

ERRATUM TO THE PAPER [BDSKL20]

PÉTER BÁLINT, JACOPO DE SIMOI, VADIM KALOSHIN, AND MARTIN LEGUIL

In the paper [BDSKL20], we initiated the study of the Marked Length Spectral rigidity for open dispersing billiards of the form $\mathcal{D} = \mathbb{R}^2 \setminus \cup_{i=1}^3 \mathcal{O}_i$, where each obstacle $\mathcal{O}_i$ is strongly convex and has C^3 boundary. Such billiards exhibit a non-trivial hyperbolic trapped set Λ , consisting of points whose orbits under the billiard map $\mathcal{F}$ remain in a compact part of $\mathbb{R}^2$. Moreover, assuming a non-eclipse condition is satisfied, it has been known since the work of Morita [Mor91] that the trapped set admits a natural symbolic coding. In other words, when restricted to Λ , the dynamics of $\mathcal{F}$ is conjugated to the subshift of finite type given by:

$$\Sigma := \{(\sigma_i)_{i \in \mathbb{Z}} \in \{1, 2, 3\}^{\mathbb{Z}} : \sigma_{i+1} \neq \sigma_i, \forall i \in \mathbb{Z}\}.$$

This allows us to define the *Marked Length Spectrum* of $\mathcal{D}$ as the map $\mathcal{L}$ which assigns to each finite word σ , with $\sigma^\infty \in \Sigma$, the length (or period) $\mathcal{L}(\sigma)$ of the periodic orbit for $\mathcal{F}$ that it encodes.

Given a periodic orbit represented by such a word σ , we defined a sequence $(h_n)_{n \in \mathbb{N}}$ of periodic orbits with palindromic symmetry, shadowing some orbit homoclinic to σ . We investigated the asymptotic expansions of the lengths $\mathcal{L}(h_n)$ as $n \rightarrow +\infty$ in order to extract local information about the orbit represented by σ . In particular, we showed that the Lyapunov exponents of periodic orbits are completely encoded by the length data $\mathcal{L}$ (see [BDSKL20, Theorem D, Corollaries B and E]), which was one of the key ingredients used in our subsequent work [DSKL23] on the Marked Length Spectral determination of such open dispersing billiards with analytic boundaries and a partial $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetry.

A mistake in one of the estimates leading to [BDSKL20, Theorem A] was later pointed out to us by Otto Vaughn Osterman. In this note, we provide the corrected statement of Theorem A, along with explanations about its proof (which remains essentially unchanged, except for a computational error). The main difference is that there is no asymmetry in $\mathcal{L}(h_n)$ depending on the parity of n , contrary to our initial claim. This affects [BDSKL20, Corollary C], where we had claimed that curvatures at two-periodic points could be recovered from the Marked Length Spectrum $\mathcal{L}$, relying on this asymmetry. Indeed, to recover such curvatures, we now need to impose a partial $\mathbb{Z}_2$ -symmetry, namely, that the two end-points of the periodic orbit have the same curvature; the question of whether these curvatures can be recovered when they are distinct remains open.

On the other hand, [BDSKL20, Theorem D and Corollary E], the statements on the recovery of periodic Lyapunov exponents — important for the subsequent work [DSKL23] — remain unaffected (and were reproven in [DSKL23, Appendix A]). Below, we use the same notation as in the original paper [BDSKL20].

1. CORRECTED STATEMENTS

The corrected version of [BDSKL20, Theorem A] is as follows:

Theorem A. *Let us consider a 2-periodic orbit for $\mathcal{F}$, encoded by a word σ ; we denote by $\lambda^{-1} < 1 < \lambda$ its multipliers. Let $(h_n)_n$ be the sequence of shadowing periodic orbits defined in [BDSKL20], where h_n is encoded by a finite word $\tau\sigma^n$. For $n \gg 1$, the following asymptotic expansion holds:*

$$(1.1) \quad \mathcal{L}(\tau\sigma^n) - (n+1)\mathcal{L}(\sigma) - \mathcal{L}^\infty = -C\lambda^n + O(\lambda^{\frac{3}{2}n}),$$

for some real number $\mathcal{L}^\infty = \mathcal{L}^\infty(\sigma, \tau) \in \mathbb{R}$, and some constant $C = C(\sigma, \tau) > 0$. Moreover, if $\mathcal{K}_1, \mathcal{K}_2 > 0$ denote the curvatures at the bouncing points of σ , then

$$(1.2) \quad \lambda(\mathcal{L}(\sigma)\mathcal{K}_1 + 2)(\mathcal{L}(\sigma)\mathcal{K}_2 + 2) = (1 + \lambda)^2.$$

In particular, [BDSKL20, Corollary B] regarding the recovery of the multiplier λ from the Marked Length Spectrum $\mathcal{L}$ still holds, by (1.1), and since $C > 0$. The constant $C(\sigma, \tau)$ is also determined by the Marked Length Spectrum. In contrast, [BDSKL20, Corollary C] must be replaced with the following, weaker, result:

Corollary C. *If $\mathcal{K}_1 = \mathcal{K}_2$, then their common value is determined by the Marked Length Spectrum.*

Indeed, it follows immediately from (1.2), since both $\mathcal{L}(\sigma)$ and λ are determined by $\mathcal{L}$.

2. ABOUT THE PROOF OF THEOREM A

Given an open dispersing billiard $\mathcal{D}$, and an arclength parametrization γ of its boundary, we denote by h the distance function between two boundary points, $h: (s, s') \mapsto \|\gamma(s) - \gamma(s')\|$. Recall that h is a generating function for the billiard map $\mathcal{F}: \mathcal{M} \rightarrow \mathcal{M}$. Recall also that $\mathcal{M} \ni (s, \varphi)$, where s denotes the arc-length parameter and φ is the angle between the outgoing trajectory and the normal to the boundary at the point parametrized by s .

Lemma 2.1. *Given two consecutive iterates $(s, \varphi) \in \mathcal{M}$ and $(s', \varphi') = \mathcal{F}(s, \varphi)$, we have*

$$(2.1) \quad \partial_1 h(s, s') = -\sin \varphi, \quad \partial_2 h(s, s') = \sin \varphi'.$$

Letting $\mathcal{K}, \mathcal{K}' > 0$ be the respective curvatures at the points $\gamma(s), \gamma(s')$, and $\tau = h(s, s')$, we have

$$(2.2) \quad \mathbf{H}_h(s, s') = \begin{pmatrix} \partial_{11}h(s, s') & \partial_{12}h(s, s') \\ \partial_{21}h(s, s') & \partial_{22}h(s, s') \end{pmatrix} = \begin{pmatrix} \mathcal{K} \cos \varphi + \frac{(\cos \varphi)^2}{\tau} & \frac{\cos \varphi \cos \varphi'}{\tau} \\ \frac{\cos \varphi \cos \varphi'}{\tau} & \mathcal{K}' \cos \varphi' + \frac{(\cos \varphi')^2}{\tau} \end{pmatrix}.$$

Proof. First-order derivatives in (2.1) are computed by a direct Taylor expansion. For instance,

$$\begin{aligned} h(s + \varepsilon, s') &= \sqrt{\|\gamma(s + \varepsilon) - \gamma(s')\|^2} = h(s, s') \sqrt{1 + 2\varepsilon \gamma'(s) \cdot \frac{\gamma(s) - \gamma(s')}{\|\gamma(s) - \gamma(s')\|^2} + O(\varepsilon^2)} \\ &= h(s, s') + \varepsilon \gamma'(s) \cdot \frac{\gamma(s) - \gamma(s')}{\|\gamma(s) - \gamma(s')\|} + O(\varepsilon^2) = h(s, s') - \varepsilon \sin \varphi + O(\varepsilon^2). \end{aligned}$$

Let us now turn to the second-order derivatives in (2.2); we only provide details for $\partial_{11}h$, as the others are analogous. From (2.1), we have $\partial_{11}h(s, s') = -\cos \varphi \partial_1 \varphi(s, s')$, where φ is considered as a function of (s, s') . Fix now $(s, \varphi) \in \mathcal{M}$, let $(s', \varphi') = \mathcal{F}(s, \varphi)$, let $\mathcal{K}$ be the curvature at $\gamma(s)$, and $\tau = h(s, s')$. Using the formula for the differential $D\mathcal{F}(s, \varphi)$ in [CM06, (2.26), p.35], we have

$$ds' = -\frac{1}{\cos \varphi'}(\tau \mathcal{K} + \cos \varphi)ds - \frac{\tau}{\cos \varphi'}d\varphi.$$

Now, consider φ as a function of (s, s') . Setting $ds' = 0$ above, we obtain $\partial_1 \varphi(s, s') = -(\mathcal{K} + \frac{\cos \varphi}{\tau})$, which, combined with the previous expression for $\partial_{11}h(s, s')$, yields the desired identity. $\square$

As in [BDSKL20], let $s(0), s(1)$ denote the s -coordinates of the collision points of the two-periodic orbit σ ; let $(s_\infty(k))_{k \in \mathbb{Z}}$ denote the s -coordinates of the collision points in the homoclinic orbit under consideration, and let $(s_n(k))_{k=0, \dots, 2n+1}$ denote the s -coordinates of the points in the shadowing periodic orbits $h_n = \tau \sigma^n$, $n \geq 0$. To prove (1.1), we observe that (see [BDSKL20, Section 5.3]) due to the palindromic symmetry of h_n ,

$$(2.3) \quad \mathcal{L}(h_n) = (n+1)\mathcal{L}(\sigma) + \mathcal{L}^\infty - \Sigma_n^1 - \Sigma_n^2,$$

where

$$\begin{aligned} \Sigma_n^1/2 &= \sum_{k=0}^n h(s_\infty(k), s_\infty(k+1)) - h(s_n(k), s_n(k+1)), \\ \Sigma_n^2/2 &= \sum_{k=n+1}^{+\infty} h(s_\infty(k), s_\infty(k+1)) - h(s(k), s(k+1)). \end{aligned}$$

We consider Taylor expansions of the above sums. Using (2.1), the first-order variation of $\Sigma_n^1/2$ is

$$\begin{aligned} & \sum_{k=0}^n \partial_1 h(s_n(k), s_n(k+1)) (s_\infty(k) - s_n(k)) + \\ & \sum_{k=0}^n \partial_2 h(s_n(k), s_n(k+1)) (s_\infty(k+1) - s_n(k+1)) = \\ & = - \sum_{k=0}^n \sin \varphi_n(k) (s_\infty(k) - s_n(k)) + \sum_{\ell=1}^{n+1} \sin \varphi_n(\ell) (s_\infty(\ell) - s_n(\ell)) = 0, \end{aligned}$$

after the change of variables $\ell = k + 1$ in the second sum, and because $\varphi_n(0) = \varphi_n(n+1) = 0$. Similarly, the first-order variation in $\Sigma_n^2/2$ vanishes because $\varphi(0) = \varphi(1) = 0$.

Let us now consider second-order terms of $\Sigma_n^1/2$. Recall that for $\ell \geq 0$ (see [BDSKL20, Proposition 4.1, Corollary 4.5]),

$$(2.4) \quad s_\infty(2\ell) - s(2\ell) = C_{1,s}^s \lambda^\ell \xi_\infty + O(\lambda^{\frac{3\ell}{2}}), \quad s_\infty(2\ell+1) - s(2\ell+1) = C_{2,s}^s \lambda^\ell \xi_\infty + O(\lambda^{\frac{3\ell}{2}}),$$

and for $\ell \in \{0, \dots, \lfloor \frac{n+1}{2} \rfloor\}$,

$$(2.5) \quad \begin{aligned} s_n(2\ell) - s_\infty(2\ell) &= C_{1,s}^u \lambda^{n+1-\ell} \xi_\infty + O(\lambda^{n-\frac{\ell}{2}}), \\ s_n(2\ell+1) - s_\infty(2\ell+1) &= C_{2,s}^u \lambda^{n+1-\ell} \xi_\infty + O(\lambda^{n-\frac{\ell}{2}}). \end{aligned}$$

For $k \in \{0, \dots, n\}$, and $i, j \in \{1, 2\}$, we abbreviate $a_{ij}^n(k) := \partial_{ij} h(s_n(k), s_n(k+1))$.

Let us first assume that $n = 2m + 1$ is odd. By (2.5), the second-order variation of $\Sigma_n^1/2$ is

$$\begin{aligned} & \frac{\xi_\infty^2}{2} \sum_{\ell=0}^m \left[\sum_{\epsilon=0,1} \left(a_{11}^n(2\ell+\epsilon) (C_{1+\epsilon,s}^u)^2 + 2a_{12}^n(2\ell+\epsilon) C_{1,s}^u C_{2,s}^u \lambda^{-\epsilon} + \right. \right. \\ & \quad \left. \left. + a_{22}^n(2\ell+\epsilon) (C_{2-\epsilon,s}^u)^2 \right) \right] \lambda^{2(n+1-\ell)} \end{aligned}$$

up to an error that is $O(\lambda^{\frac{3}{2}n})$. The main variation in the above sum comes from terms with $\ell \simeq m$, for which the constants $a_{ij}^n(2\ell+\epsilon)$ can be replaced, up to higher-order terms, by their counterparts $a_{ij}(\epsilon) := \partial_{ij} h(s(\epsilon), s(\epsilon+1))$. Indeed, by (2.4)-(2.5), and since the boundary is of class C^3 ,

$$|a_{ij}^n(2\ell+\epsilon) - a_{ij}(\epsilon)| = O(\max(\lambda^\ell, \lambda^{n+1-\ell})) \asymp \lambda^m \text{ when } \ell \simeq m.$$

Let then

$$\begin{aligned} C^1 &:= \sum_{\epsilon=0,1} a_{11}(\epsilon)(C_{1+\epsilon,s}^u)^2 + 2a_{12}(\epsilon)C_{1,s}^u C_{2,s}^u \lambda^{-\epsilon} + a_{22}(\epsilon)(C_{2-\epsilon,s}^u \lambda^{-\epsilon})^2 \\ &= \sum_{\epsilon=0,1} (V_\epsilon^u)^\top \mathbf{H}_h(s(\epsilon), s(\epsilon+1)) V_\epsilon^u, \end{aligned}$$

for the vectors $(V_\epsilon^u)^\top = (C_{1+\epsilon,s}^u, C_{2-\epsilon,s}^u \lambda^{-\epsilon})$, $\epsilon = 0, 1$. Note that $V_\epsilon^u \neq 0$ (see e.g. [BDSKL20, Corollary 5.1]). By (2.2), and Sylvester's criterion (recall that the curvature $\mathcal{K}$ is positive), the Hessian $\mathbf{H}_h(s(\epsilon), s(\epsilon+1))$ is positive definite. From the above expression, we deduce that $C^1 > 0$. Since the error term in the Taylor formula is also of order $O(\lambda^{\frac{3}{2}n})$, we thus obtain

$$\Sigma_n^1 = C^1 \lambda^2 \frac{\xi_\infty^2}{\lambda^{-1} - \lambda} \lambda^n + O(\lambda^{\frac{3}{2}n}).$$

Now, we consider second-order terms of Σ_n^2 . By (2.4), the second-order variation of Σ_n^2 is

$$C^2 \xi_\infty^2 \sum_{\ell=m+1}^{+\infty} \lambda^{2\ell} = C^2 \frac{\xi_\infty^2}{\lambda^{-1} - \lambda} \lambda^n$$

up to an error of order $O(\lambda^{\frac{3}{2}n})$, for the constant

$$C^2 := \sum_{\epsilon=0,1} a_{11}(\epsilon)(C_{1+\epsilon,s}^s)^2 + 2a_{12}(\epsilon)C_{1,s}^s C_{2,s}^s \lambda^\epsilon + a_{22}(\epsilon)(C_{2-\epsilon,s}^s \lambda^\epsilon)^2.$$

By the same reasoning as before, we have $C^2 > 0$. Similarly, the error term in the Taylor formula is of order $O(\lambda^{\frac{3}{2}n})$. By summing the expansions of Σ_n^1 , Σ_n^2 , we conclude the proof of (1.1) when $n = 2m + 1$ is odd, with $C := \frac{\xi_\infty^2}{\lambda^{-1} - \lambda} (C^1 \lambda^2 + C^2) > 0$.

Now, if $n = 2m$ is even, we obtain

$$\Sigma_n^1 + \Sigma_n^2 = \xi_\infty^2 \left(\sum_{\ell=0}^m \hat{C}^1 \lambda^{2(n+1-\ell)} + \sum_{\ell=m+1}^{+\infty} \hat{C}^2 \lambda^{2\ell} \right) + O(\lambda^{\frac{3}{2}n}) = \hat{C} \lambda^n + O(\lambda^{\frac{3}{2}n}),$$

for the constant $\hat{C} := \frac{\xi_\infty^2}{\lambda^{-1} - \lambda} \lambda (\hat{C}^1 + \hat{C}^2) > 0$, where

$$\hat{C}^1 := \sum_{\epsilon=0,1} a_{11}(\epsilon)(C_{1+\epsilon,s}^u \lambda^\epsilon)^2 + 2a_{12}(\epsilon)C_{1,s}^u C_{2,s}^u \lambda^\epsilon + a_{22}(\epsilon)(C_{2-\epsilon,s}^u)^2 > 0,$$

$$\hat{C}^2 := \sum_{\epsilon=0,1} a_{11}(\epsilon)(C_{1+\epsilon,s}^s \lambda^{-\epsilon})^2 + 2a_{12}(\epsilon)C_{1,s}^s C_{2,s}^s \lambda^{-\epsilon} + a_{22}(\epsilon)(C_{2-\epsilon,s}^s)^2 > 0.$$

Let $\ell := \mathcal{L}(\sigma)/2$, and let $\mathcal{K}_1, \mathcal{K}_2$ be the curvatures at the bouncing points of σ . By (2.2), we have

$$a_{11}(0) = a_{22}(1) = \mathcal{K}_1 + \frac{1}{\ell}, \quad a_{11}(1) = a_{22}(0) = \mathcal{K}_2 + \frac{1}{\ell}, \quad a_{12}(0) = a_{12}(1) = \frac{1}{\ell}.$$

For $i = 1, 2$, define $\alpha_i = \ell a_{ii}(0) = \ell \mathcal{K}_i + 1$. Equation (1.2) is shown in [BDSKL20]; indeed, from a direct computation that relates the coefficients of the differential $D\mathcal{F}^2(s(0), 0)$ (using [CM06, (2.26), p.35]) to its eigenvalues, we have $(1 + \lambda)^2 = 4\lambda\alpha_1\alpha_2$. From [BDSKL20, Corollaries 5.1, 5.2], we also have

$$C_{1,s}^u = C_{1,s}^s, \quad C_{2,s}^s = -\frac{1+\lambda}{2\alpha_2} C_{1,s}^s, \quad C_{2,s}^u = -\frac{1+\lambda}{2\alpha_2\lambda} C_{1,s}^s.$$

We can then verify that

$$\hat{C} = C = 2\lambda \frac{1-\lambda}{1+\lambda} \left(\mathcal{K}_1 + \frac{1}{\ell} \right) (\xi_\infty C_{1,s}^s)^2.$$

In other words, (1.1) holds regardless of the parity of n , which concludes the proof of Theorem A.

REFERENCES

- [BDSKL20] Péter Bálint, Jacopo De Simoi, Vadim Kaloshin, and Martin Leguil. Marked length spectrum, homoclinic orbits and the geometry of open dispersing billiards. *Communications in Mathematical Physics*, 374(3):1531–1575, 2020.
- [CM06] Nikolai Chernov and Roberto Markarian. *Chaotic billiards*. Number 127. American Mathematical Soc., 2006.
- [DSKL23] Jacopo De Simoi, Vadim Kaloshin, and Martin Leguil. Marked length spectral determination of analytic chaotic billiards with axial symmetries. *Inventiones mathematicae*, 233(2):829–901, 2023.
- [Mor91] Takehiko Morita. The symbolic representation of billiards without boundary condition. *Transactions of the American Mathematical Society*, 325(2):819–828, 1991.

PÉTER BÁLINT, DEPARTMENT OF STOCHASTICS, INSTITUTE OF MATHEMATICS, BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS, MŰEGYETEM RKP. 3, H-1111 BUDAPEST, HUNGARY

Email address: `balint.peter@ttk.bme.hu`

JACOPO DE SIMOI, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, 40 ST GEORGE ST. TORONTO, ON, CANADA M5S 2E4

Email address: `jacopods@math.utoronto.ca`

URL: <http://www.math.utoronto.ca/jacopods>

INSTITUTE OF SCIENCE AND TECHNOLOGY AUSTRIA, KLOSTERNEUBURG, AUSTRIA

Email address: `vadim.kaloshin@gmail.com`

ÉCOLE POLYTECHNIQUE, CMLS, ROUTE DE SACLAY, 91128 PALAISEAU CEDEX, FRANCE

Email address: `martin.leguil@polytechnique.edu`