_2 are positive, and then, by (A.4), for $i = 1,2$, it holds $P(\mu_i) < 0$, where $P(X) = X^2 - (1+\lambda^2)X + \lambda^2$. Since the roots of P are $\{\lambda^2, 1\}$, we deduce that $\mu_1, \mu_2 \in (\lambda^2, 1)$, and then the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a sink as well.

(d) If $k_{1,2} = 0$, then $\chi_{p,\lambda}(x) = x^2 + 2\lambda x + \lambda^2 = (x+\lambda)^2$, hence $\mu_1 = \mu_2 = -\lambda$.

(e)–(f) Finally, assume that $k_{1,2} < 0$. In that case, it is easy to check that $\lambda_\pm = -1 + \frac{2}{k_{1,2}}\big(1 \mp \sqrt{1 - k_{1,2}}\big) < 0$, and then, for $\lambda \in (0,1)$, $\Delta > 0$, hence the eigenvalues of $Df_\lambda^2(p)$ are real. By (A.2), μ_1, μ_2 have the same sign, and $0 < \mu_1^2 \leq \mu_1\mu_2 = \lambda^2$, thus $|\mu_1| \in (0, \lambda)$. Moreover, as $k_{1,2} < 0$, and by (A.2), for $i = 1,2$, we have $\mu_i + \frac{\lambda^2}{\mu_i} = \mathrm{tr} Df_\lambda^2(p) < -2\lambda$, hence $\mu_2 < -\lambda < \mu_1 < 0$. We have

$$\mu_2 = \mu_2(\lambda) = \frac{1}{2}\Big((1+\lambda)^2 k_{1,2} - 2\lambda - \sqrt{(1+\lambda)^2 k_{1,2}\big((1+\lambda)^2 k_{1,2} - 4\lambda\big)}\Big)\,.$$

Observe that

$$\lim_{\lambda \to 0^+} \mu_2(\lambda) = k_{1,2}, \qquad \lim_{\lambda \to 1^-} \mu_2(\lambda) = -1 + 2k_{1,2} - 2\sqrt{k_{1,2}\big(k_{1,2} - 1\big)} < -1\,.$$

By direct computation, we see that the equation $\mu_2(\lambda) = -1$ admits a solution in $(0,1)$ if and only if $k_{1,2} \in (-1, 0)$; in that case, the only solution in $(0,1)$ is $\lambda = \bar{\lambda}$, with

$$\bar{\lambda} = \bar{\lambda}(p) := \frac{1 - \sqrt{-k_{1,2}}}{1 + \sqrt{-k_{1,2}}} \in (0,1)\,.$$

We conclude that

(e) if $k_{1,2} \in (-1, 0)$, then $\bar{\lambda} \in (0,1)$, and
   - (i) for any $\lambda \in (0, \bar{\lambda})$, $-1 < \mu_2 < -\lambda < \mu_1 < 0$, and the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a sink;
   - (ii) for $\lambda = \bar{\lambda}$, $\mu_1 = -\lambda^2$, $\mu_2 = -1$, and the 2-periodic orbit $\{p, f_\lambda(p)\}$ is parabolic;
   - (iii) for any $\lambda \in (\bar{\lambda}, 1)$, $\mu_2 < -1 < -\lambda^2 < \mu_1 < 0$, and the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a saddle;

(f) if $k_{1,2} \leq -1$, then for any $\lambda \in (0,1)$, $\mu_2 < -1 < -\lambda^2 < \mu_1 < 0$, and the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a saddle. □

Proof of Lemma 3.7. We now consider the case where $\lambda\colon \mathbb{A} \to (0,1)$ is a general C^{k-1} function such that $f_\lambda := \mathcal{H}_\lambda \circ f$ is a dissipative billiard map in the sense of Definition A, where $\mathcal{H}_\lambda\colon (s,r) \mapsto (s, \lambda(s,r)r)$.

Fix a 2-periodic orbit $\{p, f_\lambda(p)\}$ for f_λ. Let us denote by $\mu_1 = \mu_1(\lambda), \mu_2 = \mu_2(\lambda)$ the eigenvalues of $Df_\lambda^2(p)$, with $|\mu_1| \leq |\mu_2|$. By (A.1), the characteristic polynomial $\chi_{p,\lambda}(x) = \det(Df_\lambda^2(p) - x\,\mathrm{id})$ is equal to

$$\chi_{p,\lambda}(x) = x^2 - \big((1+\lambda_1)(1+\lambda_2)k_{1,2} - (\lambda_1 + \lambda_2)\big)x + \lambda_1\lambda_2\,,$$

with $k_{1,2} = (1 + \tau\mathcal{K}_1)(1 + \tau\mathcal{K}_2)$. Recall that we assume that $k_{1,2} \geq 0$.

On the one hand, if the eigenvalues μ_1, μ_2 are not real, then they are complex conjugate, and as $\mu_1\mu_2 = \lambda_1\lambda_2 \in (0,1)$, their modulus is strictly less than 1, and $\{p, f_\lambda(p)\}$ is a sink.

On the other hand, if $\mu_1, \mu_2 \in \mathbb{R}$, then as $|\mu_1| \leq |\mu_2|$, and $\mu_1\mu_2 = \lambda_1\lambda_2 \in (0,1)$, we deduce that $|\mu_1| < 1$. Thus, the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a saddle or a sink, unless $\mu_2 = 1$ or $\mu_2 = -1$. But

$$\chi_{p,\lambda}(1) = 1 - (1+\lambda_1)(1+\lambda_2)k_{1,2} + (\lambda_1+\lambda_2) + \lambda_1\lambda_2 = (1+\lambda_1)(1+\lambda_2)(1-k_{1,2}), \tag{A.5}$$

with $(1+\lambda_1)(1+\lambda_2) > 0$, hence $\chi_{p,\lambda}(1) = 0$ if and only if $k_{1,2} = 1$. In that case, we have

$$\chi_{p,\lambda}(x) = (x-1)(x-\lambda_1\lambda_2),$$

hence 1 is an eigenvalue no matter which λ we choose. In particular, $\{p, f_\lambda(p)\}$ is parabolic for the conservative billiard map f_1.

Moreover, as $k_{1,2} \geq 0$, we have

$$\chi_{p,\lambda}(-1) = 1 + (1+\lambda_1)(1+\lambda_2)k_{1,2} - (\lambda_1+\lambda_2) + \lambda_1\lambda_2 \geq (1-\lambda_1)(1-\lambda_2) > 0,$$

thus -1 is never an eigenvalue.

To conclude the proof, it remains to show that under the assumption that there is no parabolic 2-periodic orbit, then for any 2-periodic point p, for the dissipative billiard map f_λ, the point p is a saddle if and only if $k_{1,2} > 1$, and a sink if and only if $k_{1,2} < 1$.

Indeed, on the one hand, if $k_{1,2} > 1$, then (A.5) above shows that $\chi_{p,\lambda}(1) < 0$. Since $\lim_{x\to+\infty} \chi_{p,\lambda}(x) = +\infty$, we deduce that $\chi_{p,\lambda}$ vanishes somewhere on $(1, +\infty)$, hence $\mu_2 > 1$, and then the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a saddle.

On the other hand, if $k_{1,2} < 1$, then (A.5) above shows that $\chi_{p,\lambda}(1) > 0$. But $1 > x_{\min}$, where $x_{\min} \in \mathbb{R}$ is the point at which the quadratic polynomial $\chi_{p,\lambda}$ attains its minimum; indeed, as $k_{1,2} < 1$, we have

$$x_{\min} = \frac{1}{2}((1+\lambda_1)(1+\lambda_2)k_{1,2} - (\lambda_1+\lambda_2)) < \frac{1}{2}(1+\lambda_1\lambda_2) < 1.$$

Thus, $\chi_{p,\lambda}(1) > 0$ implies that $\chi_{p,\lambda}$ is positive on $[1, +\infty)$, hence no eigenvalue has modulus > 1 (recall that if it were the case, then $\mu_2 > 1$ would be a real zero of $\chi_{p,\lambda}$), and the 2-periodic orbit $\{p, f_\lambda(p)\}$ is a sink. □

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