SMOOTH CONJUGACY CLASSES OF 3D AXIOM A FLOWS AND SPECTRAL RIGIDITY OF HYPERBOLIC BILLIARDS

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Abstract. We show a rigidity result for 3-dimensional contact *Axiom A* flows: given two 3D contact Axiom A flows Φ_1, Φ_2 whose restrictions $\Phi_1|_{\Lambda_1}, \Phi_2|_{\Lambda_2}$ to basic sets Λ_1, Λ_2 are orbit equivalent, we prove that if periodic orbits in correspondence have the same length, then the conjugacy is as regular as the flows and respects the contact structure, extending a previous result due to Feldman-Ornstein [19]. Some of the ideas are reminiscent of the work of Otal [43]. We show how this can be applied to the study of spectral rigidity of dispersing billiards.

Contents

1. Introduction, statement of the results

The concept of *rigidity* arises in several ways in dynamics; one of them is the problem of knowing when two smooth systems which are topologically conjugated are actually *smoothly* conjugated. It appears for instance in the framework of diffeomorphisms of the circle. In [1] Arnold proved the first $\mathcal{C}^\omega$-linearization result.

More precisely, he showed that an analytic diffeomorphism with Diophantine rotation number α and sufficiently close to the rotation R_α is analytically conjugated to R_α. A global result in the $\mathcal{C}^\infty$ category is due to Herman, in [26], where he also proved the optimality of the Diophantine condition in the smooth case; see also [55], [31] for related works.

For low dimensional *Anosov* systems, the question of rigidity has been investigated in many works, see for instance the series of papers by de la Llave, Marco and Moriyón [35, 11, 36, 14], [13], and [12]. While renormalization is one of the main tools behind the study of rigidity for circle diffeomorphisms, the approach for *hyperbolic* systems is quite different. Indeed, for such systems, *periodic orbits* are abundant, and each of them carries with itself an obstruction to smooth conjugacy, namely the associated eigenvalues of the differential. In the aforementioned works of de la Llave-Marco-Moriyón, it is shown that those obstructions are actually complete invariants for smooth conjugacy classes. The Anosov assumption can be relaxed, namely, we may consider systems where hyperbolicity is only observed on a subset of the phase space. In particular, when the *non-wandering set* is hyperbolic, this leads to the notion of *Axiom A* systems. In [49], Pinto-Rand showed that Lipschitz conjugacy classes of hyperbolic basic sets on surfaces, which possess an invariant measure absolutely continuous with respect to Hausdorff measure, can be characterised in many ways, in particular, in terms of eigenvalues at periodic points. Let us also mention the works [48] and [3], where other rigidity results for hyperbolic sets have been obtained. In the context of expanding maps in any dimension, Gogolev and Rodriguez-Hertz [21] have shown that, open and densely, smooth conjugacy classes are determined by the value of the Jacobian of the return maps at periodic points.

Let us now say a few words on rigidity questions in *geometric* frameworks. A natural setting is that of hyperbolic *geodesic flows*. In this case, the general hope is that periodic data, in particular, the *length spectrum*, may be sufficient to characterize not only smooth conjugacy classes, but also to recover some *geometric* information. The question of *spectral rigidity* asks whether the (marked) length spectrum is sufficient to determine the metric up to isometry. There exist various instances of this problem, both local and non-local. Guillemin-Kazhdan [25] have shown that compact negatively curved surfaces are spectrally rigid in the *deformative* sense: a family $(g_s)_{s\in(0,1)}$ of isospectral negatively curved metrics is *isometric*, that is, for each $s \in (0,1)$, there exists a diffeomorphism ϕ_s such that $g_s = \phi_s^* g_0$. Later, Paternain-Salo-Uhlmann [44] proved that any Anosov surface is spectrally rigid in the deformative sense. Let us recall that for hyperbolic surfaces, periodic trajectories can be naturally marked by free homotopy classes. The question of spectral rigidity for hyperbolic surfaces was addressed by Otal [43] and independently by Croke [9], who obtained the following *global* result: two negatively curved metrics g_0 and g_1 on a closed surface with the same *marked length spectrum* are isometric (see also [10] for the multidimensional case). Recently, Guillarmou-Lefeuvre [24] proved that in all dimensions, the marked length spectrum of a Riemannian manifold with Anosov geodesic flow and non-positive curvature locally determines the metric. See also the recent work [22] where a sharpened version of Otal and Croke's result was obtained. Other works have also investigated the case where the hyperbolic set is not the whole manifold. For instance, in [23], Guillarmou considers a smooth one-parameter family $(g_s)_{s\in(0,1)}$ of metrics on a smooth connected compact manifold with strictly convex boundary. When the metrics have no conjugate points, and the trapped set is a hyperbolic set for the geodesic flow, he proved that if all the metrics in the family are lens equivalent, then they are isometric. Following this work, Lefeuvre [34] studied the X-ray transform on a smooth compact connected Riemannian manifold with hyperbolic trapped set. Other results in this direction have been recently obtained also by Chen, Erchenko and Gogolev in [5].

Another setting where rigidity questions for the length spectrum have been investigated is the case of planar billiards. Several results have been obtained in the convex case; De Simoi-Kaloshin-Wei [16] have proved dynamical spectral rigidity for $\mathbb{Z}_2$-symmetric strictly convex domains close to a circle. Let us also mention that recently, for smoothly conjugate billiard maps of Birkhoff billiards, Kaloshin-Koudjinan [29] study rigidity in the form of Marvizi-Melrose invariants. Yet, more than convex billiards, the framework of *dispersing billiards* is the most natural analogue of hyperbolic geodesic flows; indeed, although convex billiards may exhibit some hyperbolicity, for dispersing billiards, hyperbolicity is present on the whole phase space. The case of *Sinai billiards* is very interesting, due to the abundance of periodic orbits; yet, the complicated structure of the set of periodic orbits as well as the presence of singularities make them hard to deal with. Several works have been dedicated to the study of *open dispersing billiards* (see [38, 39, 40, 51, 42, 15], and also [46]). Recall that their dynamics is of *Axiom A* type, and that their non-wandering set can be described symbolically (see [38] for instance), which allows to define a marked length spectrum. In [15], the question of marked length spectral determination was solved for such billiards with two symmetries, when the boundary is $\mathcal{C}^\omega$, and under some non-degeneracy condition. Observe that in the $\mathcal{C}^k$ category, $k \in \mathbb{N}_{\geq 3} \cup \{+\infty\}$, the marked length spectrum is insufficient to fully determine the geometry of such tables; indeed, periodic orbits are not dense in the whole phase space, so it is possible to deform the geometry of the arcs of the table which are not "seen" by the trapped set, i.e., which come from "gaps" of the projection on the table of the Cantor set on which we have information through periodic orbits.

In the present work, we generalize the result of Feldman-Ornstein [19] from contact Anosov flows on 3-manifolds to contact Axiom A flows on 3-manifolds. More precisely, equality of the length data allows us to upgrade an orbit equivalence to a flow conjugacy as regular as the flows, see Theorem A. We apply this result to the study of spectral rigidity of open dispersing billiards: for $k \geq 3$, we show that two $\mathcal{C}^k$ open dispersing billiards[1] whose billiard maps are topologically conjugated on some horseshoe and have the same length data are actually *smoothly* conjugated, in a *canonical* way. Under the additional assumption that the differential of the canonical conjugacy between the horseshoes *preserves vertical fibers* – see Definition 1.13 – (or if the tables have the same *marked stable action spectrum*, see Remark 1.12) we show that the billiards have the same "geometry" at the corresponding points on the table, see Theorem D, Corollary F, and Corollary E; in particular, for non-eclipsing billiards with analytic boundary, this implies that the two tables are *isometric*. Our assumption about the infinitesimal preservation of fibers is the analogue in the Axiom A setting of what Otal [43] shows in the case of two geodesic flows on compact negatively curved surfaces with the same marked length spectrum.

[1] Actually, the same result also holds for more general billiards, see Theorem D.

Question 1. *Given two Axiom A billiard flows/geodesic flows on negatively curved surfaces which are smoothly conjugated on certain basic sets (in a canonical way), is it possible to show that the conjugacy infinitesimally preserves vertical fibers?*

1.1. **Preliminaries.** Let $\Phi = (\Phi^t)_{t\in\mathbb{R}}$ be a continuous flow defined on a manifold M. For each point $x \in M$, we denote by $\mathcal{O}_\Phi(x) := \{\Phi^t(x)\}_{t\in\mathbb{R}}$ the Φ-orbit of x. We denote by $\mathrm{Fix}(\Phi) := \{x \in M : \Phi^t(x) = x \text{ for all } t \in \mathbb{R}\}$ the set of *fixed points* of Φ, and we denote by $\mathrm{Per}(\Phi) := \{y \in M : \Phi^T(y) = y \text{ for some } T > 0\}$ the set of *periodic points* of Φ; for any $x \in \mathrm{Per}(\Phi)$, we let $T_\Phi(x) = T_\Phi(\mathcal{O}_\Phi(x)) > 0$ be the prime period of x. Recall that the *non-wandering set* $\Omega(\Phi) \subset M$ is the set of points $x \in M$ such that for any open set $U \ni x$, any $T_0 > 0$, there exists $T > T_0$ such that $\Phi^T(U) \cap U \neq \emptyset$. When Φ is a differentiable flow on some smooth manifold M, we denote by $X_\Phi(\cdot) := \frac{d}{dt}|_{t=0}\Phi(\cdot, t)$ its *flow vector field.*

In the following, given an integer $n \geq 1$, and $\beta \in (0, 1)$, we say that a function f is of class $\mathcal{C}^{n,\beta}$ if f is $\mathcal{C}^n$, and its n^{th} derivative is β-Hölder continuous.

Definition 1.1 (Orbit equivalence)**.** For $i = 1, 2$, let $\Phi_i = (\Phi_i^t)_{t\in\mathbb{R}}$ be a flow defined on a manifold M_i, and let $\Lambda_i \subset M_i$ be a Φ_i-invariant subset. We say that the flows Φ_1, Φ_2 are *orbit equivalent* on Λ_1, Λ_2 if there exists a homeomorphism $\Psi\colon \Lambda_1 \to \Lambda_2$ such that for some continuous function $\theta\colon \Lambda_1 \times \mathbb{R} \to \mathbb{R}$, we have for each $x \in \Lambda_1$:

- $\theta(x, 0) = 0$, and $\theta(x, \cdot)$ is an increasing $\mathcal{C}^{1,\beta}$ homeomorphism of $\mathbb{R}$, for some $\beta \in (0, 1)$;
- $\Psi \circ \Phi_1^t(x) = \Phi_2^{\theta(x,t)} \circ \Psi(x)$, for all $t \in \mathbb{R}$.

In other words, Ψ sends Φ_1-orbits to Φ_2-orbits:

$$\Psi(\mathcal{O}_{\Phi_1}(x)) = \mathcal{O}_{\Phi_2}(\Psi(x)), \quad \text{for all } x \in \Lambda_1.$$

Recall that Ψ is automatically $\mathcal{C}^\delta$ for some $\delta \in (0, 1)$, if Λ_1, Λ_2 are compact hyperbolic sets (see Katok-Hasselblatt [30, Theorem 19.1.2]).

Moreover, we say that Ψ is *iso-length-spectral* if

$$T_{\Phi_1}(x) = T_{\Phi_2}(\Psi(x)), \quad \forall\, x \in \mathrm{Per}(\Phi_1) \cap \Lambda_1,$$

i.e., the flows Φ_1, Φ_2 have the same periodic length data.

If M_1, M_2 are smooth, and Φ_1, Φ_2 are differentiable flows, we abbreviate as $X_i := X_{\Phi_i}$ the flow vector field, for $i = 1, 2$, and we say that Ψ is *differentiable along* Φ_1*-orbits* (in Λ_1) if the Lie derivative

$$\Lambda_1 \ni x \mapsto L_{X_1}\Psi(x) := \lim_{t\to 0} \frac{1}{t}\big(\Psi \circ \Phi_1^t(x) - \Psi(x)\big) \in \mathbb{R}X_2 \circ \Psi(x)$$

is a well-defined continuous function.

Definition 1.2 (Adapted contact form)**.** Given a smooth (connected) 3-manifold M, recall that a *contact form* is a smooth differential 1-form that satisfies the *non-integrability condition* $\alpha \wedge d\alpha > 0$.

Let $k \geq 2$, and let $\Phi = (\Phi^t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^k$ Axiom A flow defined on a smooth 3-manifold M. Given a basic set $\Lambda \subset M$ for Φ, we say that a contact form α is *adapted to* Λ if it satisfies the following Reeb conditions:

(a) $\imath_X\alpha|_\Lambda \equiv 1$;
(b) $X|_{\mathcal{W}_\Phi^{cs}(\Lambda)} \in \ker d\alpha|_{\mathcal{W}_\Phi^{cs}(\Lambda)}$ and $X|_{\mathcal{W}_\Phi^{cu}(\Lambda)} \in \ker d\alpha|_{\mathcal{W}_\Phi^{cu}(\Lambda)}$.

In the following, we fix a $\mathcal{C}^\infty$ smooth Riemannian manifold M, and we consider a $\mathcal{C}^2$ flow $\Phi = (\Phi^t)_{t\in\mathbb{R}}$ on M.

Definition 1.3 (Hyperbolic set)**.** A Φ-invariant compact subset $\Lambda \subset M \setminus \mathrm{Fix}(\Phi)$ is called a *(uniformly) hyperbolic* set (for Φ) if there exists a $D\Phi$-invariant splitting

$$T_xM = E^s(x) \oplus \mathbb{R}X(x) \oplus E^u(x), \qquad \forall\, x \in \Lambda,$$

where the *(strong) stable bundle* E^s_Φ, resp. the *(strong) unstable bundle* E^u_Φ is uniformly contracted, resp. expanded, i.e., there exist $C > 0$, $\lambda \in (0,1)$ such that

$$\|D\Phi^t(x) \cdot v\| \le C\lambda^t \|v\|, \qquad \forall\, x \in \Lambda,\ \forall\, v \in E^s_\Phi(x),\ \forall\, t \ge 0,$$
$$\|D\Phi^{-t}(x) \cdot v\| \le C\lambda^t \|v\|, \qquad \forall\, x \in \Lambda,\ \forall\, v \in E^u_\Phi(x),\ \forall\, t \ge 0.$$

We also denote by E^{cs}_Φ, resp. E^{cu}_Φ, the *weak stable bundle* $E^{cs}_\Phi := E^s_\Phi \oplus \mathbb{R}X$, resp. the *weak unstable bundle* $E^{cu}_\Phi := \mathbb{R}X \oplus E^u_\Phi$.

Let us recall the definition of an *Axiom A* flow:

Definition 1.4 (Axiom A flow)**.** A flow $\Phi\colon M \times \mathbb{R} \to M$ is called *Axiom A* if the non-wandering set $\Omega(\Phi) \subset M$ can be written as a disjoint union $\Omega(\Phi) = \Lambda \cup F$, where Λ is a closed hyperbolic set such that periodic orbits are dense in Λ, and $F \subset \mathrm{Fix}(\Phi)$ is a finite union of hyperbolic fixed points.

Definition 1.5 (Lamination)**.** Let $n \ge 1$, $\beta \in (0,1)$. A $\mathcal{C}^{n,\beta}$-lamination of a set $\Lambda \subset M$ is a disjoint collection of $\mathcal{C}^{n,\beta}$ submanifolds of a given same dimension, which vary continuously in the $\mathcal{C}^{n,\beta}$-topology, and whose union contains the set Λ.

Let $\Phi\colon M \times \mathbb{R} \to M$ be an Axiom A flow with a decomposition $\Omega(\Phi) = \Lambda \cup F$ as in Definition 1.4. The stable bundle E^s_Φ, resp. the unstable bundle E^u_Φ, over Λ integrates to a continuous lamination $\mathcal{W}^s_\Phi$, resp. $\mathcal{W}^u_\Phi$, called the *(strong) stable lamination*, resp. the *(strong) unstable lamination*. Similarly, E^{cs}_Φ, resp. E^{cu}_Φ integrates to a continuous lamination $\mathcal{W}^{cs}_\Phi$, resp. $\mathcal{W}^{cu}_\Phi$, called the *weak stable lamination*, resp. the *weak unstable lamination*. For each point $x \in \Lambda$, a local orbit segment in $\mathcal{O}_\Phi(x)$ containing x will also be denoted as $\mathcal{W}^c_{\Phi,\mathrm{loc}}(x) = \mathcal{W}^{cs}_{\Phi,\mathrm{loc}}(x) \cap \mathcal{W}^{cu}_{\Phi,\mathrm{loc}}(x)$. Each of these laminations is invariant under the dynamics, i.e., $\Phi^t(\mathcal{W}^*_\Phi(x)) = \mathcal{W}^*_\Phi(\Phi^t(x))$, for all $x \in M$ and $* = s, u, c, cs, cu$. For each subset $S \subset \Lambda$, we also denote $\mathcal{W}^*_\Phi(S) := \cup_{x \in S} \mathcal{W}^*_\Phi(x)$, for $* = s, u, c, cs, cu$.

Besides, we have $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_m$ for some integer $m \ge 1$, where for each $i \in \{1, \ldots, m\}$, Λ_i is a hyperbolic set such that $\Phi|_{\Lambda_i}$ is transitive, and $\Lambda_i = \cap_{t \in \mathbb{R}} \Phi^t(U_i)$ for some open set $U_i \supset \Lambda_i$. The set Λ_i is called a *basic set* of Φ.

Remark 1.6. In general, the stable/unstable distributions $E^{s/u}_{\mathcal{F}}$ at a hyperbolic invariant set Λ of some diffeomorphism $\mathcal{F}$ are only Hölder continuous, but according to Pinto-Rand [47], when the stable, resp. unstable leaves are one-dimensional, and Λ has local product structure, then the stable holonomies, resp. unstable holonomies are of class $\mathcal{C}^{1,\beta}$, $\beta \in (0,1)$. In our case, both distributions are one-dimensional, so the holonomies will be $\mathcal{C}^{1,\beta}$, for some $\beta \in (0,1)$.

Let us recall the following version of the extension theorem due to Whitney [52]. It legitimates the notion of differentiability in Whitney sense.

Theorem 1.7. *Fix an integer $k \ge 1$. Let $A \subset \mathbb{R}^n$ be a closed subset, $n \ge 1$, and let $f_0, \ldots, f_k\colon A \to \mathbb{R}$ be continuous functions such that for some $\beta \in (0,1)$, it holds*

$$f_0(y) - f_0(x) = \sum_{j=1}^{k} \frac{f_j(x)}{j!}(y-x)^j + O(|y-x|^{k+\beta}), \quad \forall\, x, y \in A. \tag{1.1}$$

Then, there exists a $\mathcal{C}^{k,\beta}$ *function* $f\colon \mathbb{R}^n \to \mathbb{R}$ *such that* $f|_A = f_0|_A$, $f^{(j)}|_A = f_j|_A$ *for* $j = 1,\dots,k$, *and* $f|_{\mathbb{R}^n\setminus A}$ *is* $\mathcal{C}^\omega$. *A function* $f_0\colon A \to \mathbb{R}$ *which satisfies* (1.1) *for some functions* $f_1,\dots,f_k\colon A \to \mathbb{R}$ *is said to be* $\mathcal{C}^{k,\beta}$ *in Whitney sense.*

1.2. **Dynamical spectral rigidity of contact Axiom A flows.** Our main dynamical result is the following.

Theorem A (Length spectral rigidity on basic sets)**.** *Fix* $k \geq 2$. *For* $i = 1, 2$, *let* $\Phi_i = (\Phi_i^t)_{t\in\mathbb{R}}$ *be a* $\mathcal{C}^k$ *Axiom A flow defined on a* 3*-manifold* M_i. *Let* Λ_i *be a basic set for* Φ_i, *and assume that there exists a smooth contact form* α_i *on* M_i *that is adapted to* Λ_i. *If there exists an orbit equivalence* $\Psi_0\colon \Lambda_1 \to \Lambda_2$ *between* $\Phi_1|_{\Lambda_1}$ *and* $\Phi_2|_{\Lambda_2}$ *that is differentiable along* Φ_1*-orbits and iso-length-spectral, then*

(1) $\Phi_1|_{\Lambda_1}$, $\Phi_2|_{\Lambda_2}$ *are* $\mathcal{C}^k$*-conjugate; more precisely, there exists a Hölder continuous homeomorphism* $\Psi\colon \Lambda_1 \to \Lambda_2$ *that is* $\mathcal{C}^k$ *in Whitney sense, such that*
$$\Psi \circ \Phi_1^t(x) = \Phi_2^t \circ \Psi(x), \quad \textit{for all } (x,t) \in \Lambda_1 \times \mathbb{R};$$
(2) Ψ *preserves the contact form, i.e.,* $\Psi^*\alpha_2|_{\Lambda_1} = \alpha_1|_{\Lambda_1}$.

In other terms, iso-length-spectral orbit equivalence classes between basic sets of $\mathcal{C}^k$ Axiom A flows with an adapted contact form are in one-to-one correspondence with $\mathcal{C}^k$ flow conjugacy classes between these basic sets, where the conjugacy preserves the contact form. Besides, it will be clear from the proof that the $\mathcal{C}^k$-regularity is actually needed on Λ_i (in Whitney sense).

Remark 1.8. Let Φ_1, Φ_2, and let Λ_1, Λ_2 be as in Theorem A. The flow conjugacy $\Psi\colon \Lambda_1 \to \Lambda_2$ between $\Phi_1|_{\Lambda_1}$ and $\Phi_2|_{\Lambda_2}$ given by Theorem A is essentially unique. Indeed, for any other flow conjugacy $\widetilde{\Psi}\colon \Lambda_1 \to \Lambda_2$, it holds
$$(\Psi^{-1} \circ \widetilde{\Psi}) \circ \Phi_1^t = \Phi_1^t \circ (\Psi^{-1} \circ \widetilde{\Psi}) \quad \text{on } \Lambda_1,$$
that is, $\Psi^{-1} \circ \widetilde{\Psi}$ is in the diffeomorphism centralizer of $\Phi_1|_{\Lambda_1}$. By [2, Theorem 1.4], the centralizer is trivial, hence $\widetilde{\Psi} = \Psi \circ \Phi_1^T$, for some $T \in \mathbb{R}$. In Subsection 3.1, we explain that in some cases (when the system has a time-reversal symmetry) there is a natural way to choose T so as to make the conjugacy *canonical.*

Since the Hausdorff dimension is preserved by Lipschitz continuous homeomorphisms, and since the stable/unstable Hausdorff dimensions are constant on Λ (see for instance [45]), we deduce from Theorem A the following result:

Corollary B. *Let* $\Phi = (\Phi^t)_{t\in\mathbb{R}}$ *be a* $\mathcal{C}^k$ *Axiom A flow defined on a smooth* 3*-manifold* M, $k \geq 2$. *Let* Λ *be a basic set for* Φ *with an adapted smooth contact form* α. *Then, the Hausdorff dimensions* $\dim_H(\Lambda)$, $\delta^{(s)}(\Lambda)$, $\delta^{(u)}(\Lambda)$ *are invariant under iso-length-spectral orbit equivalences, where for* $* = s, u$, *we let* $\delta^{(*)}(\Lambda) = \delta^{(*)} := \dim_H(\Lambda \cap \mathcal{W}_\Phi^*(x))$, *for any* $x \in \Lambda$.

1.3. **Open dispersing billiards.** We consider a billiard table $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^{\ell} \mathcal{O}_i$ obtained by removing from the plane $\ell \geq 3$ *obstacles* $\mathcal{O}_1,\dots,\mathcal{O}_\ell$, each $\mathcal{O}_i$ being a convex domain with $\mathcal{C}^k$ boundary $\partial\mathcal{O}_i$, for some $k \geq 3$, such that $\overline{\mathcal{O}_1},\dots,\overline{\mathcal{O}_\ell}$ are pairwise disjoint. For each $i \in \{1,\dots,\ell\}$, we let $|\partial\mathcal{O}_i|$ be the corresponding perimeter, and parametrize each $\partial\mathcal{O}_i$ counterclockwisely in arc-length by some map $\Upsilon_i \in \mathcal{C}^k(\mathbb{T}_i, \mathbb{R}^2)$, $s \mapsto \Upsilon_i(s)$, where $\mathbb{T}_i := \mathbb{R}/(|\partial\mathcal{O}_i|\mathbb{Z})$. The set of all such billiard tables will be denoted by $\mathbf{B}$, and for each $\ell \geq 3$, we let $\mathbf{B}(\ell) \subset \mathbf{B}$ be the subset of tables with ℓ obstacles.

Let $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^{\ell} \mathcal{O}_i \in \mathbf{B}$, for some $\ell \geq 3$. We denote the collision space by

$$\mathcal{M} := \bigcup_i \mathcal{M}_i, \qquad \mathcal{M}_i := \{(q,v),\ q \in \partial\mathcal{O}_i,\ v \in \mathbb{R}^2,\ \|v\| = 1,\ \langle v, n\rangle \geq 0\},$$

where n is the unit normal vector to $\partial\mathcal{O}_i$ pointing outside $\mathcal{O}_i$. For each $x = (q,v) \in \mathcal{M}$, we have $q = \Upsilon_i(s)$, for some $i \in \{1,\dots,\ell\}$ and some arclength parameter $s \in \mathbb{T}_i$; we let $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ be the oriented angle between n and v, and set $r := \sin\varphi$. Therefore, each $\mathcal{M}_i$ can be seen as a cylinder $\mathbb{T}_i \times [-1,1]$ endowed with coordinates (s,r). In the following, given a point $x = (s,r) \in \mathcal{M}$, we let $\Upsilon(s) := q$ be the associated point of $\partial\mathcal{D}$.

For each pair $(s_1,r_1),(s_2,r_2) \in \mathcal{M}$, we denote by

$$h(s_1,s_2) := \|\Upsilon(s_1) - \Upsilon(s_2)\| \tag{1.2}$$

the Euclidean length of the segment connecting the associated points of the table.

Let $\mathfrak{M} := \{(q,v) \in \mathcal{D} \times \mathbb{S}^1\}/\sim$ be the quotient of $\mathcal{D} \times \mathbb{S}^1$ by the relation $\sim$:

$$(q_1,v_1) \sim (q_2,v_2) \quad \iff \quad q_1 = q_2 \in \partial\mathcal{D} \text{ and } v_2 = \mathcal{R}_{q_1}(v_1),$$

where $\mathcal{R}_{q_1}$ is the reflection in $\mathbb{R}^2$ with respect to the tangent line $T_{q_1}\partial\mathcal{D}$. An element of $\mathfrak{M}$ will be denoted as $[(q,v)]$. In the following, we identify a point $[(q,v)] \in \mathfrak{M}$, $q \in \partial\mathcal{D}$, with the corresponding element $(q,v) \in \mathcal{M}$. Let $\Phi = (\Phi^t)_{t\in\mathbb{R}}$ be the associated billiard flow on $\mathfrak{M}$. We can describe this flow with coordinates (x,y,ω), where $(x,y) \in \mathbb{R}^2$ are the Cartesian coordinates of some point $q \in \mathcal{D}$ on the table and $\omega \in [0,2\pi)$ denotes the couterclockwise angle between the positive x axis and the velocity vector v. For each $x \in \mathcal{M}$, we let $\tau(x) \in \mathbb{R}_+ \cup \{+\infty\}$ be the first return time of the Φ-orbit of x to $\mathcal{M}$, and denote by

$$\mathcal{F} = \mathcal{F}(\mathcal{D})\colon \mathcal{M} \cap \{\tau \neq +\infty\} \to \mathcal{M}, \quad x \mapsto \Phi^{\tau(x)}(x)$$

the associated billiard map, which we see as a map $\mathcal{F}\colon (s,r) \mapsto (s',r')$, with $s' = s'(s,r)$ and $r' = r'(s,r)$.

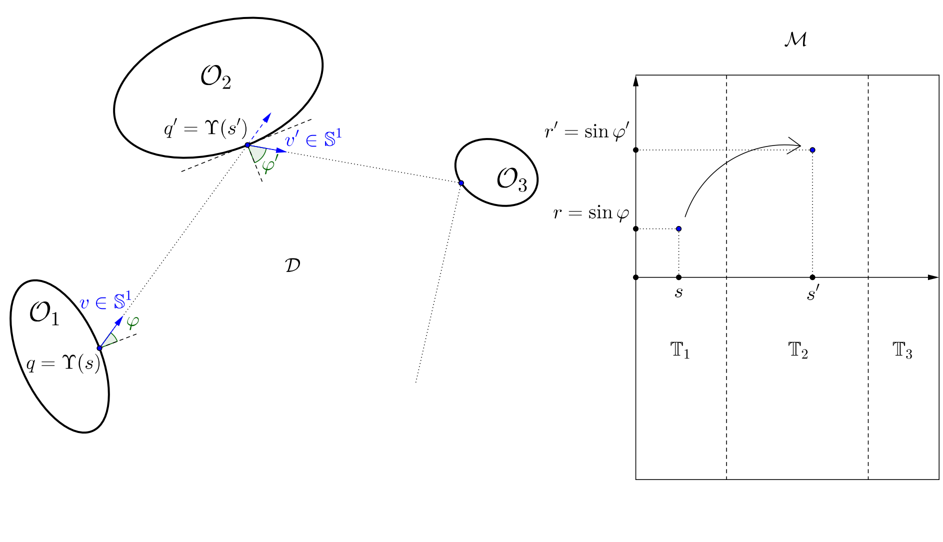


FIGURE 1. An open dispersing billiard and its phase space.

For any point $x = (s, r) \in \mathcal{M}$ with a well-defined image $(s', r') = \mathcal{F}(s, r)$, recall that $h = h(s, s')$ is the distance between the two points of collision. Note that $h(s, s') = h(s, s'(s, r)) = \tau(s, r)$ is the first return time of $(s, r) \in \mathcal{M}$ to $\mathcal{M}$. Let $\mathcal{K} := \mathcal{K}(s)$, $\mathcal{K}' := \mathcal{K}(s')$ be the respective curvatures, and set $\nu = \nu(r) := \sqrt{1 - r^2}$, $\nu' := \nu(r') = \sqrt{1 - (r')^2}$. By the formulas in Chernov-Markarian [7], the differential of the billiard map is

$$D\mathcal{F}(s, r) = -\begin{bmatrix} \frac{1}{\nu'}(h\mathcal{K} + \nu) & \frac{h}{\nu\nu'} \\ h\mathcal{K}\mathcal{K}' + \mathcal{K}\nu' + \mathcal{K}'\nu & \frac{1}{\nu}(h\mathcal{K}' + \nu') \end{bmatrix}. \tag{1.3}$$

The map $\mathcal{F}$ is exact symplectic for the Liouville form $\lambda = -r ds$:

$$\mathcal{F}^*\lambda - \lambda = d\tau. \tag{1.4}$$

Fix a lift $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ to $\mathbb{R} \times [-1, 1]$. We let $|\partial\mathcal{D}| := |\partial\mathcal{O}_1| + \cdots + |\partial\mathcal{O}_m|$ be the total perimeter, and extend the definition of h by letting $h(s+p|\partial\mathcal{D}|, s'+q|\partial\mathcal{D}|) = h(s, s')$, for any $p, q \in \mathbb{Z}$. Then, h is a *generating function* for the dynamics of $\widetilde{\mathcal{F}}$ (or $\mathcal{F}$):

$$\begin{cases} r &= \frac{\partial h(s,s')}{\partial s}, \\ r' &= -\frac{\partial h(s,s')}{\partial s'}. \end{cases}$$

Observe that $\mathcal{F}$ is a negative twist map, i.e., $\frac{\partial s'}{\partial r}(s, r) < 0$, and that $-\frac{\partial^2 h}{\partial s \partial s'}(s, s') > 0$. Let us also recall that the time-reversal involution $\mathcal{I}\colon (s, r) \mapsto (s, -r)$ conjugates the billiard map with its inverse, i.e., $\mathcal{F} \circ \mathcal{I} = \mathcal{I} \circ \mathcal{F}^{-1}$.

Due to the strict convexity of the obstacles, the dynamics is of Axiom A type (see [39, 40] or [51, Subsection 2.1] for more details). In connection with Remark 1.6, let us also recall that several works have been dedicated to the smoothness of stable/unstable laminations of open dispersing billiards (see Morita [39] and Stoyanov [51]). Besides, if the non-wandering set

$$\Omega(\mathcal{F}) := \bigcap_{j \in \mathbb{Z}} \mathcal{F}^j(\mathcal{M})$$

has no tangential collisions, then it is a hyperbolic set; moreover, we have $\Omega(\mathcal{F}) = \Lambda \cup F$, $\Lambda \cap F = \emptyset$, where F is a finite union of periodic points, and Λ can be written as a disjoint union $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_m$, $m \geq 1$, each Λ_i being a horseshoe such that $\mathcal{F}|_{\Lambda_i}$ is conjugated to a non-trivial subshift of finite type. In the following, for each point $x \in \Omega(\mathcal{F})$, we denote by $\mathcal{W}^s_{\mathcal{F}}(x)$, resp. $\mathcal{W}^u_{\mathcal{F}}(x)$, its stable, resp. unstable manifold for the map $\mathcal{F}$. The non-wandering set $\Omega(\Phi)$ of the billiard flow Φ is the set of all points in the orbit of some $x \in \Omega(\mathcal{F})$. Similarly, when speaking about a basic set for Φ in the following, we mean the union of orbits of all the points in a set Λ_i as above, for some $i \in \{1, \ldots, m\}$. Let us define the quotient set $\Lambda_i^\tau := \{(s, r, t) \in \Lambda_i \times \mathbb{R} : 0 \leq t \leq \tau(s, r)\}/\approx$, where

$$((s, r), \tau(s, r)) \approx (\mathcal{F}(s, r), 0).$$

We can identify Λ_i^τ with the set $\{(s, r, t) \in \Lambda_i \times \mathbb{R} : 0 \leq t < \tau(s, r)\}$, and define the projection $\Pi\colon \Lambda_i^\tau \to \partial\mathcal{D}$ as[2]

$$\Pi(s, r, t) := s \simeq \Upsilon(s). \tag{1.5}$$

The billiard flow Φ restricted to the orbits of points in Λ_i is defined at all times and can be seen as a *special flow* induced by the vertical vector field $X = \frac{\partial}{\partial t} = (0, 0, 1)$

[2]By a slight abuse of notation, we will also denote by $\Pi\colon \Lambda_i \to \partial\mathcal{D}$ the projection $(s, r) \mapsto s$.

on Λ_i^τ. The billiard flow is $\mathcal{C}^{k-1}$ on Λ_i^τ in the following sense: for any $T>0$, there exist an integer $n=n(T)>0$ (with $\lim_{T\to+\infty} n(T)=+\infty$) and a neighborhood $\mathcal{U}_T$ of Λ_i such that $\mathcal{U}_T\subset\cap_{j=-n}^n\mathcal{F}^j(\mathcal{M})$, and such that the time-t map Φ^t of the flow induced by X on $\big(\mathcal{U}_T\times[-T,T]\big)/\approx$ is well-defined on $\mathcal{U}_T\times\{0\}$, for any $t\in[-T,T]$, and of class $\mathcal{C}^{k-1}$ (see [7, Lemma 2.24]). Actually, it is more convenient to see this in the (x,y,ω)-coordinates introduced above, as this coordinate representation also allows to describe points which are not in $\Omega(\Phi)$: for any point $(s,r,t)\in\Lambda_i^\tau$, with $r=\sin\varphi\in(-1,1)$, $t\in[0,\tau(s,r))$, we let $U(s,r,t):=(x,y,\omega)\in\mathfrak{M}$ be the corresponding (x,y,ω)-coordinates, with $x=x(s,r,t)$, $y=y(s,r,t)$, and

$$\omega=\omega(s,r)=\angle\big(R_{-\frac{\pi}{2}+\varphi}\big(\Upsilon'(s)\big),(1,0)\big)=\angle\big(R_{-\frac{\pi}{2}+\arcsin r}\big(\Upsilon'(s)\big),(1,0)\big),$$
$$\big(x(s,r,t),y(s,r,t)\big)=\Upsilon(s)+t(\cos\omega,\sin\omega),$$

where $\Upsilon(s)$ is the associated point of $\partial\mathcal{D}$, and for $\theta\in\mathbb{R}$, R_θ is the rotation of angle θ. The map Υ is $\mathcal{C}^k$, hence the change of coordinates U is of class $\mathcal{C}^{k-1}$.

Claim 1.9. *The contact form $\alpha=\lambda+dt$ is adapted to Λ_i^τ (recall Definition 1.2).*

Proof. Let us verify that $\imath_X\alpha=1$ and $\imath_X d\alpha=0$. Indeed, for any $(s,r,t)=((s,r),t)\in\mathcal{M}\times\mathbb{R}$, we have

$$\alpha(s,r,t)\big(X(s,r,t)\big)=(\lambda(s,r)+dt)\frac{\partial}{\partial t}=1,$$

and

$$d\alpha(s,r,t)(X(s,r,t))=d\lambda(s,r)\frac{\partial}{\partial t}=0.$$

Besides, for $W\colon(s,r,t)\mapsto(\mathcal{F}(s,r),t-\tau(s,r))$, we have

$$\begin{aligned}W^*\alpha(s,r,t)&=\alpha\circ W(s,r,t)=\alpha(\mathcal{F}(s,r),t-\tau(s,r))\\&=\lambda(\mathcal{F}(s,r))+d(t-\tau(s,r))=\mathcal{F}^*\lambda(s,r)+dt-d\tau(s,r)\\&=\lambda(s,r)+d\tau(s,r)+dt-d\tau(s,r)=\alpha(s,r,t).\end{aligned}$$

Therefore, α descends to an adapted contact form on Λ_i^τ. □

Let us also recall how the contact structure looks like in (x,y,ω)-coordinates. For each point $X=(x,y,\omega)\in\mathfrak{M}$, we let

$$\begin{aligned}T_X\mathfrak{M}\supset T_X^0\mathfrak{M}&:=\ker\big(-\sin\omega dx+\cos\omega dy\big)\cap\ker\big(d\omega\big),\\T_X\mathfrak{M}\supset T_X^\perp\mathfrak{M}&:=\ker\big(\cos\omega dx+\sin\omega dy\big).\end{aligned}$$

The one-dimensional subbundle $T^0\mathfrak{M}\subset T\mathfrak{M}$ and the two-dimensional subbundle $T^\perp\mathfrak{M}\subset T\mathfrak{M}$ are $D\Phi$-invariant. More precisely, for any $t\in\mathbb{R}$, the differential $D\Phi^t$ acts on $T^0\mathfrak{M}\oplus T^\perp\mathfrak{M}$ as follows (see Chernov-Markarian [7] for more details):

$$D\Phi^t(X)=\begin{bmatrix}1&0\\0&D^\perp\Phi^t(X)\end{bmatrix},\quad\forall X\in\mathfrak{M}.$$

In particular, $\cos\omega dx+\sin\omega dy$ is the contact form in (x,y,ω)-coordinates, and $T^\perp\mathfrak{M}$ is the associated contact distribution.

Definition 1.10. Let $\mathcal{D}_1,\mathcal{D}_2\in\mathbf{B}$ be two open dispersing billiards with $\mathcal{C}^k$ boundaries, for some $k\geq 3$, and let Φ_1,Φ_2 be the associated billiard flows. Given two basic sets $\Lambda_1^{\tau_1}\subset\Omega(\Phi_1)$, $\Lambda_2^{\tau_2}\subset\Omega(\Phi_2)$, we say that $\mathcal{D}_1,\mathcal{D}_2$ are *iso-length-spectral* on $\Lambda_1^{\tau_1},\Lambda_2^{\tau_2}$ if there exists an iso-length-spectral orbit equivalence between $\Lambda_1^{\tau_1}$ and $\Lambda_2^{\tau_2}$.

Theorem C. *Let $\mathcal{D}_1, \mathcal{D}_2 \in \mathbf{B}$ be two open dispersing billiards with $\mathcal{C}^k$ boundaries, for some $k \geq 3$, and billiard flows Φ_1, Φ_2. Let us consider a basic set $\Lambda_i^{\tau_i}$ for Φ_i, $i = 1, 2$, and let Λ_i be the projection of $\Lambda_i^{\tau_i}$ onto the first two coordinates (s_i, r_i). If $\mathcal{D}_1, \mathcal{D}_2$ are iso-length-spectral on $\Lambda_1^{\tau_1}$, $\Lambda_2^{\tau_2}$, then there exists a map $\widetilde{\Psi}\colon (s_1, r_1, t_1) \mapsto (s_2, r_2, t_2)$ that is $\mathcal{C}^{k-1}$ in Whitney sense, except at collisions (when $t_1 = 0$ or $t_2 = 0$), which conjugates Φ_1, Φ_2 on $\Lambda_1^{\tau_1}, \Lambda_2^{\tau_2}$ respectively. The map $\widetilde{\Psi}$ induces a conjugacy $\Psi\colon \Lambda_1 \to \Lambda_2$ between the respective billiard maps $\mathcal{F}_1|_{\Lambda_1}, \mathcal{F}_2|_{\Lambda_2}$ which is $\mathcal{C}^{k-1}$ in Whitney sense, and such that $\Psi^*(ds_2 \wedge dr_2) = ds_1 \wedge dr_1$ on Λ_1. Besides, let $\mathcal{I}_i\colon (s_i, r_i) \mapsto (s_i, -r_i)$, $i = 1, 2$. If $\Psi^{-1} \circ \mathcal{I}_2 \circ \Psi \circ \mathcal{I}_1|_{\Lambda_1}$ fixes $\mathcal{F}_1$-orbits, i.e., $\mathcal{I}_2 \circ \Psi(x_1)$ and $\Psi \circ \mathcal{I}_1(x_1)$ are in the same $\mathcal{F}_2$-orbit, for all $x_1 \in \Lambda_1$, then Ψ can be chosen in a unique way such that*

$$\Psi \circ \mathcal{I}_1|_{\Lambda_1} = \mathcal{I}_2 \circ \Psi|_{\Lambda_1}. \tag{1.6}$$

Moreover, the respective generating functions τ_1, τ_2 of $\mathcal{F}_1, \mathcal{F}_2$ satisfy

$$\tau_2 \circ \Psi - \tau_1 = \chi \circ \mathcal{F}_1 - \chi \quad \text{on } \Lambda_1, \tag{1.7}$$

for some function $\chi\colon \Lambda_1 \to \mathbb{R}$ which is $\mathcal{C}^{k-1}$ in Whitney sense, such that

$$\Psi^* \lambda_2 - \lambda_1 = d\chi \quad \text{on } \Lambda_1, \quad \text{where } \lambda_i = -r_i ds_i,\ i = 1, 2, \tag{1.8}$$

and which satisfies $\chi \circ \mathcal{I}_1 = -\chi$; in particular, the function χ and its differential $d\chi$ vanish identically on $\Lambda_1 \cap \{r_1 = 0\}$.

The proof of Theorem C is given in Section 3.

Remark 1.11. In Theorem C, we consider the case of dispersing billiards, as those exhibit naturally uniformly hyperbolic dynamics. Yet, even in the case of convex billiards, generically, hyperbolic dynamics arises from *Aubry-Mather* periodic orbits with transverse heteroclinic intersections (see for instance [27] for more details). Thus, our result may also be applied to the associated horseshoes.

Remark 1.12. The function χ in Theorem C can be interpreted as the difference between stable (or unstable) actions for the billiard maps $\mathcal{F}_1, \mathcal{F}_2$. Indeed, fix a 2-periodic point $p_1 \in \Lambda_1$, and let $p_2 := \Psi(p_1) \in \Lambda_2$. Let us consider a point $x_1 \in \Lambda_1$ in the stable manifold $\mathcal{W}^s_{\mathcal{F}_1}(p_1)$ of p_1, and let $x_2 := \Psi(x_1) \in \mathcal{W}^s_{\mathcal{F}_2}(p_2) \cap \Lambda_2$. For $i = 1, 2$, we define the *stable action* of x_i as the sum of the following convergent series:

$$\mathcal{A}^s_{p_i,\mathcal{F}_i}(x_i) = \mathcal{A}^s_i(x_i) := \sum_{k=0}^{+\infty} \big(\tau_i \circ \mathcal{F}_i^k(x_i) - \tau_i \circ \mathcal{F}_i^k(p_i)\big).$$

Since the two billiards have the same periodic length data, and since p_1, p_2 are 2-periodic, we have $\tau_1 \circ \mathcal{F}_1^k(p_1) = \tau_2 \circ \mathcal{F}_2^k(p_2)$, for each $k \in \mathbb{Z}$. Observe that $\lim_{k\to+\infty} \chi \circ \mathcal{F}_1^k(x_1) = \chi(p_1) = 0$, as χ is odd in the r_1-variable. By (1.7), we thus conclude that

$$\mathcal{A}^s_1(x_1) - \mathcal{A}^s_2(x_2) = \sum_{k=0}^{+\infty} \big(\tau_1 \circ \mathcal{F}_1^k(x_1) - \tau_2 \circ \mathcal{F}_2^k(x_2)\big) = \chi(x_1) - \lim_{k\to+\infty} \chi \circ \mathcal{F}_1^k(x_1) = \chi(x_1),$$

i.e., $\chi(x_1)$ is the difference between the stable actions $\mathcal{A}^s_1(x_1)$ and $\mathcal{A}^s_2(\Psi(x_1))$. We say that $\mathcal{D}_1, \mathcal{D}_2$ have the same *marked stable action spectrum* if

$$\mathcal{A}^s_2 \circ \Psi(x_1) = \mathcal{A}^s_1(x_1), \quad \forall\, x_1 \in \mathcal{W}^s_{\mathcal{F}_1}(p_1) \cap \Lambda_1.$$

By the above discussion, if $\mathcal{D}_1, \mathcal{D}_2$ have the same marked stable action spectrum, then χ vanishes at every point in the stable manifold of p_1; as $\mathcal{W}^s_{\mathcal{F}_1}(p_1)$ is dense in Λ_1, the function χ vanishes at every point of Λ_1, thus, $\tau_2 \circ \Psi|_{\Lambda_1} = \tau_1|_{\Lambda_1}$.

Let $\mathcal{D}_1, \mathcal{D}_2$ be two billiards with $\mathcal{C}^k$ boundaries, $k \geq 3$, such that $\mathcal{D}_1, \mathcal{D}_2$ are iso-length-spectral on basic sets $\Lambda_1^{\tau_1}$, $\Lambda_2^{\tau_2}$; let Λ_1, Λ_2 be the respective projections of $\Lambda_1^{\tau_1}, \Lambda_2^{\tau_2}$ onto the first two coordinates. In the same way, we can show (see Section 3) that the restrictions of the billiard maps $\mathcal{F}_1, \mathcal{F}_2$ to Λ_1, Λ_2 are conjugated by a $\mathcal{C}^{k-1}$ map $\Psi\colon (s_1, r_1) \mapsto (s_2, r_2)$ such that $\Psi^*(ds_2 \wedge dr_2) = ds_1 \wedge dr_1$. Thus, there exist functions $a, b, c, d\colon \Lambda_1 \to \mathbb{R}$ which are $\mathcal{C}^{k-2}$ in Whitney sense, and such that for any $(s_1, r_1) \in \Lambda_1$,

$$D\Psi(s_1, r_1) = \begin{bmatrix} a(s_1, r_1) & c(s_1, r_1) \\ b(s_1, r_1) & d(s_1, r_1) \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R}), \tag{1.9}$$

as Ψ preserves the area form. Besides, if (1.6) holds, then the functions b, c satisfy $b(s_1, -r_1) = -b(s_1, r_1)$, $c(s_1, -r_1) = -c(s_1, r_1)$, for any $(s_1, r_1) \in \Lambda_1$. In particular, for any point $(s_1, 0) \in \Lambda_1 \cap \{r_1 = 0\}$ with a perpendicular bounce, we have

$$b(s_1, 0) = c(s_1, 0) = 0, \quad \text{hence} \quad d(s_1, 0) = a^{-1}(s_1, 0).$$

Definition 1.13 (Infinitesimal fiber preservation)**.** We say that the differential $D\Psi$ preserves *vertical fibers* if for each $(s_1, r_1) \in \Lambda_1$, $D\Psi(s_1, r_1)$ is lower-triangular, i.e., the function c in (1.9) vanishes identically on Λ_1; in particular, $d = a^{-1}$ on Λ_1.

Let us now describe how the previous dynamical result can be applied to the study of spectral rigidity of hyperbolic billiards.

Theorem D (Spectral rigidity of hyperbolic billiards)**.** *Let $\mathcal{D}_1, \mathcal{D}_2$ be two billiards with $\mathcal{C}^k$ boundaries, for some $k \geq 3$, and let $\mathcal{F}_1, \mathcal{F}_2$ be the associated billiard maps. Assume that there exists a horseshoe*[3] *Λ_1, resp. Λ_2 for $\mathcal{F}_1$, resp. $\mathcal{F}_2$, such that $\mathcal{F}_1|_{\Lambda_1}$ and $\mathcal{F}_2|_{\Lambda_2}$ are topologically conjugated and have the same periodic length data. Then, there exists a conjugacy $\Psi\colon \Lambda_1 \to \Lambda_2$ between $\mathcal{F}_1|_{\Lambda_1}, \mathcal{F}_2|_{\Lambda_2}$ which is $\mathcal{C}^{k-1}$ in Whitney sense and such that $\Psi^*(ds_2 \wedge dr_2) = ds_1 \wedge dr_1$ on Λ_1. Let us further assume that*

(1) $\Psi^{-1} \circ \mathcal{I}_2 \circ \Psi \circ \mathcal{I}_1|_{\Lambda_1}$ fixes $\mathcal{F}_1$-orbits, where $\mathcal{I}_i\colon (s_i, r_i) \mapsto (s_i, -r_i)$, for $i = 1, 2$;
(2) there exists a point $x_1 \in \Lambda_1 \cap \{r_1 = 0\}$ whose orbit is dense in Λ_1, and such that $\Psi(x_1) \in \mathcal{F}_2^{-k}(\{r_2 = 0\})$ for some $k \in \mathbb{Z}$;
(3) $\mathcal{F}_1^2|_{\Lambda_1}$ is transitive;
(4) $D\Psi$ preserves vertical fibers.

Then, after replacing Ψ with $\mathcal{F}_2^k \circ \Psi$, it holds:

(1) $\Psi \circ \mathcal{I}_1 = \mathcal{I}_2 \circ \Psi$;
(2) $D\Psi(s_1, r_1) = \mathrm{id}$, for any $(s_1, r_1) \in \Lambda_1$;
(3) the conjugacy Ψ induces a homeomorphism $\mathcal{Z} = \mathcal{Z}_{\Lambda_1, \Lambda_2}$ between the projections $\Pi(\Lambda_1), \Pi(\Lambda_2)$ that is $\mathcal{C}^{k-1}$ in Whitney sense (where $\Pi\colon (s, r) \mapsto s$);
(4) if $\mathcal{K}_1$, $\mathcal{K}_2$ denote the (Gaussian) curvature functions, then the $(k-2)$-jets of $\mathcal{K}_1$ and $\mathcal{K}_2 \circ \mathcal{Z}$ coincide on $\Pi(\Lambda_1)$;
(5) the lengths of orbit segments between two consecutives bounces in Λ_1, Λ_2 coincide. More precisely, the generating functions h_1, h_2 satisfy $h_2(\mathcal{Z}(s_1), \mathcal{Z}(s_1')) = h_1(s_1, s_1')$, for any $s_1 \in \Pi(\Lambda_1)$;
(6) the angles between orbit segments in Λ_1, Λ_2 coincide.

Remark 1.14. In other words, (4) means that the tables $\mathcal{D}_1$, $\mathcal{D}_2$ have the same "local" geometry at corresponding points of the projections $\Pi(\Lambda_1) \subset \partial \mathcal{D}_1$,

[3] Let us recall that a *horseshoe* for a diffeomorphism f is a transitive, locally maximal hyperbolic set that is totally disconnected and not finite.

$\Pi(\Lambda_2) \subset \partial\mathcal{D}_2$, while (5)-(6) mean that the *traces* T_1, T_2 of $\Pi(\Lambda_1), \Pi(\Lambda_2)$ on $\mathcal{D}_1, \mathcal{D}_2$ are *isometric*, where T_1, T_2 are respectively the sets of all (infinite) broken lines obtained by connecting consecutive bounces in Λ_1, Λ_2 (see Figure 2). Here, T_1 being isometric to T_2 means that each of these sets can be obtained one from another by applying a rotation and a translation.

Indeed, as Λ_1, Λ_2 are transitive sets, we can select a point $x_1 \in \Lambda_1$ whose orbit is dense in Λ_1; the orbit of $x_2 := \Psi(x_1) \in \Lambda_2$ is also dense in Λ_2. Thus, for $i = 1, 2$, the trace T_i can be seen as the closure (in $\mathcal{D}_i \subset \mathbb{R}^2$) of the (infinite) broken line L_i obtained by connecting consecutive bounces on the obstacles of the orbit of x_i. By (5)-(6), associated segments in T_1, T_2 have the same lengths, and the angles between consecutive segments coincide. Therefore, L_1, L_2 have the same geometry, as well as $T_1 = \overline{L_1}$, $T_2 = \overline{L_2}$.

Remark 1.15. Assumptions (1)-(2)-(3) in Theorem D can be checked in the case of open dispersing billiards (see Subsection 3.4). In other words, (4) is the only additional hypothesis needed to show that open dispersing billiards are spectrally rigid. Besides, by inspecting the proof given in Subsection 3.3 (see in particular Lemmas 3.9 and 3.10), we see that assumption (4) could be replaced with the assumption that χ in the cohomological equation (1.7) vanishes (or $d\chi = 0$ in (1.8)); indeed, in that case, $\Psi^*\lambda_2 - \lambda_1 = 0$ in (1.8), and then, assumption (4) is satisfied. In particular:

Corollary E. *Let $\mathcal{D}_1, \mathcal{D}_2 \in \mathbf{B}$ be two open dispersing billiards with the same periodic length data on basic sets Λ_1, Λ_2 as in Theorem C. If, moreover, $\mathcal{D}_1, \mathcal{D}_2$ have the same marked stable action spectrum in the sense of Remark 1.12, then the conclusion of Theorem D is true, i.e., $\mathcal{D}_1, \mathcal{D}_2$ are "isometric" on the projections $\Pi(\Lambda_1), \Pi(\Lambda_2)$.*

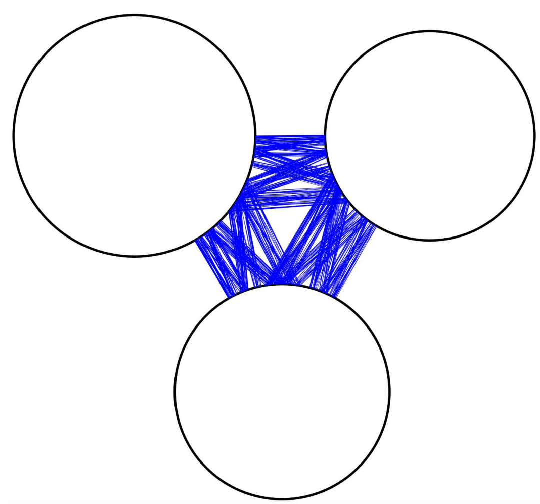
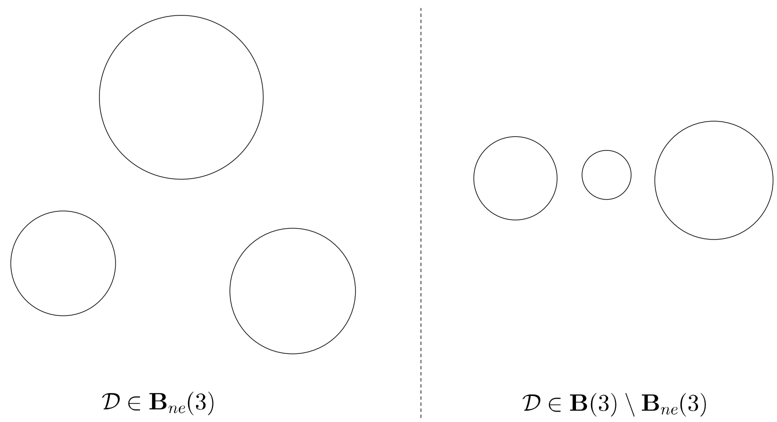


FIGURE 2. Trace on the table of the non-wandering set (picture by S. Dyatlov [18]) for the 3-disk model.

1.4. **Non-eclipsing billiards.** We now discuss the following important example (see [38, 15] for more details). Fix an integer $\ell \geq 3$. We let $\mathbf{B}_{ne}(\ell) \subset \mathbf{B}(\ell)$ be the set of all billiards $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^{\ell} \mathcal{O}_i \in \mathbf{B}(\ell)$ which satisfy the following

Non-eclipse condition: The convex hull of any two obstacles is disjoint from any other obstacle.



$\mathcal{D} \in \mathbf{B}_{ne}(3)$ $\qquad$ $\mathcal{D} \in \mathbf{B}(3) \setminus \mathbf{B}_{ne}(3)$

Let $\mathcal{F}$, resp. Φ be the associated billiard map, resp. billiard flow. The non-wandering set $\Omega(\mathcal{F})$ is reduced to a single basic set Λ. Moreover, $\mathcal{F}|_\Lambda$ is conjugated by some Hölder homeomorphism to the subshift of finite type associated with the transition matrix $(1-\delta_{i,j})_{1\leq i,j\leq \ell}$, where $\delta_{i,j}=1$, when $i=j$, and $\delta_{i,j}=0$ otherwise, when $i\neq j$. In other words, any *admissible* word $\varsigma \in \mathrm{Adm}_\infty$, i.e., such that $\varsigma = (\varsigma_j)_j \in \{1,\dots,\ell\}^{\mathbb{Z}}$ with $\varsigma_{j+1}\neq \varsigma_j$, for all $j\in\mathbb{Z}$, can be realized by an orbit, and by hyperbolicity of the dynamics, this orbit is unique. We denote by $x(\varsigma)\in\Omega(\mathcal{F})$ the point with symbolic coding ς. Let $\mathrm{Adm} \subset \cup_{j\geq 2}\{1,\dots,\ell\}^j$ be the set of all finite words $\sigma=\sigma_1\dots\sigma_j$, $j\geq 2$, such that $\sigma^\infty := \cdots\sigma\sigma\sigma\cdots \in \mathrm{Adm}_\infty$. It is the set of symbolic codings of periodic orbits. In particular, we may thus define the *marked length spectrum* $\mathcal{MLS}(\mathcal{D})$ as the map

$$\mathcal{MLS}(\mathcal{D})\colon \mathrm{Adm}\to\mathbb{R},\quad \sigma\mapsto \mathcal{A}(\sigma),$$

where $\mathcal{A}(\sigma)=T_\Phi(x(\sigma^\infty))$ is the perimeter of the periodic orbit encoded by σ.

We also define $\mathrm{Stab}\subset\mathrm{Adm}_\infty$ as the subset of words $(\sigma_j)_{j\in\mathbb{Z}}\in\mathrm{Adm}_\infty$ such that for some $k_0\in\mathbb{Z}$, we have $\sigma_{k_0+2j}=1$ and $\sigma_{k_0+(2j+1)}=2$, for all $j\geq 0$. We extend the definition of $\mathcal{A}$ by letting $\mathcal{A}(\sigma):=\mathcal{A}^s_{p,\mathcal{F}}(x(\sigma))+\mathcal{A}(12)$, for any $\sigma\in\mathrm{Stab}$, where $\mathcal{A}^s_{p,\mathcal{F}}(x(\sigma))$ is defined as in Remark 1.12, for the 2-periodic point $p:=x((12)^\infty)$. We may then define the *marked action spectrum* $\mathcal{MAS}(\mathcal{D})$ as the map

$$\mathcal{MAS}(\mathcal{D})\colon \mathrm{Adm}\cup\mathrm{Stab}\to\mathbb{R},\quad \sigma\mapsto\mathcal{A}(\sigma).$$

For any billiards $\mathcal{D}_1,\mathcal{D}_2\in\mathbf{B}_{ne}(\ell)$ with respective billiard maps $\mathcal{F}_1,\mathcal{F}_2$, the restrictions $\mathcal{F}_1|_{\Omega(\mathcal{F}_1)}$, $\mathcal{F}_2|_{\Omega(\mathcal{F}_2)}$ are topologically conjugated in a canonical way, by sending a point $x_1\in\Omega(\mathcal{F}_1)$ to the point $x_2\in\Omega(\mathcal{F}_2)$ with the same coding. The billiard flows Φ_1,Φ_2 are thus orbit equivalent through some Hölder continuous orbit equivalence. As a consequence of Theorem D and Corollary E, we obtain:

Corollary F (Spectral rigidity of open dispersing billiards without eclipse)**.** *Fix $\ell\geq 3$, and let $\mathcal{D}_1,\mathcal{D}_2\in\mathbf{B}_{ne}(\ell)$ with $\mathcal{C}^k$ boundaries, for some $k\geq 3$. If $\mathcal{D}_1,\mathcal{D}_2$ have the same marked length spectrum, then the respective billiards maps $\mathcal{F}_1,\mathcal{F}_2$ are conjugated on $\Omega(\mathcal{F}_1),\Omega(\mathcal{F}_2)$ by a map $\Psi\colon\Omega(\mathcal{F}_1)\to\Omega(\mathcal{F}_2)$ that is $\mathcal{C}^{k-1}$ in Whitney* *sense, such that* $\Psi^*(ds_2 \wedge dr_2) = ds_1 \wedge dr_1$ *and* $\Psi \circ \mathcal{I}_1 = \mathcal{I}_2 \circ \Psi$ *on* $\Omega(\mathcal{F}_1)$*, where* $\mathcal{I}_i\colon (s_i, r_i) \mapsto (s_i, -r_i)$*, for* $i = 1, 2$*, is the time-reversal involution.*

If $D\Psi$ *preserves vertical fibers – in particular, if* $\mathcal{MAS}(\mathcal{D}_1) = \mathcal{MAS}(\mathcal{D}_2)$ *– then the* $(k-2)$*-jets of the curvature functions* $\mathcal{K}_1$*,* $\mathcal{K}_2 \circ \mathcal{Z}$ *coincide on* $\Pi(\Omega(\mathcal{F}_1))$*, where* $\Pi\colon (s, r) \mapsto s$*, and* $\mathcal{Z} = \mathcal{Z}_{\Omega(\mathcal{F}_1),\Omega(\mathcal{F}_2)}$ *is the map given by Theorem D. Besides, the traces of* $\Omega(\mathcal{F}_1), \Omega(\mathcal{F}_2)$ *on* $\mathcal{D}_1, \mathcal{D}_2$ *are isometric in the sense of Remark 1.14.*

In particular, if the boundaries $\partial\mathcal{D}_1, \partial\mathcal{D}_2$ *are* $\mathcal{C}^\omega$ *(or, more generally,* quasi-analytic*, see* [17] *for instance), then* $\mathcal{D}_1, \mathcal{D}_2$ *are isometric, i.e., they can be obtained one from another by composition of a translation and a rotation.*

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2. Smooth conjugacy classes for 3D Axiom A flows on basic sets

2.1. **Synchronization of the flows using periodic data.** Let us start by recalling the fact that an orbit equivalence between two hyperbolic flows can be upgraded to a flow conjugacy as long as the lengths of associated periodic orbits coincide.

Proposition 2.1. *Let* $k \geq 2$*, and let* $\Phi_1 = (\Phi_1^t)_{t\in\mathbb{R}}$*, resp.* $\Phi_2 = (\Phi_2^t)_{t\in\mathbb{R}}$ *be a* $\mathcal{C}^k$ *Axiom A flow defined on a smooth manifold* M_1*, resp.* M_2*, and let* Λ_1*, resp.* Λ_2 *be a basic set for* Φ_1*, resp.* Φ_2*. Assume that there exists an orbit equivalence* $\Psi_0\colon \Lambda_1 \to \Lambda_2$ *differentiable along* Φ_1*-orbits, and that*

$$T_{\Phi_1}(x) = T_{\Phi_2}(\Psi(x)), \qquad \text{for each } x \in \mathrm{Per}(\Phi_1) \cap \Lambda_1. \tag{2.1}$$

Then the flows Φ_1, Φ_2 *are topologically conjugate, i.e., there exists a homeomorphism* $\Psi\colon \Lambda_1 \to \Lambda_2$ *such that*

$$\Psi \circ \Phi_1^t(x) = \Phi_2^t \circ \Psi(x), \quad \text{for all } (x,t) \in \Lambda_1 \times \mathbb{R}.$$

Proof. The proof is classical but we recall it here for completeness.

We fix an orbit equivalence $\Psi_0\colon \Lambda_1 \to \Lambda_2$ that is differentiable along Φ_1-orbits. Let X_1, X_2 be the respective flow vector fields of Φ_1, Φ_2, and let $L_{X_1}\Psi_0$ be the Lie derivative of Ψ_0 along Φ_1. As Ψ_0 sends Φ_1-orbits to Φ_2-orbits, it holds

$$L_{X_1}\Psi_0(x) = v_{\Psi_0}(x) X_2(\Psi_0(x)), \quad \text{for all } x \in \Lambda_1,$$

for some function $v_{\Psi_0}\colon \Lambda_1 \to \mathbb{R}$ which measures the "speed" of Ψ_0 along the flow direction. Observe that $v_{\Psi_0}(x) = \frac{d}{dt}|_{t=0}\theta(x,t)$.

By (2.1), for each $x \in \mathrm{Per}(\Phi_1) \cap \Lambda_1$ we have

$$\int_0^{T_{\Phi_1}(x)} dt = T_{\Phi_1}(x) = T_{\Phi_2}(\Psi_0(x)) = \int_0^{T_{\Phi_1}(x)} \frac{d}{ds}|_{s=0}\theta(\Phi_1^t(x), s)dt = \int_0^{T_{\Phi_1}(x)} v_{\Psi_0}(\Phi_1^t(x))dt,$$

hence

$$\frac{1}{T_{\Phi_1}(x)} \int_0^{T_{\Phi_1}(x)} \big(v_{\Psi_0}(\Phi_1^t(x)) - 1\big)dt = 0, \qquad \text{for each } x \in \mathrm{Per}(\Phi_1) \cap \Lambda_1.$$

We deduce from Livsic's theorem (see [30, Subsection 19.2]) that there exists a continuous function $u\colon \Lambda_1 \to \mathbb{R}$ differentiable along Φ_1-orbits such that $v_{\Psi_0} - 1 =$

$L_{X_1}u$. Let us set $\Psi\colon x\mapsto \Phi_2^{-u(x)}\circ\Psi_0(x)$. Given any $x\in\Lambda_1$, we compute

$$\begin{aligned}
v_\Psi(x)X_2(\Psi(x)) &= L_{X_1}\big(\Phi_2^{-u(x)}\circ\Psi_0\big)(x)\\
&= \lim_{t\to 0}\frac{1}{t}\Big(\Phi_2^{\theta(x,t)-u(\Phi_1^t(x))}\circ\Psi_0(x)-\Phi_2^{-u(x)}\circ\Psi_0(x)\Big)\\
&= X_2(\Phi_2^{-u(x)}\circ\Psi_0(x))\lim_{t\to 0}\frac{1}{t}\Big(\theta(x,t)-u(\Phi_1^t(x))+u(x)\Big)\\
&= \big(v_{\Psi_0}(x)-L_{X_1}u(x)\big)X_2\big(\Phi_2^{-u(x)}\circ\Psi_0(x)\big)=X_2(\Psi(x)),
\end{aligned}$$

i.e., $v_\Psi\equiv 1$ on Λ_1.

As a result, the homeomorphism Ψ is a flow conjugacy between Φ_1 and Φ_2 on Λ_1:

$$\Psi\circ\Phi_1^t(x)=\Phi_2^t\circ\Psi(x),\quad \text{for all } (x,t)\in\Lambda_1\times\mathbb{R}.$$

□

2.2. **Markov families for Axiom A flows on basic sets.** In this part, we recall some classical facts about Markov families for Axiom A flows on basic sets, following the presentation given in [6].

Let $k\geq 2$, and let $\Phi=(\Phi^t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^k$ Axiom A flow defined on a smooth manifold M.

Definition 2.2 (Rectangle, proper family)**.** A closed subset $R\subset M$ is called a *rectangle* if there is a small closed codimension one smooth disk $D\subset M$ transverse to the flow Φ such that $R\subset D$, and for any $x,y\in R$, the point

$$[x,y]_R:=D\cap\mathcal{W}^{cs}_{\Phi,\mathrm{loc}}(x)\cap\mathcal{W}^{cu}_{\Phi,\mathrm{loc}}(y)$$

exists and also belongs to R. A rectangle R is called *proper* if $R=\overline{\mathrm{int}(R)}$ in the topology of D. For any rectangle R and any $x\in R$, we let

$$\mathcal{W}^s_R(x):=R\cap\mathcal{W}^{cs}_{\Phi,\mathrm{loc}}(x),\quad \mathcal{W}^u_R(x):=R\cap\mathcal{W}^{cu}_{\Phi,\mathrm{loc}}(x).$$

A finite collection of proper rectangles $\mathcal{R}=\{R_1,\dots,R_m\}$, $m\geq 1$, is called a *proper family of size* $\varepsilon>0$ if:

(1) $M=\{\Phi^t(\mathcal{S}):t\in[-\varepsilon,0]\}$, where $\mathcal{S}:=R_1\cup\cdots\cup R_m$;
(2) $\mathrm{diam}(D_i)<\varepsilon$, for each $i=1,\dots,m$, where $D_i\supset R_i$ is a disk as above;
(3) for any $i\neq j$, $D_i\cap\{\Phi^t(D_j):t\in[0,\varepsilon]\}=\emptyset$ or $D_j\cap\{\Phi^t(D_i):t\in[0,\varepsilon]\}=\emptyset$.

The set $\mathcal{S}$ is called a *cross-section* of the flow Φ.

Notation 2.3. Let $\mathcal{R}=\{R_1,\dots,R_m\}$ be a proper family with $m\geq 1$ elements.

The cross-section $\mathcal{S}:=R_1\cup\cdots\cup R_m$ is associated with a Poincaré map $\mathcal{F}\colon\mathcal{S}\to\mathcal{S}$, where for any $x\in\mathcal{S}$, we let $\mathcal{F}(x):=\Phi^{\tau_\mathcal{S}(x)}(x)$, the function $\tau_\mathcal{S}\colon\mathcal{S}\to\mathbb{R}_+$ being the first return time on $\mathcal{S}$, i.e., $\tau_\mathcal{S}(x):=\inf\{t>0:\Phi^t(x)\in\mathcal{S}\}>0$, for all $x\in\mathcal{S}$.

Besides, for $*=s,u$ and $x\in R_i$, $i\in\{1,\dots,m\}$, we also let $\mathcal{W}^*_\mathcal{F}(x):=\mathcal{W}^*_{R_i}(x)$.

Definition 2.4 (Markov family)**.** Given some small $\varepsilon>0$, and some integer $m\geq 1$, a proper family $\mathcal{R}=\{R_1,\dots,R_m\}$ of size ε, with Poincaré map $\mathcal{F}$, is called a *Markov family* if it satisfies the following Markov property: for any $x\in\mathrm{int}(R_i)\cap\mathcal{F}^{-1}(\mathrm{int}(R_j))\cap\mathcal{F}(\mathrm{int}(R_k))$, with $i,j,k\in\{1,\dots,m\}$, it holds

$$\mathcal{W}^s_{R_i}(x)\subset\overline{\mathcal{F}^{-1}(\mathcal{W}^s_{R_j}(\mathcal{F}(x)))}\quad\text{and}\quad \mathcal{W}^u_{R_i}(x)\subset\overline{\mathcal{F}(\mathcal{W}^u_{R_k}(\mathcal{F}^{-1}(x)))}.$$

Theorem 2.5 (see Theorem 4.2 in [6])**.** *The restriction of an Axiom A flow to any basic set has a Markov family of arbitrary small size.*

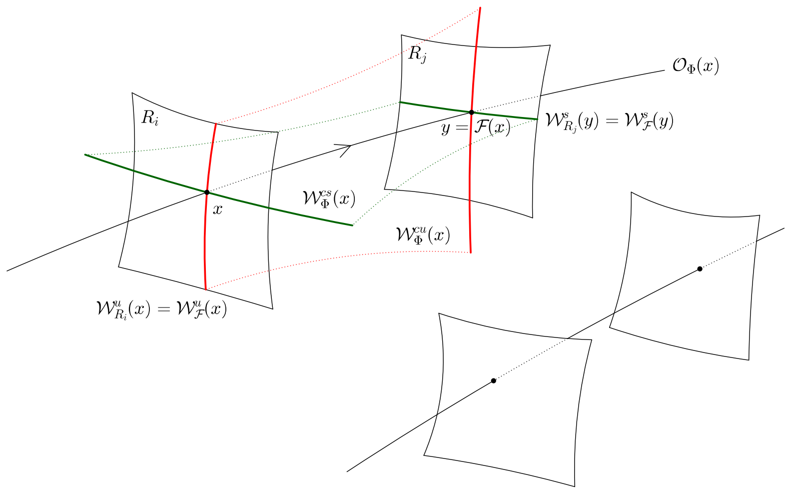

FIGURE 3. Markov family for the flow Φ.

2.3. **Quadrilaterals and temporal displacements.** Let $\Phi = (\Phi^t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^k$ Axiom A flow on a smooth manifold M, with $k \geq 2$, and fix a basic set Λ for Φ.

Definition 2.6 (Quadrilaterals)**.** A *quadrilateral* is a quadruple $\mathscr{Q} = (x_0, x_1, x_2, x_3) \subset \Lambda^4$ such that $x_1 \in \mathcal{W}^s_{\Phi,\mathrm{loc}}(x_0)$, $x_2 \in \mathcal{W}^u_{\Phi,\mathrm{loc}}(x_1)$ and $x_3 \in \mathcal{W}^s_{\Phi,\mathrm{loc}}(x_2) \cap \mathcal{W}^{cu}_{\Phi,\mathrm{loc}}(x_0)$. We let $x_4 = x_4(\mathscr{Q}) := \mathcal{W}^c_{\Phi,\mathrm{loc}}(x_0) \cap \mathcal{W}^u_{\Phi,\mathrm{loc}}(x_3)$. In particular, $x_4 = \Phi^t(x_0)$, for some time $t = t(\mathscr{Q}) \in \mathbb{R}$.

Let us consider a proper Markov family $\mathcal{R} = \{R_1, \dots, R_m\}$ for $\Phi_{|\Lambda}$ of size ε, for some integer $m \geq 1$ and some small $\varepsilon > 0$. Let $\mathcal{F}$ be the associated Poincaré map, and set $\mathcal{S} := R_1 \cup \dots \cup R_m$. We denote by $\overline{\Lambda} := \Lambda \cap \mathcal{S}$ the trace of Λ on $\mathcal{S}$.

We say that a quadrilateral $\mathscr{Q} = (x_0, x_1, x_2, x_3) \subset \Lambda^4$ is $\mathcal{R}$-*good* if $x_0 \in R_i$ for some $i = i(\mathscr{Q}) \in \{1, \dots, m\}$, and $x_j \in \cup_{t\in(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})} \Phi^t(R_i)$, for each $j \in \{1, \dots, 4\}$. Note that, up to time translation, there is no loss of generality to assume that $x_0 \in \mathcal{S}$. For any such quadrilateral, and for $j \in \{1, \dots, 4\}$, we denote by $\bar{x}_j$ the projection along the flow line of x_j on R_i, and we let $\overline{\mathscr{Q}} := (\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$. Note that $\bar{x}_0, \dots, \bar{x}_3 \in \overline{\Lambda}$; besides, $\bar{x}_1 \in \mathcal{W}^s_{R_i}(\bar{x}_0)$, $\bar{x}_2 \in \mathcal{W}^u_{R_i}(\bar{x}_1)$, and $\bar{x}_3 \in \mathcal{W}^s_{R_i}(\bar{x}_2) \cap \mathcal{W}^u_{R_i}(\bar{x}_0)$.

Definition 2.7 (s/u-holonomies)**.** Fix $i \in \{1, \dots, m\}$, and let $z_0, z_1 \in R_i \cap \Lambda$ be such that $z_1 \in \mathcal{W}^s_{R_i}(z_0)$. We define the *stable holonomy* $H^s_{\mathcal{S}}(z_0, z_1) \in \mathbb{R}$ as the time $t \in \mathbb{R}$ with smallest absolute value $|t|$ such that $\Phi^t(z_1) \in \mathcal{W}^s_{\Phi,\mathrm{loc}}(z_0)$. Similarly, for any $z_0, z_1 \in R_i \cap \Lambda$, $z_1 \in \mathcal{W}^u_{R_i}(z_0)$, we define the *unstable holonomy* $H^u_{\mathcal{S}}(z_0, z_1) \in \mathbb{R}$ as the time $t \in \mathbb{R}$ with smallest absolute value $|t|$ such that $\Phi^t(z_1) \in \mathcal{W}^u_{\Phi,\mathrm{loc}}(z_0)$.

Lemma 2.8. *For any $i \in \{1, \dots, m\}$, and for any $z_0, z_1 \in \mathcal{W}^s_{R_i}(z_0)$, it holds*

$$H^s_{\mathcal{S}}(z_0, z_1) = \sum_{j=0}^{+\infty} \tau_{\mathcal{S}}(\mathcal{F}^j(z_1)) - \tau_{\mathcal{S}}(\mathcal{F}^j(z_0)).$$

Proof. Fix $i \in \{1, \dots, m\}$, and let $z_0, z_1 \in R_i \cap \Lambda$ be such that $z_1 \in \mathcal{W}^s_{R_i}(z_0)$. We abbreviate $H := H^s_{\mathcal{S}}(z_0, z_1)$ and set $z_2 = \Phi^H(z_1)$. Fix $\varepsilon > 0$ arbitrarily small. As $z_1 \in \mathcal{W}^s_{R_i}(z_0)$ and $z_2 \in \mathcal{W}^s_{\Phi,\mathrm{loc}}(z_0)$, for $n \gg 1$ sufficiently large, it holds

$$\begin{aligned} d(\mathcal{F}^n(z_0), \mathcal{F}^n(z_1)) &< \varepsilon, \\ d(\Phi^{t_n}(z_0), \Phi^{t_n}(z_2)) &< \varepsilon, \end{aligned} \tag{2.2}$$

with $\mathcal{F}^n(z_0) = \Phi^{t_n}(z_0)$ and $t_n := \sum_{j=0}^{n-1} \tau_{\mathcal{S}}(\mathcal{F}^j(z_0))$. Set $u_n := \sum_{j=0}^{n-1} \tau_{\mathcal{S}}(\mathcal{F}^j(z_1))$, so that $\mathcal{F}^n(z_1) = \Phi^{u_n}(z_1)$. The points $\mathcal{F}^n(z_0), \mathcal{F}^n(z_1)$ are exponentially close, and $\tau_{\mathcal{S}}$ is Lipschitz, hence the sequence $(u_n - t_n)_{n \geq 1}$ converges to some limit $\ell \in \mathbb{R}$. Since $z_2 = \Phi^H(z_1)$, and by the triangular inequality, (2.2) yields

$$d(\Phi^{u_n}(z_1), \Phi^{t_n + H}(z_1)) < 2\varepsilon.$$

As we are considering local manifolds, we deduce that $|u_n - t_n - H| < C\varepsilon$, for some uniform constant $C > 0$. Letting $n \to +\infty$, we get $\ell = H$, i.e.,

$$H = \sum_{j=0}^{+\infty} \tau_{\mathcal{S}}(\mathcal{F}^j(z_1)) - \tau_{\mathcal{S}}(\mathcal{F}^j(z_0)).$$

□

Using the same ideas as in Lemma 2.8, we have the following

Lemma 2.9. *For any $i \in \{1, \dots, m\}$, and for any $z_0, z_1 \in \mathcal{W}^u_{R_i}(z_0)$, it holds*

$$H^u_{\mathcal{S}}(z_0, z_1) = \sum_{j=-\infty}^{-1} \tau_{\mathcal{S}}(\mathcal{F}^j(z_0)) - \tau_{\mathcal{S}}(\mathcal{F}^j(z_1)).$$

Let $\mathscr{Q} = (x_0, x_1, x_2, x_3) \subset \Lambda^4$ be a $\mathcal{R}$-good quadrilateral, with $x_0 \in R_i$, $i \in \{1, \dots, m\}$. Let $x_4 = x_4(\mathscr{Q})$, and let $\overline{\mathscr{Q}} := (\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$. As $\bar{x}_1 \in \mathcal{W}^s_{R_i}(\bar{x}_0)$, $\bar{x}_2 \in \mathcal{W}^u_{R_i}(\bar{x}_1)$, and $\bar{x}_3 \in \mathcal{W}^s_{R_i}(\bar{x}_2) \cap \mathcal{W}^u_{R_i}(\bar{x}_0)$, we may define the *temporal displacement* $H(\mathscr{Q}) \in \mathbb{R}$ as

$$H(\mathscr{Q}) := H^s_{\mathcal{S}}(\bar{x}_0, \bar{x}_1) + H^u_{\mathcal{S}}(\bar{x}_1, \bar{x}_2) + H^s_{\mathcal{S}}(\bar{x}_2, \bar{x}_3) + H^u_{\mathcal{S}}(\bar{x}_3, \bar{x}_0). \tag{2.3}$$

By Lemma 2.8 and 2.9, we have:

$$\begin{aligned} H(\mathscr{Q}) &= \lim_{n \to +\infty} \Big[\sum_{j=-n}^{n} -\tau_{\mathcal{S}}(\mathcal{F}^j(\bar{x}_0)) + \tau_{\mathcal{S}}(\mathcal{F}^j(\bar{x}_1)) - \tau_{\mathcal{S}}(\mathcal{F}^j(\bar{x}_2)) + \tau_{\mathcal{S}}(\mathcal{F}^j(\bar{x}_3)) \Big] \\ &= \lim_{n \to +\infty} \big[-\tau^n_{\mathcal{S}}(\bar{x}_0) + \tau^n_{\mathcal{S}}(\bar{x}_1) - \tau^n_{\mathcal{S}}(\bar{x}_2) + \tau^n_{\mathcal{S}}(\bar{x}_3) \big], \end{aligned} \tag{2.4}$$

where for any point $z \in \mathcal{S}$, and for any integer $n \geq 0$, we let

$$\tau^n_{\mathcal{S}}(z) := \sum_{j=-n}^{n} \tau_{\mathcal{S}}(\mathcal{F}^j(z)). \tag{2.5}$$

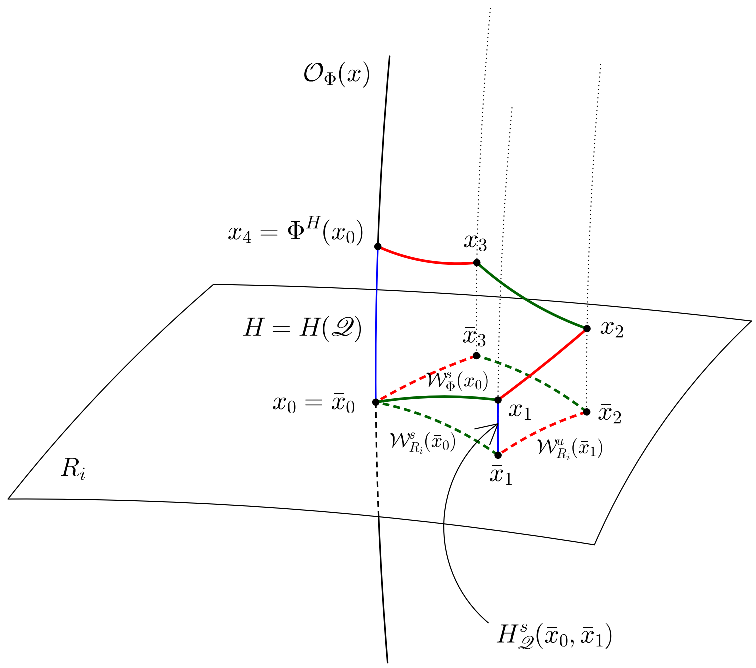

Figure 4. Quadrilaterals and temporal displacements.

2.4. **Periodic approximations of temporal displacements.** Let us recall the following fact.

Lemma 2.10. *For each $i \in \{1, \dots, m\}$, the stable holonomies $H^s_{\mathcal{S}}(y_0, y_1)$, resp. unstable holonomies $H^u_{\mathcal{S}}(z_0, z_1)$, depend continuously on the points $y_0, y_1 \in R_i \cap \Lambda$, $y_1 \in \mathcal{W}^s_{R_i}(y_0)$, resp. on the points $z_0, z_1 \in R_i \cap \Lambda$, $z_1 \in \mathcal{W}^u_{R_i}(z_0)$.*

Proof. Let us consider the case where $y_0, y_1 \in R_i \cap \Lambda$, $y_1 \in \mathcal{W}^s_{R_i}(y_0)$, the other case is analogous. By definition, the stable holonomy $H^s_{\mathcal{S}}(y_0, y_1)$ satisfies

$$\mathcal{W}^s_{\Phi,\mathrm{loc}}(y_0) \cap \mathcal{W}^c_{\Phi,\mathrm{loc}}(y_1) = \{\Phi^{H^s_{\mathcal{S}}(y_0,y_1)}(y_1)\}.$$

As the invariant manifolds vary continuously, the intersection of the two sets on the left hand side depends continuously on the pair y_0, y_1, with $y_1 \in \mathcal{W}^s_{R_i}(y_0)$. By looking at the right hand side, we conclude that the holonomies are continuous. □

The main goal of this section is to show the following proposition, whose content already appears in the work of Otal [43].

Proposition 2.11. *For any $\mathcal{R}$-good quadrilateral $\mathscr{Q} = (x_0, x_1, x_2, x_3) \in \Lambda^4$, the quantity $H(\mathscr{Q})$ is determined by the lengths of periodic orbits.*

Proposition 2.11 is a direct outcome of Lemma 2.12 and Proposition 2.13 below.

Lemma 2.12. *For any $\mathcal{R}$-good quadrilateral $\mathscr{Q} = (x_0, x_1, x_2, x_3) \in \Lambda^4$, there exists a sequence $(\mathscr{Q}^n)_{n\in\mathbb{N}} \in (\Lambda^4)^{\mathbb{N}}$ of $\mathcal{R}$-good quadrilaterals $\mathscr{Q}^n = (x^n_0, x^n_1, x^n_2, x^n_3)$ with*

$x_0^n, x_2^n \in \mathrm{Per}(\Phi)$ *such that* $\lim_{n\to+\infty} \mathscr{Q}^n = \mathscr{Q}$*, i.e.,* $\lim_{n\to\infty} x_j^n = x_j$*, for each* $j = 0,\dots,3$*. In particular, it holds*

$$H(\mathscr{Q}) = \lim_{n\to+\infty} H(\mathscr{Q}^n).$$

Proof. Fix a $\mathcal{R}$-good quadrilateral $\mathscr{Q} = (x_0, x_1, x_2, x_3) \in \Lambda^4$, with $x_0 \in R_i$, $i \in \{1,\dots,m\}$, and let $\overline{\mathscr{Q}} := (\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$ be the projection of $\mathscr{Q}$ on R_i as before.

As periodic points are dense in Λ, for $j = 0, 2$, there exists a sequence $(\bar{x}_j^n)_{n\in\mathbb{N}} \in \big(\mathrm{Per}(\Phi) \cap R_i\big)^{\mathbb{N}}$ of periodic points such that $\lim_{n\to+\infty} \bar{x}_j^n = \bar{x}_j$. Let $\bar{x}_1^n := [\bar{x}_0^n, \bar{x}_2^n]_{R_i}$ and $\bar{x}_3^n := [\bar{x}_2^n, \bar{x}_0^n]_{R_i}$, so that the lift $\mathscr{Q}^n := (x_0^n, x_1^n, x_2^n, x_3^n)$ of $\overline{\mathscr{Q}}^n := (\bar{x}_0^n, \bar{x}_1^n, \bar{x}_2^n, \bar{x}_3^n)$ is a $\mathcal{R}$-good quadrilateral, where

$$x_0^n := \bar{x}_0^n, \qquad x_1^n := \Phi^{H^s_{\mathcal{S}}(\bar{x}_0^n, \bar{x}_1^n)}(\bar{x}_1^n),$$
$$x_2^n := \Phi^{H^s_{\mathcal{S}}(\bar{x}_0^n, \bar{x}_1^n) + H^u_{\mathcal{S}}(\bar{x}_1^n, \bar{x}_2^n)}(\bar{x}_2^n), \qquad x_3^n := \Phi^{H^s_{\mathcal{S}}(\bar{x}_0^n, \bar{x}_1^n) + H^u_{\mathcal{S}}(\bar{x}_1^n, \bar{x}_2^n) + H^s_{\mathcal{S}}(\bar{x}_2^n, \bar{x}_3^n)}(\bar{x}_3^n),$$

and $x_0^n, x_2^n \in \mathrm{Per}(\Phi)$. Clearly, we have $\lim_{n\to+\infty} \mathscr{Q}^n = \mathscr{Q}$. By the definition (2.3) of temporal displacements in terms of holonomies, and by Lemma 2.10, the function $\widetilde{\mathscr{Q}} \mapsto H(\widetilde{\mathscr{Q}})$ is continuous. Thus, we conclude that $H(\mathscr{Q}) = \lim_{n\to+\infty} H(\mathscr{Q}^n)$. □

Proposition 2.13. *For any* $\mathcal{R}$*-good quadrilateral* $\mathscr{Q} = (x_0, x_1, x_2, x_3) \in \Lambda^4$ *such that* $x_0, x_2 \in \mathrm{Per}(\Phi)$*, the quantity* $H(\mathscr{Q})$ *is determined by the lengths of periodic orbits. More precisely, there exists a sequence* $(\bar{x}^n)_{n\in\mathbb{N}} \in \mathrm{Per}(\Phi)^{\mathbb{N}}$ *of periodic points such that for any* $\varepsilon > 0$*, there exists an integer* $N_0(\varepsilon) \in \mathbb{N}$ *such that*

$$\Big| H(\mathscr{Q}) - \big[T_\Phi(\bar{x}^n) - (4n+1)T_\Phi(x_0) - (4n+1)T_\Phi(x_2)\big] \Big| < \varepsilon, \quad \forall\, n \geq N_0(\varepsilon).$$

Proof. Fix a $\mathcal{R}$-good quadrilateral $\mathscr{Q} = (x_0, x_1, x_2, x_3) \in \Lambda^4$, with $x_0 \in R_i$, $i \in \{1,\dots,m\}$ and $x_0, x_2 \in \mathrm{Per}(\Phi)$, and let $\overline{\mathscr{Q}} := (\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$ be the projection of $\mathscr{Q}$ on R_i. Let us choose $p \geq 1$ sufficiently large such that $\mathcal{F}^p(\bar{x}_0) = \bar{x}_0$ and $\mathcal{F}^p(\bar{x}_2) = \bar{x}_2$. After replacing $\tau_{\mathcal{S}}$ with

$$\tau^p_{+,\mathcal{S}}(\cdot) := \sum_{j=0}^{p-1} \tau_{\mathcal{S}} \circ \mathcal{F}^j(\cdot),$$

we may thus assume that x_0, x_2 are fixed points under the Poincaré map $\mathcal{F}$.

Note that $\bar{x}_1 = [\bar{x}_0, \bar{x}_2]_{R_i}$ and $\bar{x}_3 = [\bar{x}_2, \bar{x}_0]_{R_i}$ are heteroclinic intersections between the invariant manifolds of the fixed points $\bar{x}_0, \bar{x}_2$.

As Λ is a basic set, the dynamics can be coded symbolically, using some finite alphabet $\mathscr{A}$. In the following, for each finite word σ in $\mathscr{A}$, we denote by $|\sigma| \in \mathbb{N}$ the length of σ.

The fixed (periodic) points $\bar{x}_0$, $\bar{x}_2$, correspond to a symbol (a finite sequence of symbols) σ_0, σ_2 respectively. The point $\bar{x}_1$ is a heteroclinic intersection between $\mathcal{W}^s_{R_i}(\bar{x}_0)$ and $\mathcal{W}^u_{R_i}(\bar{x}_2)$, hence there exist a symbol $\sigma_0^1 \in \mathscr{A}$ and two finite words σ_-^1, σ_+^1 in $\mathscr{A}$ such that the symbolic coding of $\bar{x}_1$ is

$$\bar{x}_1 \longleftrightarrow \dots \sigma_2\sigma_2\sigma_-^1 \underset{\uparrow}{\sigma_0^1} \sigma_+^1\sigma_0\sigma_0 \dots$$

Similarly, there exist a symbol $\sigma_0^3 \in \mathscr{A}$ and two finite words σ_-^3, σ_+^3 in $\mathscr{A}$ such that the symbolic coding of $\bar{x}_3$ is

$$\bar{x}_3 \longleftrightarrow \dots \sigma_0\sigma_0\sigma_-^3 \underset{\uparrow}{\sigma_0^3} \sigma_+^3\sigma_2\sigma_2 \dots$$

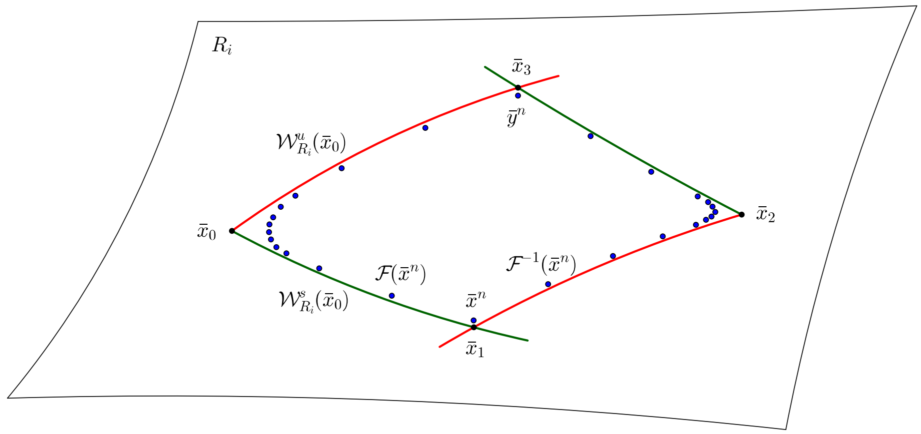

FIGURE 5. Approximating periodic orbits with a prescribed combinatorics.

Up to redefining σ_+^1 as $\sigma_+^1 \underbrace{\sigma_0 \dots \sigma_0}_{n-|\sigma_+^1|}$, without loss of generality, we can assume that $|\sigma_+^1| = n$. Similarly, we can assume that $|\sigma_-^1| = |\sigma_+^3| = |\sigma_-^3| = n$.

For each integer $n \geq 0$, we define a periodic point $\bar{x}^n$ whose symbolic coding is given by the infinite word $\dots(\sigma_0^1\sigma^n)(\underset{\uparrow}{\sigma_0^1}\sigma^n)(\sigma_0^1\sigma^n)\dots$, where

$$\sigma^n := \sigma_+^1 \underbrace{\sigma_0 \dots \sigma_0}_{2n} \sigma_-^3 \sigma_0^3 \sigma_+^3 \underbrace{\sigma_2 \dots \sigma_2}_{2n} \sigma_-^1.$$

Thus, the point of $\bar{x}^n$ is a periodic point, of period

$$2 + 4n + |\sigma_-^1| + |\sigma_+^1| + |\sigma_-^3| + |\sigma_+^3| = 2 + 8n.$$

Lemma 2.14. *For any $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$ such that for each integer $n \geq N$, the following inequalities hold:*

$$\left| \sum_{k=-2n}^{2n} \Big[\tau_{\mathcal{S}}(\mathcal{F}^k(\bar{x}^n)) - \tau_{\mathcal{S}}(\mathcal{F}^k(\bar{x}_1)) \Big] \right| < \varepsilon, \tag{2.6}$$

$$\left| \sum_{k=-2n}^{2n} \Big[\tau_{\mathcal{S}}(\mathcal{F}^k(\bar{y}^n)) - \tau_{\mathcal{S}}(\mathcal{F}^k(\bar{x}_3)) \Big] \right| < \varepsilon, \tag{2.7}$$

where we have set $\bar{y}^n := \mathcal{F}^{4n+1}(\bar{x}^n)$, and for each $M \geq 2n+1$, we have

$$\left| \sum_{k=2n+1}^{M} \Big[\tau_{\mathcal{S}}(\mathcal{F}^k(\bar{x}_1)) - T_\Phi(\bar{x}_0) \Big] \right| + \left| \sum_{k=-M}^{-2n-1} \Big[\tau_{\mathcal{S}}(\mathcal{F}^k(\bar{x}_1)) - T_\Phi(\bar{x}_2) \Big] \right| < \varepsilon, \tag{2.8}$$

$$\left| \sum_{k=2n+1}^{M} \Big[\tau_{\mathcal{S}}(\mathcal{F}^k(\bar{x}_3)) - T_\Phi(\bar{x}_2) \Big] \right| + \left| \sum_{k=-M}^{-2n-1} \Big[\tau_{\mathcal{S}}(\mathcal{F}^k(\bar{x}_3)) - T_\Phi(\bar{x}_0) \Big] \right| < \varepsilon. \tag{2.9}$$

Observe that the two sums in (2.6) *and in* (2.7) *add up to*

$$\sum_{k=-2n}^{2n} \tau_{\mathcal{S}}(\mathcal{F}^k(\bar{x}^n)) + \sum_{k=-2n}^{2n} \tau_{\mathcal{S}}(\mathcal{F}^k(\bar{y}^n)) = T_\Phi(\bar{x}^n). \tag{2.10}$$

Proof of Lemma 2.14. By looking at the symbolic codings of $\bar{x}^n$ and $\bar{x}_1$, we see that they have the same symbolic past (resp. future) for at least $3n$ steps of iterations under $\mathcal{F}$. By hyperbolicity of $\mathcal{F}$, for some constant $\lambda \in (0,1)$, we thus have

$$d\big(\mathcal{F}^k(\bar{x}^n), \mathcal{F}^k(\bar{x}_1)\big) = O(\lambda^n), \quad \forall\, k \in \{-2n, \dots, 2n\}. \tag{2.11}$$

Indeed, without loss of generality (after possibly iterating n_0 times, for some integer $n_0 \geq 1$ independent of n), we may assume that each of these points belongs to some small neighborhood of $\bar{x}_0$ where the dynamics is conjugated to the differential $D\mathcal{F}(\bar{x}_0)$. More precisely, by Lemma 23 in [27], for any $\delta > 0$, and for $j = 0, 2$, there exist a neighborhood $\mathcal{U}_j$ of $\bar{x}_j$, a neighborhood $\mathcal{V}_j \subset \mathbb{R}^2$ of $(0,0)$, and a $\mathcal{C}^{1,\frac{1}{2}}$-diffeomorphism $\chi_j \colon \mathcal{U}_j \to \mathcal{V}_j$, such that

$$\chi_j \circ \mathcal{F} \circ \chi_j^{-1} = D\mathcal{F}(\bar{x}_j), \qquad \|\chi_j - \mathrm{id}\|_{\mathcal{C}^1} \leq \delta, \qquad \|\chi_j^{-1} - \mathrm{id}\|_{\mathcal{C}^1} \leq \delta.$$

By (2.11), summing over all the indices $k \in \{-2n, \dots, 2n\}$, and as $\tau_{\mathcal{S}}$ is Lipschitz continuous, the left hand side in (2.6) is of order at most $O(n\lambda^n)$; therefore, for n sufficiently large, this term is smaller than ε. Inequality (2.7) is proved similarly.

Finally, (2.8) is proved using the same linearizing coordinates near $\bar{x}_0$ and $\bar{x}_2$, noting that $\bar{x}_1 \in \mathcal{W}^s_{R_i}(\bar{x}_0)$, resp. $\bar{x}_1 \in \mathcal{W}^u_{R_i}(\bar{x}_2)$, so that $\bar{x}_1$ has the same future as $\bar{x}_0$, resp. the same past as $\bar{x}_2$, hence all of its future, resp. past iterates (after iterating finitely many times) belong to the neighborhood $\mathcal{U}_0$ of $\bar{x}_0$, resp. to the neighborhood $\mathcal{U}_2$ of $\bar{x}_2$, endowed with linearizing coordinates. We argue similarly for (2.9), which concludes the proof of Lemma 2.14. □

Let us now conclude the proof of Proposition 2.13. Fix some small $\varepsilon > 0$. By (2.4), for $m \geq 1$ sufficiently large, we have

$$\left| H(\mathscr{Q}) - \sum_{j=-m}^{m} \Big[-\tau_{\mathcal{S}}(\mathcal{F}^j(\bar{x}_0)) + \tau_{\mathcal{S}}(\mathcal{F}^j(\bar{x}_1)) - \tau_{\mathcal{S}}(\mathcal{F}^j(\bar{x}_2)) + \tau_{\mathcal{S}}(\mathcal{F}^j(\bar{x}_3)) \Big] \right| < \frac{\varepsilon}{2}.$$

By Lemma 2.14, there exists a periodic point $\bar{x}^n \in \mathrm{Per}(\Phi)$ such that inequalities (2.6),(2.7),(2.8),(2.9) hold for $\bar{x}^n$ and $\frac{\varepsilon}{8}$ in place of ε. Splitting the different sums of return times to match these inequalities, and thanks to (2.10), we conclude that

$$\Big| H(\mathscr{Q}) - \big[T_\Phi(\bar{x}^n) - (4n+1)T_\Phi(x_0) - (4n+1)T_\Phi(x_2) \big] \Big| < \varepsilon,$$

as desired. □

2.5. **Temporal displacements and areas of quadrilaterals.** Assume that there exists a smooth contact form α on M that is adapted to the basic set Λ in the sense of Definition 1.2. Recall the following fact:

Lemma 2.15. *We have $E^s_\Phi(x) \subset \ker \alpha(x)$, for all $x \in \mathcal{W}^s_\Phi(\Lambda)$, and $E^u_\Phi(x) \subset \ker \alpha(x)$, for all $x \in \mathcal{W}^u_\Phi(\Lambda)$. In particular, it holds*

$$E^s_\Phi(x) \oplus E^u_\Phi(x) = \ker \alpha(x), \quad \forall\, x \in \Lambda. \tag{2.12}$$

Proof. Let $\Gamma = \{\gamma(t) \in t \in [0,1]\} \subset \mathcal{W}^s_{\Phi,\mathrm{loc}}(x)$ be an arc in the local stable manifold of some point $x \in \Lambda$. For each $T > 0$, we have

$$\int_\Gamma \alpha = \int_0^1 \alpha(\gamma(t))(\gamma'(t))dt = \int_0^1 \alpha(\Phi^T \circ \gamma(t))(D\Phi^T(\gamma(t)) \cdot \gamma'(t))dt = \int_{\Phi^T \circ \Gamma} \alpha.$$

As α is uniformly bounded, and $\lim_{T\to+\infty} D\Phi^T(\gamma(t)) \cdot \gamma'(t) \to 0$, for each $t \in [0,1]$, we deduce that $\int_\Gamma \alpha = 0$. Therefore, we have $E^s_\Phi(y) \subset \ker \alpha(y)$, for any $y \in \mathcal{W}^s_\Phi(x)$, $x \in \Lambda$. We argue similarly for the unstable direction.

Let $x \in \Lambda$. The identity (2.12) follows from the inclusions $E^s_\Phi(x) \subset \ker \alpha(x)$, $E^u_\Phi(x) \subset \ker \alpha(x)$, and the equality of the dimensions of the two subspaces. □

Let $\mathscr{Q} = (x_0, x_1, x_2, x_3) \in \Lambda^4$ be a $\mathcal{R}$-good quadrilateral, with $x_0 \in R_i$, for some $i \in \{1, \dots, m\}$, and let $\overline{\mathscr{Q}} := (\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$ be the projection of $\mathscr{Q}$ on R_i. We define $\widehat{\mathscr{Q}}$ as the set of all points $x \in R_i$ in the closed region bounded by the arcs $\bar{\Gamma}_0$, $\bar{\Gamma}_1$, $\bar{\Gamma}_2$, $\bar{\Gamma}_3$, where for $j = 0, 2$, $\bar{\Gamma}_j \subset \mathcal{W}^s_{R_i}(\bar{x}_j)$ is the stable arc connecting $\bar{x}_j$ to $\bar{x}_{j+1}$, while $\bar{\Gamma}_{j+1} \subset \mathcal{W}^u_{R_i}(\bar{x}_{j+1})$ is the unstable arc connecting $\bar{x}_{j+1}$ to $\bar{x}_{j+2}$, with $\bar{x}_4 := \bar{x}_0$. The set $\widehat{\mathscr{Q}} \subset R_i$ is transverse to the flow direction, i.e.,

$$X(x) \notin T_x \widehat{\mathscr{Q}}, \text{ for each } x \in \widehat{\mathscr{Q}}, \tag{2.13}$$

which ensures that $d\alpha|_{\widehat{\mathscr{Q}}}$ is non-degenerate. Let us define

$$\mathrm{Area}(\mathscr{Q}) := \int_{\widehat{\mathscr{Q}}} d\alpha.$$

Proposition 2.16. *Let $\mathscr{Q} = (x_0, x_1, x_2, x_3) \in \Lambda^4$ be a small quadrilateral, so that $\mathscr{Q}$ is $\mathcal{R}$-good and (2.13) is satisfied. Then*

$$\mathrm{Area}(\mathscr{Q}) = -H(\mathscr{Q}).$$

Proof. By Stokes theorem, we have

$$\mathrm{Area}(\mathscr{Q}) = \int_{\widehat{\mathscr{Q}}} d\alpha = \sum_{j=0,\dots,3} \int_{\bar{\Gamma}_j} \alpha.$$

By the definition (2.3) of $H(\mathscr{Q})$ in terms of holonomies, it is sufficient to show that $\int_{\bar{\Gamma}_0} \alpha = -H^s_{\mathcal{S}}(\bar{x}_0, \bar{x}_1)$, $\int_{\bar{\Gamma}_1} \alpha = -H^u_{\mathcal{S}}(\bar{x}_1, \bar{x}_2)$, $\int_{\bar{\Gamma}_2} \alpha = -H^s_{\mathcal{S}}(\bar{x}_2, \bar{x}_3)$, and $\int_{\bar{\Gamma}_3} \alpha = -H^u_{\mathcal{S}}(\bar{x}_3, \bar{x}_0)$. Let us prove the formula for $\bar{\Gamma}_0$, the others are proved similarly.

Let Γ^s_0 be the arc of the stable manifold $\mathcal{W}^s_{\Phi,\mathrm{loc}}(x_0)$ connecting x_0 to x_1, and let Γ^c_1 be the orbit segment $\Gamma^c_1 := \{\Phi^t(x_1)\}_{t\in[0,\mathcal{H}^s_{\mathcal{S}}(\bar{x}_0,\bar{x}_1)]} \subset \mathcal{W}^c_{\Phi,\mathrm{loc}}(x_1)$. We define $\mathcal{T}_0 \subset M$ as the set of all points $x \in \mathcal{W}^{cs}_{\Phi,\mathrm{loc}}(x_0)$ in the closed region bounded by the arcs $\bar{\Gamma}_0, \Gamma^s_0, \Gamma^c_1$, see Figure 6. By Stokes theorem, we have

$$\int_{\mathcal{T}_0} d\alpha = \int_{\bar{\Gamma}_0} \alpha - \int_{\Gamma^s_0} \alpha + \int_{\Gamma^c_1} \alpha.$$

Since $X|_{\mathcal{W}^{cs}_\Phi(\Lambda)} \in \ker d\alpha|_{\mathcal{W}^{cs}_\Phi(\Lambda)}$, it holds that $\int_{\mathcal{T}_0} d\alpha = 0$. By Lemma 2.15, we have $\int_{\Gamma^s_0} \alpha = 0$, hence,

$$\int_{\bar{\Gamma}_0} \alpha = -\int_{\Gamma^c_1} \alpha.$$

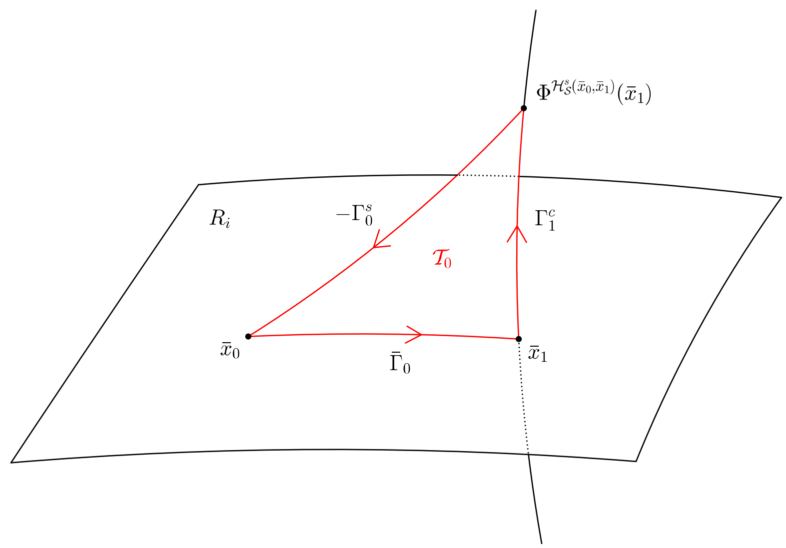

FIGURE 6. $\mathcal{T}_0$ is the closed region bounded by the arcs $\bar{\Gamma}_0, -\Gamma_0^s, \Gamma_1^c$.

Moreover,

$$\int_{\Gamma_1^c} \alpha = \int_0^{H^s_{\mathcal{S}}(\bar{x}_0,\bar{x}_1)} \alpha(X(\Phi^t(\bar{x}_1)))dt.$$

Since $\imath_X\alpha|_\Lambda \equiv 1$, we also have $\int_0^{H^s_{\mathcal{S}}(\bar{x}_0,\bar{x}_1)} \alpha(X(\Phi^t(\bar{x}_1)))dt = H^s_{\mathcal{S}}(\bar{x}_0,\bar{x}_1)$, which concludes. $\square$

As an immediate consequence of Proposition 2.11 and Proposition 2.16, we thus obtain:

Corollary 2.17. *For any small quadrilateral $\mathscr{Q} = (x_0, x_1, x_2, x_3) \in \Lambda^4$, the quantity* $\mathrm{Area}(\mathscr{Q})$ *is determined by the lengths of periodic orbits.*

Corollary 2.18. *Fix $k \geq 2$. For $i = 1, 2$, let $\Phi_i = (\Phi_i^t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^k$ Axiom A flow defined on a smooth 3-manifold M_i. Let Λ_i be a basic set for Φ_i, and let α_i be a smooth contact form adapted to Λ_i. If there exists an iso-length-spectral flow conjugacy $\Psi\colon \Lambda_1 \to \Lambda_2$ between $\Phi_1|_{\Lambda_1}$ and $\Phi_2|_{\Lambda_2}$, then, for any point $x_0 \in \Lambda_1$, and for any small quadrilateral $\mathscr{Q} = (x_0, x_1, x_2, x_3) \in \Lambda_1^4$, it holds*

$$\mathrm{Area}(\mathscr{Q}) = \mathrm{Area}(\Psi(\mathscr{Q})),$$

where $\Psi(\mathscr{Q})$ is the quadrilateral $\Psi(\mathscr{Q}) := (\Psi(x_0), \Psi(x_1), \Psi(x_2), \Psi(x_3)) \in \Lambda_2^4$.

2.6. **Smoothness of the conjugacy.** In the following, we fix a point $x_0 \in \Lambda \cap R_i$, for some $i \in \{1, \dots, m\}$. Let Q_0 be the set of all sufficiently small quadrilaterals $\mathscr{Q} = (x_0, x_1, x_2, x_3) \in \Lambda^4$ based at x_0. The goal of this part is to show that the set of areas $\{\mathrm{Area}(\mathscr{Q})\}_{\mathscr{Q}\in Q_0}$ determines the "infinitesimal" shape of the set $\Lambda \cap \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0)$, resp. $\Lambda \cap \mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0)$.

In particular, given another Axiom A flow whose restriction to some basic set is conjugate to $\Phi|_\Lambda$ by some homeomorphism Ψ, and such that, for any small quadrilateral $\mathscr{Q}$, it holds $\mathrm{Area}(\mathscr{Q}) = \mathrm{Area}(\Psi(\mathscr{Q}))$, we show that Ψ is differentiable at any point of Λ, with Hölder continuous differential.

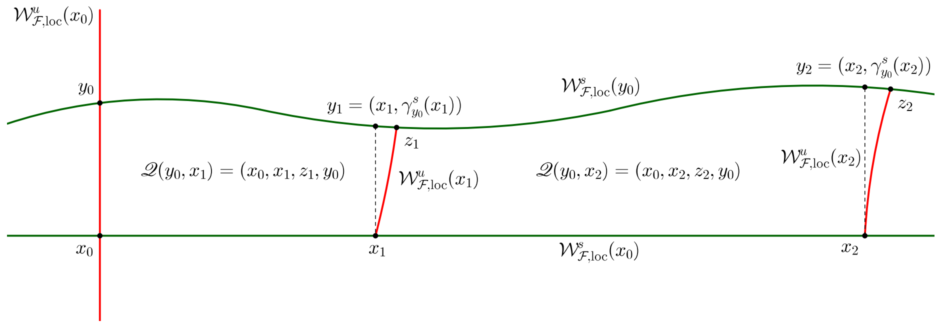

FIGURE 7. Small quadrilaterals.

We take a chart $\mathcal{R} = \mathcal{R}_{x_0} : \mathcal{U}_0 \to \mathcal{V}_0$ from a neighborhood $\mathcal{U}_0 \subset R_i$ of x_0 to a neighborhood $\mathcal{V}_0 \subset \mathbb{R}^2$ of $\{0_{\mathbb{R}^2}\}$ such that $\mathcal{R}(\mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0) \cap \mathcal{U}_0) \subset (\mathbb{R} \times \{0\}) \cap \mathcal{V}_0$ and $\mathcal{R}(\mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0) \cap \mathcal{U}_0) \subset (\{0\} \times \mathbb{R}) \cap \mathcal{V}_0$. In the following, we thus identify $\mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0)$, resp. $\mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0)$ with the horizontal, resp. vertical coordinate axis of $\mathbb{R}^2$. Moreover, for any point $v = \mathcal{R}(u) \in \mathcal{V}_0$, we denote by $\rho(v) d\xi \wedge d\eta := \mathcal{R}_*(d\alpha_u)$ the corresponding area form. For each point $y_0 \in \mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0)$, we see $\mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(y_0)$ as the graph of some function $\gamma^s_{y_0}$ over the horizontal axis. By an abuse of notation, in the following, we identify an object and its image in the chart $\mathcal{R}$. For instance, a point $x_1 \in \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0)$ will be identified with the point $(x_1, 0) \in \mathbb{R}^2$; besides, we will denote by $V_{x_1} := \big(\{x_1\} \times \mathbb{R}\big) \cap \mathcal{V}_0$ the vertical segment in $\mathcal{V}_0$ passing through $x_1 \simeq (x_1, 0)$.

Definition 2.19 (Holonomy maps for $\mathcal{W}^s_{\mathcal{F},\mathrm{loc}}$)**.** For any points $y_0 \in \mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0) \cap \Lambda$ and $x_1 \in \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0)$, we define the point $\mathcal{H}^s_{x_0,x_1}(y_0) \in V_{x_1}$ as

$$\{\mathcal{H}^s_{x_0,x_1}(y_0)\} := \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(y_0) \cap V_{x_1} = (x_1, \gamma^s_{y_0}(x_1)).$$

In other words, the map $\mathcal{H}^s_{x_0,x_1}$ is the holonomy map along $\mathcal{W}^s_{\mathcal{F},\mathrm{loc}}$ from $\mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0) \cap \Lambda \simeq V_{x_0}$ to V_{x_1}. To ease the notation, we also abbreviate $y_1 = y_1(y_0) := \mathcal{H}^s_{x_0,x_1}(y_0)$. Note that, *a priori*, $y_1 \notin \Lambda$.

Lemma 2.20. *There exists a continuous function $C = C_{x_0} : \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0) \to \mathbb{R}$ such that for any $x_1 \in \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0)$, it holds*

$$\lim_{\mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0) \cap \Lambda \ni y_0 \to x_0} \frac{d(y_1(y_0), x_1)}{d(y_0, x_0)} = \lim_{\mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0) \cap \Lambda \ni y_0 \to x_0} \frac{\gamma^s_{y_0}(x_1)}{\gamma^s_{y_0}(x_0)} = C(x_1). \tag{2.14}$$

Moreover, it holds

$$\lim_{\mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0) \ni x_1 \to x_0} C(x_1) = 1.$$

Proof. According to Remark 1.6, stable holonomy maps are $\mathcal{C}^{1,\beta}$, for some $\beta \in (0,1)$. For any $y_0 \in \mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0) \cap \Lambda$, we let $y_1 = y_1(y_0) \in V_{x_1}$ be defined as above.

As $\mathcal{H}^s_{x_0,x_1}(x_0) = x_1$ and $\mathcal{H}^s_{x_0,x_1}(y_0) = y_1$, the quotient in (2.14) can be written as $\frac{d(\mathcal{H}^s_{x_0,x_1}(y_0),\mathcal{H}^s_{x_0,x_1}(x_0))}{d(y_0,x_0)}$. From the definition of $\gamma^s_{y_0}$, this quantity is also equal to $\frac{\gamma^s_{y_0}(x_1)}{\gamma^s_{y_0}(x_0)}$. Moreover, it has a limit as $y_0 \to x_0$, which we denote by $C(x_1) \in \mathbb{R}$. We thus get a continuous map $C = C_{x_0} \colon \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0) \to \mathbb{R}$. Moreover, the holonomy map $\mathcal{H}^s_{x_0,x_1}$ converges to the identity in the $\mathcal{C}^1$ topology as $x_1 \to x_0$, hence $\lim_{x_1 \to x_0} C(x_1) = C(x_0) = 1$. □

For any points $y_0 \in \mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0) \cap \Lambda$, $x_1 \in \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0) \cap \Lambda$ close to x_0, we also abbreviate $z_1 = z_1(y_0) := [y_0, x_1]_{R_i}$. Recall that by local product structure, we have $z_1 \in \Lambda$. We denote by $\mathscr{Q}(y_0, x_1) := (x_0, x_1, z_1, y_0) \in \Lambda^4$ the associated quadrilateral.

Lemma 2.21. *For any points $y_0 \in \mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0) \cap \Lambda$, $x_1 \in \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0) \cap \Lambda$ close to x_0, the area of the quadrilateral $\mathscr{Q}(y_0, x_1)$ is equal to*

$$\mathrm{Area}(\mathscr{Q}(y_0, x_1)) = (y_0 - x_0)(x_1 - x_0)[\rho(x_0) + o(1)], \tag{2.15}$$

where ρ is the density function of $\mathcal{R}_*(d\alpha)$ introduced above. Therefore, for any points $y_0 \in \mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0) \cap \Lambda$, $x_1, x_2 \in \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0) \cap \Lambda$ close to x_0, we have[4]

$$\frac{\mathrm{Area}(\mathscr{Q}(y_0, x_2))}{\mathrm{Area}(\mathscr{Q}(y_0, x_1))} = \frac{x_2 - x_0}{x_1 - x_0} + o(1). \tag{2.16}$$

Proof. For any $y_0 \in \mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_0) \cap \Lambda$, $x_1 \in \mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(x_0) \cap \Lambda$ close to x_0, we have

$$\mathrm{Area}(\mathscr{Q}(y_0, x_1)) = \int_{x_0}^{x_1} \left(\int_0^{\gamma^s_{y_0}(\xi)} \rho(\xi, \eta)\, d\eta \right) d\xi + o\big((y_1 - x_1)^2\big),$$

where $y_1 = y_1(y_0)$. Here, we use the fact that the unstable lamination $\mathcal{W}^u_{\mathcal{F}}(y_0)$ is $\mathcal{C}^1$, so that the angle between $\mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_1)$ and V_{x_1} is going to 0 as $x_1 \to x_0$, and hence, the area of the missing “triangle” bounded by $\mathcal{W}^s_{\mathcal{F},\mathrm{loc}}(y_0)$, $\mathcal{W}^u_{\mathcal{F},\mathrm{loc}}(x_1)$ and V_{x_1} is a $o((y_1 - x_1)^2)$, noting that $\rho = O(1)$ on the quadrilateral. Since the argument is a local one, (2.14) guarantees that $y_1 - x_1 = O(y_0 - x_0)$. In the following, we will always assume that $y_0 - x_0 \le x_1 - x_0$, so that $o\big((y_1 - x_1)^2\big) = o\big((y_0 - x_0)(x_1 - x_0)\big)$. Therefore, we obtain

$$\begin{aligned}
\mathrm{Area}(\mathscr{Q}(y_0, x_1)) &= \int_{x_0}^{x_1} \gamma^s_{y_0}(\xi)\big(\rho(\xi,0) + O(y_0 - x_0)\big)\, d\xi + o\big((y_0 - x_0)(x_1 - x_0)\big) \\
&= \int_{x_0}^{x_1} \big(C(\xi)(y_0 - x_0) + o(y_0 - x_0)\big)\big(\rho(\xi,0) + O(y_0 - x_0)\big)\, d\xi \\
&\quad + o\big((y_0 - x_0)(x_1 - x_0)\big) \\
&= (y_0 - x_0) \int_{x_0}^{x_1} \big(C(\xi)\rho(\xi,0) + o(1)\big)\, d\xi + o\big((y_0 - x_0)(x_1 - x_0)\big) \\
&= (y_0 - x_0)(x_1 - x_0)[\rho(x_0) + o(1)],
\end{aligned}$$

since $C(\xi) = C(x_0) + o(1) = 1 + o(1)$, when $\xi \to x_0$. Observe now that (2.16) follows immediately by taking the quotient. □

[4] We thank Disheng Xu for the idea to use three points x_0, x_1, x_2 in the same leaf and consider the ratio of areas to get rid of the “width” of quadrilaterals.

For $i = 1, 2$, let $\Phi_i = (\Phi_i^t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^k$ Axiom A flow defined on a smooth 3-manifold M_i. Let Λ_i be a basic set for Φ_i, and let α_i be a smooth contact form adapted to Λ_i. Assume that there exists a flow conjugacy $\Psi\colon \Lambda_1 \to \Lambda_2$ between $\Phi_1|_{\Lambda_1}$ and $\Phi_2|_{\Lambda_2}$. For any point $x_0 \in \Lambda_1$, and for $* = s, u$, without loss of generality, because of Lemma 2.15 and up to translating along the flow direction, we can assume that $\mathcal{W}^*_{\Phi_1,\mathrm{loc}}(x_0)$, resp. $\mathcal{W}^*_{\Phi_2,\mathrm{loc}}(\Psi(x_0))$ belongs to some rectangle $R^{(1)}$ of a Markov family for Φ_1, resp. to some rectangle $R^{(2)}$ of a Markov family for Φ_2, so that $\mathcal{W}^*_{\Phi_1,\mathrm{loc}}(x_0) = \mathcal{W}^*_{R^{(1)}}(x_0)$, and $\mathcal{W}^*_{\Phi_2,\mathrm{loc}}(\Psi(x_0)) = \mathcal{W}^*_{R^{(2)}}(\Psi(x_0))$. Moreover, by using some chart as above, we see $\Psi|_{\mathcal{W}^*_{\Phi_1,\mathrm{loc}}(x_0)}$ as a map from $S_1 \subset \mathbb{R}$ to $S_2 \subset \mathbb{R}$, with $x_0 \simeq 0 \simeq \Psi(x_0)$.

Proposition 2.22. *Assume that the flow conjugacy Ψ is iso-length-spectral. Then, for any point $x_0 \in \Lambda_1$, and for $* = s, u$, the following limit exists:*

$$\partial_*\Psi(x_0) := \lim_{\mathcal{W}^*_{\Phi_1,\mathrm{loc}}(x_0)\cap\Lambda\ni x_1\to x_0} \frac{\Psi(x_1) - \Psi(x_0)}{x_1 - x_0}.$$

Moreover, the associated map $\partial_*\Psi$ is Hölder continuous on Λ_1. In other words, for some $\beta \in (0, 1)$, the conjugacy Ψ is $\mathcal{C}^{1,\beta}$ along $\mathcal{W}^s_{\Phi_1,\mathrm{loc}}, \mathcal{W}^u_{\Phi_1,\mathrm{loc}}$ in the sense of Whitney.

Proof. Let us consider the case where $* = s$; the other case is analogous. Fix $x_0 \in \Lambda_1$. Take $y_0 \in \mathcal{W}^u_{\Phi_1,\mathrm{loc}}(x_0) \cap \Lambda_1$, $x_1, x_2 \in \mathcal{W}^s_{\Phi_1,\mathrm{loc}}(x_0) \cap \Lambda_1$ close to x_0. Without loss of generality, we assume that $d(x_0, x_1) \leq d(x_0, x_2)$. By Corollary 2.18, for $i = 1, 2$, the quadrilaterals $\mathscr{Q}(y_0, x_i) = (x_0, x_i, z_i, y_0) \in \Lambda_1^4$ and $\Psi(\mathscr{Q})(y_0, x_i) := (\Psi(x_0), \Psi(x_i), \Psi(z_i), \Psi(y_0)) \in \Lambda_2^4$ have the same area; hence,

$$\frac{\mathrm{Area}(\mathscr{Q}(y_0, x_2))}{\mathrm{Area}(\mathscr{Q}(y_0, x_1))} = \frac{\mathrm{Area}(\Psi(\mathscr{Q})(y_0, x_2))}{\mathrm{Area}(\Psi(\mathscr{Q})(y_0, x_1))}.$$

We deduce from formula (2.16) that

$$\frac{\Psi(x_2) - \Psi(x_0)}{x_2 - x_0} = \frac{\Psi(x_1) - \Psi(x_0)}{x_1 - x_0} + o\left(\frac{\Psi(x_1) - \Psi(x_0)}{x_2 - x_0}\right).$$

For any $x \in \mathcal{W}^s_{\Phi_1,\mathrm{loc}}(x_0) \cap \Lambda_1$ close to x_0, we denote $q(x) := \frac{\Psi(x)-\Psi(x_0)}{x-x_0}$. Recall that $d(x_0, x_1) \leq d(x_0, x_2)$; thus, the previous identity can be written as

$$q(x_1) = q(x_2) + o\big(\max(q(x_1), q(x_2))\big).$$

Now, let us fix a sequence of points $(u_n)_{n\in\mathbb{N}} \in (\mathcal{W}^s_{\Phi_1,\mathrm{loc}}(x_0) \cap \Lambda_1)^{\mathbb{N}}$ going to x_0 as $n \to +\infty$. It is easy to see that that $(q(u_n))_{n\in\mathbb{N}}$ is bounded. Consequently, for any $n \geq 0$, $p \geq 0$, the previous identity gives

$$q(u_{n+p}) - q(u_n) = o(1).$$

We deduce that $(q(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence, hence it converges to some limit $\ell \in \mathbb{R}$. Therefore, for any sequence $(v_n)_{n\in\mathbb{N}} \in (\mathcal{W}^s_{\Phi_1,\mathrm{loc}}(x_0) \cap \Lambda_1)^{\mathbb{N}}$ converging to x_0, it holds that $q(v_n) \to \ell$ as $n \to +\infty$. This shows that Ψ is differentiable at x_0 along $\mathcal{W}^s_{\Phi_1,\mathrm{loc}}(x_0)$, thus at any point in Λ_1, along $\mathcal{W}^s_{\Phi_1,\mathrm{loc}}$.

In order to show that the map $\partial_s\Psi$ is Hölder continuous on Λ_1 along $\mathcal{W}^s_{\Phi_1,\mathrm{loc}}$, we argue as follows. Fix $x_0 \in \Lambda_1 \cap R^{(1)}$, and let $x_0' \in \mathcal{W}^s_{R^{(1)}}(x_0) \cap \Lambda_1$ be close to x_0. Let $(u_n)_{n\in\mathbb{N}} \in (\mathcal{W}^s_{R^{(1)}}(x_0) \cap \Lambda_1)^{\mathbb{N}}$, resp. $(u_n')_{n\in\mathbb{N}} \in (\mathcal{W}^s_{R^{(1)}}(x_0') \cap \Lambda_1)^{\mathbb{N}}$, be a sequence of points in Λ_1 converging to x_0, resp. x_0' along $\mathcal{W}^s_{R^{(1)}}(x_0) = \mathcal{W}^s_{R^{(1)}}(x_0')$.

For any point $y_0 \in \mathcal{W}^u_{R^{(1)}}(x_0) \cap \Lambda_1$ close to x_0, and for each integer $n \in \mathbb{N}$, we let $\overline{\mathscr{Q}}_n(y_0) = (x_0, u_n, z_n, y_0) \in (\Lambda_1 \cap R^{(1)})^4$ and $\overline{\mathscr{Q}}'_n(y_0) = (x'_0, u'_n, z'_n, y'_0) \in (\Lambda_1 \cap R^{(1)})^4$, where $z_n = [y_0, u_n]_{R^{(1)}}$, $y'_0 = [y_0, x'_0]_{R^{(1)}}$ and $z'_n = [y'_0, u'_n]_{R^{(1)}}$. Let $\mathscr{Q}_n(y_0)$, resp. $\mathscr{Q}'_n(y_0)$, be the lift of $\overline{\mathscr{Q}}_n(y_0)$, resp. $\overline{\mathscr{Q}}'_n(y_0)$, as in the proof of Lemma 2.12. We deduce from (2.15) that

$$\begin{aligned}\mathrm{Area}(\mathscr{Q}_n(y_0)) &= (y_0 - x_0)(u_n - x_0)[\rho(x_0) + o(1)],\\ \mathrm{Area}(\mathscr{Q}'_n(y_0)) &= (y'_0 - x'_0)(u'_n - x'_0)[\rho(x'_0) + o(1)]\\ &= C_{x_0}(x'_0)(y_0 - x_0)(u'_n - x'_0)[\rho(x'_0) + o(1)],\end{aligned}$$

so that

$$\frac{\mathrm{Area}(\mathscr{Q}'_n(y_0))}{\mathrm{Area}(\mathscr{Q}_n(y_0))} = C_{x_0}(x'_0)\frac{u'_n - x'_0}{u_n - x_0}\left(1 + O(x'_0 - x_0) + o(1)\right).$$

As the images of the quadrilaterals $\mathscr{Q}_n(y_0)$ and $\mathscr{Q}'_n(y_0)$ by Ψ have the same area, we deduce that

$$\begin{aligned}&C_{\Psi(x_0)}(\Psi(x'_0))\frac{\Psi(u'_n) - \Psi(x'_0)}{\Psi(u_n) - \Psi(x_0)}\left(1 + O(\Psi(x'_0) - \Psi(x_0)) + o(1)\right)\\ &= C_{x_0}(x'_0)\frac{u'_n - x'_0}{u_n - x_0}\left(1 + O(x'_0 - x_0) + o(1)\right).\end{aligned}$$

Observe that

$$\begin{aligned}C_{x_0}(x'_0) &= 1 + O(x'_0 - x_0),\\ C_{\Psi(x_0)}(\Psi(x'_0)) &= 1 + O(\Psi(x'_0) - \Psi(x_0)) = 1 + O(|x'_0 - x_0|^\beta),\end{aligned}$$

for some $\beta \in (0,1)$ since Ψ is Hölder continuous. Thus, for $y_0 \to x_0$ we obtain

$$\frac{\Psi(u'_n) - \Psi(x'_0)}{u'_n - x'_0} = \frac{\Psi(u_n) - \Psi(x_0)}{u_n - x_0}\left(1 + O(|x'_0 - x_0|^\beta)\right).$$

Letting $n \to +\infty$, we deduce that $|\partial_s\Psi(x'_0) - \partial_s\Psi(x_0)| = O(|x'_0 - x_0|^\beta)$. Thus, applying Whitney's theorem, we conclude that Ψ is $\mathcal{C}^{1,\beta}$ in the sense of Whitney along $\mathcal{W}^s_{\Phi_1,\mathrm{loc}}$, for $\beta \in (0,1)$. $\square$

Recall that roughly speaking, Journé's lemma (see [28]) says that once a function is regular along the leaves of two transverse foliations, then it is regular globally. It has been generalized by Nicol-Török [41] in the case of laminations on Cantor sets (see Theorem 1.5 and Remark 1.6 in [41]). In our case, it reads as follows.

Theorem 2.23 (Theorem 1.5 in [41])**.** *Let $\Lambda \subset \mathbb{R}^2$ be a closed, hyperbolic basic set, and for $\beta \in (0,1)$, let $\mathcal{W}^s$, $\mathcal{W}^u$ be two transverse uniformly $\mathcal{C}^{1,\beta}$ laminations of Λ. Suppose that $\Theta\colon \Lambda \to \mathbb{R}^2$ is uniformly $\mathcal{C}^{1,\beta}$ in the sense of Whitney when restricted to the leaves of $\mathcal{W}^s$, $\mathcal{W}^u$. Then Θ is $\mathcal{C}^{1,\beta}$ in the sense of Whitney on Λ.*

From Proposition 2.22 and Theorem 2.23, we then deduce the following

Corollary 2.24. *Assume that there exists an iso-length-spectral flow conjugacy $\Psi\colon \Lambda_1 \to \Lambda_2$ between $\Phi_1|_{\Lambda_1}$ and $\Phi_2|_{\Lambda_2}$. Then Ψ is $\mathcal{C}^{1,\beta}$ in the sense of Whitney on Λ, for some $\beta \in (0,1)$.*

Proof. By Proposition 2.22, we know that Ψ is $\mathcal{C}^{1,\beta}$ in the sense of Whitney along stable/unstable leaves. For $i=1,2$, let us fix a Markov family $\mathcal{R}^{(i)}=\{R_1^{(i)},\dots,R_{m(i)}^{(i)}\}$ with a cross-section $\mathcal{S}^{(i)}$ as given by Theorem 2.5. By projecting Λ_1,Λ_2 along flow lines on $\mathcal{S}^{(1)},\mathcal{S}^{(2)}$, and applying Theorem 2.23 to the projected sets, we deduce that the map $\widetilde{\Psi}$ induced by Ψ between $\Lambda_1\cap\mathcal{S}^{(1)}$ and $\Lambda_2\cap\mathcal{S}^{(2)}$ is $\mathcal{C}^{1,\beta}$ in the sense of Whitney, for some $\beta\in(0,1)$. Since the projection along the flow direction is $\mathcal{C}^k$, and since we can describe Ψ in terms of $\widetilde{\Psi}$ and the two projections along X_1,X_2, we conclude that Ψ is $\mathcal{C}^{1,\beta}$ in the sense of Whitney. □

2.7. **Upgraded regularity of the conjugation.** As previously, let us fix a homeomorphism $\Psi\colon\Lambda_1\to\Lambda_2$ that is $\mathcal{C}^{1,\beta}$ in Whitney sense, for some $\beta\in(0,1)$, which satisfies

$$\Psi\circ\Phi_1^t(x)=\Phi_2^t\circ\Psi(x),\quad \text{for all } (x,t)\in\Lambda_1\times\mathbb{R}. \tag{2.17}$$

Recall that for $i=1,2$ and $\star=s,u$, there exists $\delta_i^{(\star)}>0$ such that for any $x\in\Lambda_i$, we have

$$\delta_i^{(\star)}=\dim_H(\mathcal{W}^{\star}_{\Phi_i,\mathrm{loc}}(x)\cap\Lambda_i).$$

As Ψ is $\mathcal{C}^{1,\beta}$, we also have $\delta_1^{(\star)}=\delta_2^{(\star)}=:\delta^{(\star)}$.

Fix some small $\varepsilon>0$. By Theorem 2.5, for $i=1,2$, there exists a proper Markov family $\mathcal{R}^{(i)}=\{R_1^{(i)},\dots,R_{m(i)}^{(i)}\}$ for $\Phi_{i|\Lambda_i}$ of size ε, for some integer $m(i)\geq 1$. Let $\mathcal{S}^{(i)}:=R_1^{(i)}\cup\dots\cup R_{m(i)}^{(i)}$, resp. $\mathcal{F}_i$, be the associated cross-section, resp. Poincaré map. We also denote by $\overline{\Lambda}_i:=\Lambda_i\cap\mathcal{S}^{(i)}$ the trace of Λ_i on $\mathcal{S}^{(i)}$. The map $\widetilde{\Psi}$ induced by Ψ between $\overline{\Lambda}_1$ and $\overline{\Lambda}_2$ is $\mathcal{C}^{1,\beta}$ in the sense of Whitney. Recall that $\dim_H(\overline{\Lambda}_1)=\dim_H(\overline{\Lambda}_2)=\delta^{(s)}+\delta^{(u)}$ (see [37] for a reference).

By [45, Theorem 22.1], for $i=1,2$ and $\star=s,u$, there exists a (unique) *equilibrium state*[5] $\mu_i^\star$ such that for every $x\in\overline{\Lambda}_i$, the conditional measure $m_{i,x}^\star$ of $\mu_i^\star$ on $\mathcal{W}^\star_{\mathcal{F}_i}(x)\cap\overline{\Lambda}_i$ is equivalent to the $\delta^{(\star)}$-Hausdorff measure $H^{\delta^{(\star)}}$. More precisely, μ_i^s is the equilibrium state for the *potential*[5] $p_i^{(s)}:=\delta^{(s)}\log\|D\mathcal{F}_i|_{E^s_{\mathcal{F}_i}}\|$, and μ_i^u is the equilibrium state for the potential $p_i^{(u)}:=-\delta^{(u)}\log\|D\mathcal{F}_i|_{E^u_{\mathcal{F}_i}}\|$; besides, the *pressure*[5] $P(p_i^{(\star)})$ vanishes, for $\star=s,u$.

By (2.17), for any periodic point $x\in\Lambda_1$ of period $q(x)\geq 1$, the differentials $D\mathcal{F}_1^{q(x)}(x)$ and $D\mathcal{F}_2^{q(x)}(\widetilde{\Psi}(x))$ are conjugate, hence have the same eigenvalues, i.e.,

$$\sum_{k=0}^{q(x)-1}\left(\log\|D\mathcal{F}_1^k(x)|_{E^{(\star)}_{\mathcal{F}_1}}\|-\log\|D\mathcal{F}_2^k\big(\widetilde{\Psi}(x)\big)|_{E^{(\star)}_{\mathcal{F}_2}}\|\right)=0,\quad \star=s,u.$$

By Livsic's Theorem, we deduce that the potentials $\widetilde{\Psi}^*p_2^{(\star)}$ and $p_1^{(\star)}$ are cohomologous, and by [4, Proposition 4.5], we thus have $\widetilde{\Psi}^*\mu_2^{(\star)}|_{\overline{\Lambda}_1}=\mu_1^{(\star)}|_{\overline{\Lambda}_1}$. Consequently, $\widetilde{\Psi}^*m^\star_{2,\widetilde{\Psi}(x)}=m^\star_{1,x}$, for $\star=s,u$, and for a.e. $x\in\overline{\Lambda}_1$.

In the following we deal with the unstable case; the stable one is analogous. To ease the notation, we abbreviate $\delta:=\delta^{(u)}$. For $i=1,2$, and $x_i\in\overline{\Lambda}_i$, the conditional measure m^u_{i,x_i} is equivalent to H^δ, hence we can introduce the density function

[5]See for instance [45] for more details about equilibrium states, potentials, pressure etc.

$\rho^u_{i,x_i}\colon \mathcal{W}^u_{\mathcal{F}_i}(x_i)\to\mathbb{R}^*$, so that $dm^u_{i,x_i}=\rho^u_{i,x_i}dH^\delta$. Recall that the conditional measure m^u_{i,x_i} depends only on the leaf $\mathcal{W}^u_{\mathcal{F}_i}(x_i)$. Our goal in the following paragraph is to show that the function $\rho^u_{i,x_i}(\cdot)/\rho^u_{i,x_i}(x_i)$ is $\mathcal{C}^{k-1}$ in the sense of Whitney.

As $P(p^u_i)=0$, for any integer $n\geq 0$, and for any $y_i\in\mathcal{W}^u_{\mathcal{F}_i}(x_i)$, we have (see for instance [8, Section 3.2])

$$\frac{d((\mathcal{F}_i^{-n})_*m^u_{i,x_i})}{dm^u_{i,\mathcal{F}_i^{-n}(x_i)}}(\mathcal{F}_i^{-n}(y_i))=e^{-S_np^u_i(\mathcal{F}_i^{-n}(y_i))},\tag{2.18}$$

where $S_np^u_i$ is the n^{th} Birkhoff sum of p^u_i, i.e.,

$$S_np^u_i(\mathcal{F}_i^{-n}(y_i)):=\sum_{k=1}^n p^u_i(\mathcal{F}^{-k}(y_i))=-\sum_{k=1}^n\log\left\|D\mathcal{F}_i^{-1}(\mathcal{F}^{-k}(y_i))|_{E^u_{\mathcal{F}_i}}\right\|^\delta.$$

In terms of densities, (2.18) thus yields:

$$\frac{(\mathcal{F}_i^{-n})_*\rho^u_{i,x_i}}{\rho^u_{i,\mathcal{F}_i^{-n}(x_i)}}(\mathcal{F}_i^{-n}(y_i))=\frac{\rho^u_{i,x_i}(y_i)}{\rho^u_{i,\mathcal{F}_i^{-n}(x_i)}(\mathcal{F}_i^{-n}(y_i))}=\prod_{k=1}^n\left\|D\mathcal{F}_i^{-1}(\mathcal{F}^{-k}(y_i))|_{E^u_{\mathcal{F}_i}}\right\|^\delta.$$

Let us consider the ratio of the above quantity and the corresponding one at x_i. As the distance $d(\mathcal{F}^{-n}(x_i),\mathcal{F}^{-n}(y_i))$ decays exponentially fast with respect to n, and assuming that $\mathcal{F}^{-n}(x_i),\mathcal{F}^{-n}(y_i)$ converge to a point x_i^∞ (up to taking subsequences), letting $n\to+\infty$, we obtain

$$\rho^u_i(x_i,y_i):=\frac{\rho^u_{i,x_i}(y_i)}{\rho^u_{i,x_i}(x_i)}=\prod_{k=1}^{+\infty}\left(\frac{\|D\mathcal{F}_i^{-1}(\mathcal{F}^{-k}(y_i))|_{E^u_{\mathcal{F}_i}}\|}{\|D\mathcal{F}_i^{-1}(\mathcal{F}^{-k}(x_i))|_{E^u_{\mathcal{F}_i}}\|}\right)^\delta.\tag{2.19}$$

In particular, the infinite product on the right hand side is uniformly convergent (see [12, Lemma 4.3]), hence the function $\rho^u_i(x_i,\cdot)$ is $\mathcal{C}^{k-1}$ in the sense of Whitney.

In the rest of this section, we follow the proof of [12, Lemma 4.5]. Fix a point $x_1\in\overline{\Lambda}_1$ and let $x_2:=\widetilde{\Psi}(x_1)\in\overline{\Lambda}_2$. Since the foliations $\mathcal{W}^u_{\mathcal{F}_1},\mathcal{W}^u_{\mathcal{F}_2}$ have one dimensional leaves, we can parametrize patches of the unstable leaves by Riemannian length. Recall that $\widetilde{\Psi}^*m^u_{2,x_2}=m^u_{1,x_1}$; we deduce that for any point $y_1\in\mathcal{W}^u_{\mathcal{F}_1}(x_1)$, it holds (taking charts for $\mathcal{W}^u_{\mathcal{F}_1}(x_1),\mathcal{W}^u_{\mathcal{F}_2}(x_2)$, identifying functions on the leaves and functions of the coordinates, and seeing the Whitney extension of $\widetilde{\Psi}|_{\mathcal{W}^u_{\mathcal{F}_1}(x_1)}$ as a map from $\mathbb{R}$ to $\mathbb{R}$):

$$\int_{x_1}^{y_1}\rho^u_{1,x_1}(s)\,dH^\delta(s)=\int_{\widetilde{\Psi}(x_1)}^{\widetilde{\Psi}(y_1)}\rho^u_{2,\widetilde{\Psi}(x_1)}(s)\,dH^\delta(s).$$

By (2.19), we have

$$\rho^u_{1,x_1}(x_1)\int_{x_1}^{y_1}\rho^u_1(x_1,s)\,dH^\delta(s)=\rho^u_{2,x_2}(x_2)\int_{\widetilde{\Psi}(x_1)}^{\widetilde{\Psi}(y_1)}\rho^u_2(x_2,s)\,dH^\delta(s).$$

For y_1 very close to x_1, we thus obtain

$$\rho^u_{1,x_1}(x_1)\int_{x_1}^{y_1}(1+o(1))\,dH^\delta(s)=\rho^u_{2,x_2}(x_2)\int_{\widetilde{\Psi}(x_1)}^{\widetilde{\Psi}(y_1)}(1+o(1))\,dH^\delta(s),$$

that is

$$\frac{\rho^u_{1,x_1}(x_1)}{\rho^u_{2,x_2}(x_2)} = \frac{\int_{\widetilde{\Psi}(x_1)}^{\widetilde{\Psi}(y_1)} dH^\delta(s)}{\int_{x_1}^{y_1} dH^\delta(s)} + o(1).$$

Consequently,

$$\log\left(\frac{\rho^u_{1,x_1}}{\rho^u_{2,x_2}\circ\widetilde{\Psi}}\right)(x_1) = \log|\widetilde{\Psi}(y_1)-\widetilde{\Psi}(x_1)| \times \frac{\log\left|\int_{\widetilde{\Psi}(x_1)}^{\widetilde{\Psi}(y_1)} dH^\delta\right|}{\log|\widetilde{\Psi}(y_1)-\widetilde{\Psi}(x_1)|} - \\ - \log|y_1-x_1| \times \frac{\log\left|\int_{x_1}^{y_1} dH^\delta\right|}{\log|y_1-x_1|} + o(1).$$

When $\mathcal{W}^u_{\mathcal{F}_1}(x_1) \ni y_1 \to x_1$, both $\frac{\log\left|\int_{\widetilde{\Psi}(x_1)}^{\widetilde{\Psi}(y_1)} dH^\delta\right|}{\log|\widetilde{\Psi}(y_1)-\widetilde{\Psi}(x_1)|}$ and $\frac{\log\left|\int_{x_1}^{y_1} dH^\delta\right|}{\log|y_1-x_1|}$ tend to the dimension of the measure H^δ, namely, δ. We deduce that

$$\log\left(\frac{\rho^u_{1,x_1}}{\rho^u_{2,x_2}\circ\widetilde{\Psi}}\right)(x_1) = \delta\log\left(\frac{\widetilde{\Psi}(y_1)-\widetilde{\Psi}(x_1)}{y_1-x_1}\right) + o(1).$$

As $\widetilde{\Psi}|_{\overline{\Lambda}_1}$ is $\mathcal{C}^{1,\beta}$ in the sense of Whitney, letting $\mathcal{W}^u_{\mathcal{F}_1}(x_1) \ni y_1 \to x_1$, we get

$$\frac{\rho^u_{1,x_1}}{\rho^u_{2,x_2}\circ\widetilde{\Psi}}(x_1) = (\partial_u\widetilde{\Psi}(x_1))^\delta.$$

In other words, on Λ_1, the map $\widetilde{\Psi}$ satisfies

$$\partial_u\widetilde{\Psi}(\cdot) = \left(\frac{\rho^u_{1,(\cdot)}(\cdot)}{\rho^u_{2,\widetilde{\Psi}(\cdot)}\circ\widetilde{\Psi}(\cdot)}\right)^{\frac{1}{\delta}}. \tag{2.20}$$

We have seen that the functions $\rho^u_{1,(\cdot)}, \rho^u_{2,\widetilde{\Psi}(\cdot)}$ are $\mathcal{C}^{k-1}$ in Whitney sense. As $\widetilde{\Psi}$ is $\mathcal{C}^{1,\beta}$ on Λ_1 along $\mathcal{W}^u_{\mathcal{F}_1}$, the right hand side of (2.20) is $\mathcal{C}^{1,\beta}$ on Λ_1 along $\mathcal{W}^u_{\mathcal{F}_1}$. We deduce that $\widetilde{\Psi}$ is $\mathcal{C}^2$ on Λ_1 along $\mathcal{W}^u_{\mathcal{F}_1}$ in Whitney sense. By repeating the argument, we conclude that $\widetilde{\Psi}$ is $\mathcal{C}^k$ on Λ_1 along $\mathcal{W}^u_{\mathcal{F}_1}$ in Whitney sense. The same arguments applied at stable leaves imply that $\widetilde{\Psi}$ restricted to the leaves of $\mathcal{W}^s_{\mathcal{F}_1}$ is also $\mathcal{C}^k$ in Whitney sense. By using the version of Journé's Lemma in [41, Theorem 1.5] for laminations on hyperbolic sets, and arguing as in the proof of Corollary 2.24, we conclude that the conjugacy map $\Psi|_{\Lambda_1}$ is $\mathcal{C}^k$ in Whitney sense, as desired. □

2.8. **Preservation of contact forms: end of the proof of Theorem A.** We have just seen that the flow conjugacy Ψ is $\mathcal{C}^k$ in the sense of Whitney on Λ_1. In this subsection, we show that it implies that Ψ respects the contact structures. See Feldman-Ornstein [19] for related results in the case of contact Anosov flows on 3-manifolds.

Lemma 2.25. *We have* $\Psi^*\alpha_2|_{\Lambda_1} = \alpha_1|_{\Lambda_1}$.

Proof. By Lemma 2.15, for $i = 1, 2$, and for any $x_i \in \Lambda_i$, it holds

$$E^s_{\Phi_i}(x_i) \oplus E^u_{\Phi_i}(x_i) = \ker\alpha(x_i).$$

Recall that Ψ is a flow conjugacy, i.e.,

$$\Psi \circ \Phi_1^t(x_1) = \Phi_2^t \circ \Psi(x_1), \quad \forall\, t \in \mathbb{R},\, x_1 \in \Lambda_1. \tag{2.21}$$

Therefore, for $* = s, u$, it holds

$$D\Psi(x_1) E^*_{\Phi_1}(x_1) = E^*_{\Phi_2}(\Psi(x_1)).$$

In particular, $\ker \Psi^* \alpha_1(x_1) = \ker \alpha_2(\Psi(x_1))$. Moreover, differentiating (2.21) with respect to t, we obtain $D\Psi(x_1)X_1(x_1) = X_2(\Psi(x_1))$.

Let us show how this implies the result. We want to show that for any $x \in \Lambda_1$, it holds $\Psi^*\alpha_2(x) = \alpha_1(x)$. For any $v \in T_x M_1$, we decompose it as $v = v^s + v^u + cX_1(x)$, with $v^s \in E^s_{\Phi_1}(x)$, $v^u \in E^u_{\Phi_1}(x)$, $c \in \mathbb{R}$. We obtain

$$\begin{aligned}
\Psi^*\alpha_2(x)(v) &= \alpha_2(\Psi(x))\big(D\Psi(x)v^s + D\Psi(x)v^u + cD\Psi(x)X_1(x)\big)\\
&= c\alpha_2(\Psi(x))(D\Psi(x)X_1(x)) = c\alpha_2(\Psi(x))(X_2(\Psi(x)))\\
&= c\imath_{X_2}\alpha_2(\Psi(x)) = c = c\imath_{X_1}\alpha_1(x)\\
&= c\alpha_1(x)(X_1(x)) = \alpha_1(x)(v),
\end{aligned}$$

which concludes. □

Together with Proposition 2.1 and Corollary 2.24, this concludes the proof of Theorem A.

3. Spectral rigidity of hyperbolic billiards

In the following, we give the proof of Theorem C. Let us consider two billiards $\mathcal{D}_1, \mathcal{D}_2$ with $\mathcal{C}^k$ boundaries, $k \geq 3$, that are iso-length-spectral on two basic sets $\Lambda_1^{\tau_1}$, $\Lambda_2^{\tau_2}$. For $i = 1, 2$, we denote by Φ_i, resp. $\mathcal{F}_i$, the associated billiard flow, resp. billiard map, and we consider the contact form $\alpha_i := \lambda_i + dt_i$, where $\lambda_i := -r_i ds_i$ is the Liouville form. We also let Λ_1, Λ_2 be the respective projections of $\Lambda_1^{\tau_1}, \Lambda_2^{\tau_2}$ onto the first two coordinates, i.e.,

$$\Lambda_i := \{(s_i, r_i) : (s_i, r_i, t_i) \in \Lambda_i^{\tau_i} \text{ for some } t_i \in \mathbb{R}\}, \quad i = 1, 2. \tag{3.1}$$

By Theorem A, there exists a flow conjugacy $\widetilde{\Psi}\colon (s_1, r_1, t_1) \mapsto (s_2, r_2, t_2)$ between $\Phi_1|_{\Lambda_1^{\tau_1}}$ and $\Phi_2|_{\Lambda_2^{\tau_2}}$ that is $\mathcal{C}^{k-1}$ in Whitney sense, except possibly at collisions, i.e., when $t_1 = 0$ or $t_2 = 0$, and such that $\widetilde{\Psi}^*\alpha_2|_{\Lambda_1^{\tau_1}} = \alpha_1|_{\Lambda_1^{\tau_1}}$. Take a point $(s_1, r_1, t_1) \in \Lambda_1^{\tau_1}$, and let $(s_2, r_2, t_2) := \widetilde{\Psi}(s_1, r_1, t_1) \in \Lambda_2^{\tau_2}$. The vectors $\frac{\partial}{\partial r_i}$ and $\frac{\partial}{\partial s_i} + r_i \frac{\partial}{\partial t_i}$ form a basis of the contact plane, for $i = 1, 2$; as the differential of $\widetilde{\Psi}$ sends contact plane to contact plane and $\frac{\partial}{\partial t_1}$ to $\frac{\partial}{\partial t_2}$, we deduce that there exist functions $a, b, c, d\colon \Lambda_1 \to \mathbb{R}$ which are $\mathcal{C}^{k-2}$ in Whitney sense and such that

$$D\widetilde{\Psi}(s_1, r_1, t_1) = \begin{bmatrix} a(s_1, r_1) & c(s_1, r_1) & 0\\ b(s_1, r_1) & d(s_1, r_1) & 0\\ r_2 a(s_1, r_1) - r_1 & r_2 c(s_1, r_1) & 1 \end{bmatrix}, \quad \forall\, (s_1, r_1, t_1) \in \Lambda_1^{\tau_1},$$

where $ad - bc = 1$ (as $\widetilde{\Psi}^* d\alpha_2|_{\Lambda_1^{\tau_1}} = d\alpha_1|_{\Lambda_1^{\tau_1}}$).

Moreover, the map $\widetilde{\Psi}$ induces a conjugacy $\Psi\colon (s_1, r_1) \mapsto (s_2, r_2)$ between the billiard maps $\mathcal{F}_1|_{\Lambda_1}, \mathcal{F}_2|_{\Lambda_2}$ that is also $\mathcal{C}^{k-1}$ in Whitney sense, with

$$D\Psi(s_1, r_1) = \begin{bmatrix} a(s_1, r_1) & c(s_1, r_1)\\ b(s_1, r_1) & d(s_1, r_1) \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R}), \quad \forall\, (s_1, r_1) \in \Lambda_1. \tag{3.2}$$

Recall that for $i = 1, 2$, we denote by $\tau_i(s_i, r_i) = h_i(s_i, s_i') > 0$ the length of the segment between consecutive bounces $(s_i, r_i) \in \Lambda_i$ and $(s_i', r_i') = \mathcal{F}_i(s_i, r_i) \in \Lambda_i$, so that $\mathcal{F}_i^* \lambda_i - \lambda_i = dh_i$.

By the fact that $\mathcal{D}_1, \mathcal{D}_2$ have the same periodic length data on Λ_1 and Λ_2, it follows from Livsic's theorem that the restriction of $\tau_2 \circ \Psi - \tau_1$ to Λ_1 is a coboundary, i.e., for some continuous function $\chi \colon \Lambda_1 \to \mathbb{R}$, we have

$$\tau_2 \circ \Psi - \tau_1 = \chi \circ \mathcal{F}_1 - \chi \quad \text{on } \Lambda_1. \tag{3.3}$$

Actually, as Ψ is $\mathcal{C}^{k-1}$ in Whitney sense, by the results of Nicol-Török [41, Theorem 3.2], the function χ is also $\mathcal{C}^{k-1}$ in Whitney sense. To complete the proof of Theorem C, we still need to show (1.6)-(1.7)-(1.8), which is done in the next subsection.

3.1. **Image of the time-reversal involution by the conjugacy.** The conjugacy Ψ is not unique, as we may pre-compose, resp. post-compose it with any fixed iterate of $\mathcal{F}_1$, resp. $\mathcal{F}_2$. Yet, in some cases, there is a canonical way to choose the conjugacy in such a way that it preserves the time-reversal symmetry of the billiard dynamics; we will discuss this in the following. Recall that for $i = 1, 2$, $\mathcal{I}_i \colon (s_i, r_i) \mapsto (s_i, -r_i)$ is the time-reversal involution, so that $\mathcal{F}_i \circ \mathcal{I}_i = \mathcal{I}_i \circ \mathcal{F}_i^{-1}$. In the following, we investigate when it is actually possible to normalize the conjugacy such that it conjugates the time-reversal involutions of $\mathcal{F}_1$ and $\mathcal{F}_2$, i.e.,

$$\Psi \circ \mathcal{I}_1 = \mathcal{I}_2 \circ \Psi \quad \text{on } \Lambda_1. \tag{3.4}$$

Let us denote by $\hat{\mathcal{I}}_1 := \Psi^{-1} \circ \mathcal{I}_2 \circ \Psi|_{\Lambda_1}$ the conjugate of $\mathcal{I}_2$ under Ψ. Clearly, the map $\hat{\mathcal{I}}_1$ is involutive, and it conjugates $\mathcal{F}_1$ to its inverse $\mathcal{F}_1^{-1}$. In particular, the map $\Gamma := \hat{\mathcal{I}}_1 \circ \mathcal{I}_1$ belongs to the centralizer of the map $\mathcal{F}_1$ on the basic set Λ_1, i.e.,

$$\Gamma \circ \mathcal{F}_1 = \mathcal{F}_1 \circ \Gamma \quad \text{on } \Lambda_1.$$

The centralizer of Axiom A diffeomorphisms at basic pieces is typically trivial (see [20, 50]), hence we expect Γ to be an iterate of $\mathcal{F}_1$. It is actually the case, by [50, Theorem A], as long as the map Γ fixes the orbits of $\mathcal{F}_1$, i.e., assuming that

$$\forall\, x_1 \in \Lambda_1, \quad \Gamma(x_1) = \mathcal{F}_1^{\ell}(x_1), \quad \text{for some } \ell = \ell(x_1) \in \mathbb{Z}. \tag{3.5}$$

Actually, we can prove directly:

Lemma 3.1. *If (3.5) holds, then there exists an integer $k \in \mathbb{Z}$ such that*

$$\mathcal{I}_2 \circ \Psi|_{\Lambda_1} = \Psi \circ \mathcal{I}_1 \circ \mathcal{F}_1^k|_{\Lambda_1}. \tag{3.6}$$

Proof. Let $x_1 \in \Lambda_1$, and take $\ell = \ell(x_1) \in \mathbb{Z}$ such that (3.5) holds for x_1. We have

$$\begin{aligned}\Gamma(\mathcal{F}_1(x_1)) = \Psi^{-1} \circ \mathcal{I}_2 \circ \Psi \circ \mathcal{I}_1 \circ \mathcal{F}_1(x_1) = \Psi^{-1} \circ \mathcal{I}_2 \circ \Psi \circ \mathcal{F}_1^{-1} \circ \mathcal{I}_1(x_1) = \cdots = \\ = \mathcal{F}_1 \circ \Psi^{-1} \circ \mathcal{I}_2 \circ \Psi \circ \mathcal{I}_1(x_1) = \mathcal{F}_1 \circ \Gamma(x_1) = \mathcal{F}_1^{\ell}(\mathcal{F}_1(x_1)),\end{aligned}$$

hence the integer ℓ in (3.5) is constant along the orbits. As $\mathcal{F}_1|_{\Lambda_1}$ is transitive, considering $x_1 \in \Lambda_1$ with a dense orbit, and by continuity, this finishes the proof. ☐

Besides, in the case where $\mathcal{D}_1, \mathcal{D}_2$ are open dispersing billiards, after changing the conjugacy, it is possible to verify (3.4):

Lemma 3.2. *If furthermore, $\mathcal{D}_1, \mathcal{D}_2 \in$ **B**, then, based on (3.6), we can redefine Ψ so that (3.4) holds.*

Proof. Let us show that the integer k in (3.6) is even. Indeed, let x_1 be a 2-periodic point for $\mathcal{F}_1$. Thus, both $\Psi(x_1)$ and $\mathcal{F}_1^k(x_1)$ are 2-periodic, for $\mathcal{F}_2$ and $\mathcal{F}_1$ respectively. In particular, the point $\Psi(x_1)$, resp. $\mathcal{F}_1^k(x_1)$, is fixed under $\mathcal{I}_2$, resp. $\mathcal{I}_1$. Therefore, by (3.6), $\Psi(x_1) = \Psi(\mathcal{F}_1^k(x_1))$; by the injectivity of Ψ, we conclude that $\mathcal{F}_1^k(x_1) = x_1$, hence $k = 2\ell$, for some $\ell \in \mathbb{Z}$. Let us consider the map $\widehat{\Psi} := \Psi \circ \mathcal{F}_1^{-\ell}$. By (3.6), equation (3.4) is satisfied for $\widehat{\Psi}$ in place of Ψ, as

$$\mathcal{I}_2 \circ \widehat{\Psi}|_{\Lambda_1} = \Psi \circ \mathcal{I}_1 \circ \mathcal{F}_1^{k-\ell}|_{\Lambda_1} = \Psi \circ \mathcal{F}_1^{\ell-k} \circ \mathcal{I}_1|_{\Lambda_1} = \widehat{\Psi} \circ \mathcal{I}_1|_{\Lambda_1}.$$

Besides, the map $\widehat{\Psi}$ still conjugates $\mathcal{F}_1|_{\Lambda_1}$ to $\mathcal{F}_2|_{\Lambda_2}$, and it is also $\mathcal{C}^{k-1}$ in Whitney sense, which concludes. □

For more general billiards, under assumption (2) in Theorem D, i.e., if there exists a point $x_1 \in \Lambda_1 \cap \{r_1 = 0\}$ whose orbit is dense in Λ_1, and such that $\mathcal{F}_2^k \circ \Psi(x_1) \in \{r_2 = 0\}$ for some $k \in \mathbb{Z}$, we still have:

Lemma 3.3. *Assuming (2) in Theorem D, we can redefine Ψ so that (3.4) holds.*

Proof. Let us assume that there exists a point $x_1 \in \Lambda_1 \cap \{r_1 = 0\}$ whose orbit is dense in Λ_1, and such that $\mathcal{F}_2^k \circ \Psi(x_1) \in \{r_2 = 0\}$ for some $k \in \mathbb{Z}$. Let us consider the map $\widehat{\Psi} := \mathcal{F}_2^k \circ \Psi|_{\Lambda_1} = \Psi \circ \mathcal{F}_1^k|_{\Lambda_1}$. For any integer $\ell \in \mathbb{Z}$, we have

$$\begin{aligned}\mathcal{I}_2 \circ \widehat{\Psi}(\mathcal{F}_1^\ell(x_1)) &= \mathcal{I}_2 \circ \mathcal{F}_2^k \circ \Psi \circ \mathcal{F}_1^\ell(x_1) = \cdots = \mathcal{F}_2^{-\ell} \circ \mathcal{I}_2 \circ \mathcal{F}_2^k \circ \Psi(x_1)\\ &= \mathcal{F}_2^{-\ell} \circ \mathcal{F}_2^k \circ \Psi(x_1) = \mathcal{F}_2^k \circ \Psi \circ \mathcal{F}_1^{-\ell}(x_1) = \widehat{\Psi} \circ \mathcal{I}_1(\mathcal{F}_1^\ell(x_1)).\end{aligned}$$

In other words, (3.4) is satisfied for $\widehat{\Psi}$ in place of Ψ on the orbit of x_1; as the latter is dense, and by continuity, it is satisfied everywhere on Λ_1, which concludes. □

Remark 3.4. Note that if there exists a conjugacy map Ψ which satisfies (3.4), then it is unique in the following sense: if $\widehat{\Psi}$ is another conjugacy map which satisfies (3.4) and such that $\Psi^{-1} \circ \widehat{\Psi}$ fixes $\mathcal{F}_1$-orbits, then $\widehat{\Psi} = \Psi$. Indeed, in this case, $\Psi^{-1} \circ \widehat{\Psi}$ commutes with $\mathcal{F}_1$; arguing as above, we see that it is equal to $\mathcal{F}_1^k$, for some $k \in \mathbb{Z}$. Since $\Psi^{-1} \circ \widehat{\Psi}$ is also in the centralizer of $\mathcal{I}_1$, we deduce that $\mathcal{F}_1^k$ commutes with $\mathcal{I}_1$. But we also have $\mathcal{F}_1^k \circ \mathcal{I}_1 = \mathcal{I}_1 \circ \mathcal{F}_1^{-k}$, and hence, $k = 0$, i.e., $\widehat{\Psi} = \Psi$ on Λ_1.

Assuming that (3.4) holds, we also have the following result.

Lemma 3.5. *The function χ in (3.3) can be chosen such that $\chi \circ \mathcal{I}_1 = -\chi$, i.e.,*

$$\chi(s_1, -r_1) = -\chi(s_1, r_1), \quad \forall\, (s_1, r_1) \in \Lambda_1. \tag{3.7}$$

Proof. For $i = 1, 2$, we have $\tau_i = \tau_i \circ \mathcal{I}_i \circ \mathcal{F}_i$ (see Figure 8) and $\mathcal{F}_i \circ \mathcal{I}_i = \mathcal{I}_i \circ \mathcal{F}_i^{-1}$. Thus, by (3.4), we deduce that on Λ_1, it holds

$$\begin{aligned}\chi \circ \mathcal{F}_1 - \chi &= \tau_2 \circ \Psi - \tau_1 = \tau_2 \circ \mathcal{I}_2 \circ \mathcal{F}_2 \circ \Psi - \tau_1 \circ \mathcal{I}_1 \circ \mathcal{F}_1\\ &= \tau_2 \circ \mathcal{I}_2 \circ \Psi \circ \mathcal{F}_1 - \tau_1 \circ \mathcal{I}_1 \circ \mathcal{F}_1 = \big(\tau_2 \circ \Psi - \tau_1\big) \circ \mathcal{I}_1 \circ \mathcal{F}_1\\ &= \chi \circ \mathcal{F}_1 \circ \mathcal{I}_1 \circ \mathcal{F}_1 - \chi \circ \mathcal{I}_1 \circ \mathcal{F}_1 = \chi \circ \mathcal{I}_1 - \chi \circ \mathcal{I}_1 \circ \mathcal{F}_1,\end{aligned}$$

hence

$$(\chi + \chi \circ \mathcal{I}_1) \circ \mathcal{F}_1 = \chi + \chi \circ \mathcal{I}_1.$$

Therefore, the function $\chi + \chi \circ \mathcal{I}_1$ on Λ_1 is $\mathcal{F}_1$-invariant, hence constant, as $\mathcal{F}_1|_{\Lambda_1}$ is transitive and χ is continuous. Since χ is defined up to constant (for any $c \in \mathbb{R}$, (3.3) also holds for $\chi + c$ in place of χ), we can assume that this constant vanishes, which concludes. □

Lemma 3.6. *Let $\lambda_i = -r_i ds_i$ be the Liouville one-form, for $i = 1, 2$. It holds*

$$\Psi^*\lambda_2 - \lambda_1 = d\chi \quad \text{on } \Lambda_1. \tag{3.8}$$

Proof. Since $\mathcal{F}_i^*\lambda_i - \lambda_i = d\tau_i$, $i = 1, 2$, and deduce from (3.3) that on Λ_1, it holds

$$\begin{aligned}\mathcal{F}_1^*\big(\Psi^*\lambda_2 - \lambda_1 - d\chi\big) &= \Psi^*(\lambda_2 + d\tau_2) - \lambda_1 - d\tau_1 - \mathcal{F}_1^* d\chi \\ &= \Psi^*\lambda_2 - \lambda_1 + d(\tau_2 \circ \Psi - \tau_1 - \chi \circ \mathcal{F}_1) = \Psi^*\lambda_2 - \lambda_1 - d\chi.\end{aligned}$$

Let ϖ be the one-form $(\Psi^*\lambda_2 - \lambda_1 - d\chi)|_{\Lambda_1}$. By the above identity, for any q-periodic point $x_1 \in \Lambda_1$, $q \geq 2$, we have $\varpi(x_1) = (\mathcal{F}_1^q)^*\varpi(x_1) = \varpi(x_1) \circ D\mathcal{F}_1^q(x_1)$; as $D\mathcal{F}_1^q(x_1)$ is hyperbolic, by considering stable/unstable eigenvectors, we get that $\varpi(x_1) = 0$. Since periodic points are dense in Λ_1, we deduce that $\varpi = 0$, as desired. □

In particular, for any point $x_1 = (s_1, 0) \in \Lambda_1 \cap \{r_1 = 0\}$, (3.7) gives $\chi(x_1) = 0$, while (3.8) gives $d\chi(x_1) = 0$, as $\Psi(\Lambda_1 \cap \{r_1 = 0\}) = \Lambda_2 \cap \{r_2 = 0\}$. The proof of Theorem C is now complete.

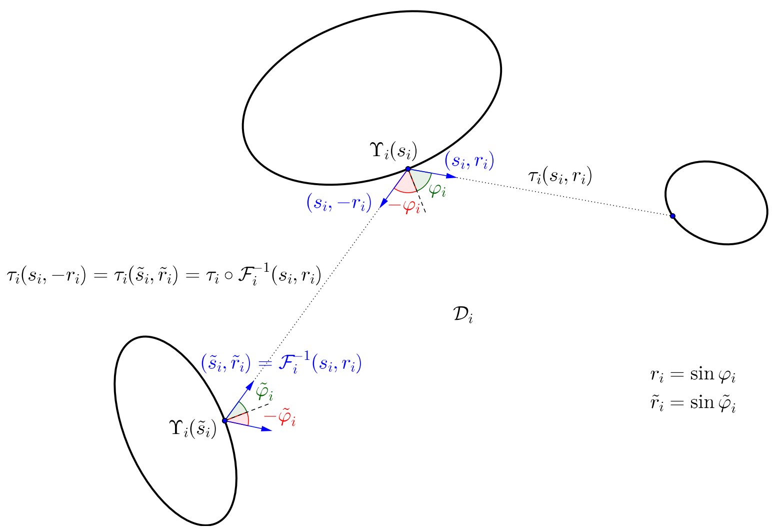

FIGURE 8. Time-reversal symmetry and generating functions.

Similarly, the conjugacy $\widetilde{\Psi}$ between the billiard flows $\Phi_1|_{\Lambda_1^{\tau_1}}, \Phi_2|_{\Lambda_2^{\tau_2}}$ is not unique, as we can pre-, resp. post-compose it with any Φ_1^t, resp. Φ_2^t, $t \in \mathbb{R}$. Yet, there is also a canonical way to choose it, which we now explain. For $i = 1, 2$, we denote by $\widetilde{\mathcal{I}}_i \colon (x_i, y_i, \omega_i) \mapsto (x_i, y_i, \omega_i + \pi)$ the time reversal involution in (x_i, y_i, ω_i)-coordinates. Let us for instance assume that for $i = 1, 2$, there exists $X_i \in \Lambda_i^{\tau_i}$ associated to a point on $\partial\mathcal{D}_i$ with a perpendicular bounce, whose orbit is dense, and such that X_2, $\widetilde{\Psi}(X_1)$ are in the same orbit. After time-translation, $\widetilde{\Psi}(X_1) = X_2$, and then,

$$\widetilde{\Psi} \circ \widetilde{\mathcal{I}}_1|_{\Lambda_1^{\tau_1}} = \widetilde{\mathcal{I}}_2 \circ \widetilde{\Psi}|_{\Lambda_1^{\tau_1}}. \tag{3.9}$$

To show this, we argue as in Lemma 3.3: indeed, as $\widetilde{\Psi}(X_1) = X_2$, we see that (3.9) holds on the orbit of X_1, hence everywhere, by the transitivity of $\Phi_1|_{\Lambda_1^{\tau_1}}$.

Although it is not clear *a priori* that $\widetilde{\Psi}$ sends points associated to bounces on $\partial\mathcal{D}_1$ to points associated to bounces on $\partial\mathcal{D}_2$, we will show that it is indeed the case when the point on $\partial\mathcal{D}_1$ has a perpendicular bounce. For $i=1,2$, we denote by $\widetilde{\Pi}_i\colon (x_i,y_i,\omega_i)\mapsto(x_i,y_i)$ the projection on the table $\mathcal{D}_i$.

Lemma 3.7. *Assume that (3.9) holds. Then, for any $Y_1\in\Lambda_1^{\tau_1}$ associated to a point $\widetilde{\Pi}_1(Y_1)\in\partial\mathcal{D}_1$ with a perpendicular bounce, its image $Y_2:=\widetilde{\Psi}(Y_1)\in\Lambda_2^{\tau_2}$ under $\widetilde{\Psi}$ is also associated to a point $\widetilde{\Pi}_2(Y_2)\in\partial\mathcal{D}_2$ with a perpendicular bounce on an obstacle.*

Proof. Let Y_1, $Y_2:=\widetilde{\Psi}(Y_1)$ be as in the lemma. As Y_1 has a perpendicular bounce, we have $\widetilde{\mathcal{I}}_1\circ\Phi_1^{-t}(Y_1)=\Phi_1^t(Y_1)$, for all $t\in\mathbb{R}$. Since $\widetilde{\Psi}$ conjugates $\Phi_1|_{\Lambda_1^{\tau_1}}$ to $\Phi_2|_{\Lambda_2^{\tau_2}}$, by (3.9), and as $\widetilde{\Pi}_i\circ\widetilde{\mathcal{I}}_i=\widetilde{\Pi}_i$, $i=1,2$, we deduce that for any $t\in\mathbb{R}$, it holds

$$\begin{aligned}\widetilde{\Pi}_2\circ\Phi_2^{-t}(Y_2)&=\widetilde{\Pi}_2\circ\widetilde{\mathcal{I}}_2\circ\widetilde{\Psi}\circ\Phi_1^{-t}(Y_1)=\widetilde{\Pi}_2\circ\widetilde{\Psi}\circ\widetilde{\mathcal{I}}_1\circ\Phi_1^{-t}(Y_1)\\&=\widetilde{\Pi}_2\circ\widetilde{\Psi}\circ\Phi_1^{t}(Y_1)=\widetilde{\Pi}_2\circ\Phi_2^{t}(Y_2).\end{aligned}$$

But $\widetilde{\Pi}_2\circ\Phi_2^{-t}(Y_2)=\widetilde{\Pi}_2\circ\Phi_2^{t}(Y_2)$, for all $t\in\mathbb{R}$, if and only if Y_2 is associated to a point on $\partial\mathcal{D}_2$ with a perpendicular bounce, which concludes. □

3.2. **Jacobi fields.** As the flow conjugacy $\widetilde{\Psi}$ between Φ_1 and Φ_2 is $\mathcal{C}^{k-1}$ in Whitney sense, it makes sense to look at how it acts on infinitesimal geodesic variations. It is related to the notion of *Jacobi fields* for billiard flows, which we now briefly recall; for more details, we refer the reader to [32, 33, 53, 54]. For that, it is more natural to work with (x,y,ω)-coordinates (see Subsection 1.3), which reflect in a better way the geometry of the table. Recall that for $i=1,2$, there exist $D\Phi_i$-invariant subbundles $T^0\mathfrak{M}_i$, $T^\perp\mathfrak{M}_i$, where for each point $X_i=(x_i,y_i,\omega_i)$ of the phase space $\mathfrak{M}_i$,

$$\begin{aligned}T_{X_i}\mathfrak{M}_i\supset T^0_{X_i}\mathfrak{M}_i&:=\ker\big(-\sin\omega dx_i+\cos\omega dy_i\big)\cap\ker\big(d\omega_i\big),\\T_{X_i}\mathfrak{M}_i\supset T^\perp_{X_i}\mathfrak{M}_i&:=\ker\big(\cos\omega_i dx_i+\sin\omega_i dy_i\big),\end{aligned}$$

$\vartheta_i:=\cos\omega_i dx_i+\sin\omega_i dy_i$ being the contact form in these coordinates, and $T^\perp\mathfrak{M}_i$ the associated contact distribution. For $i=1,2$, the contact form ϑ_i is Φ_i-invariant, adapted to $\Lambda_i^{\tau_i}$ in the sense of Definition 1.2, and $\widetilde{\Psi}^*\vartheta_2|_{\Lambda_1^{\tau_1}}=\vartheta_1|_{\Lambda_1^{\tau_1}}$; in particular, the differential $D\widetilde{\Psi}$ respects the splitting $T^0\mathfrak{M}_i\oplus T^\perp\mathfrak{M}_i$, $i=1,2$.

Let $\{\gamma_1(t,u)\}_{t,u\in\mathbb{R}}$ be a family of billiard trajectories for Φ_1 parametrized in arclength, where t is the time, and for each parameter u, $\{\gamma_1(t,u)\}_{t\in\mathbb{R}}$ is an orbit in S_1. We may then define a Jacobi field J_1 along $\gamma_1(\cdot):=\gamma_1(\cdot,0)$ as follows:

$$J_1\colon t\mapsto\frac{\partial\gamma_1(t,u)}{\partial u}|_{u=0}.$$

The Jacobi field J_1 is the infinitesimal description of the family of billiard orbits around $u=0$.

Jacobi fields split naturally into a component parallel to the trajectory, i.e., contained in $T^0\mathfrak{M}_1$, and a component perpendicular to the trajectory, i.e. contained in the contact distribution $T^\perp\mathfrak{M}_1$. At a given point, we may consider two special families of Jacobi fields: the Jacobi fields J_1 such that $J_1(0)\neq 0$ and $J_1'(0)=0$, and the Jacobi fields J_1 such that $J_1(0)=0$ and $J_1'(0)\neq 0$; we call the latter *radial*.

Let J_1 be a perpendicular Jacobi field for Φ_1. For each $t\in\mathbb{R}$, the vector $J_1'(t):=\frac{d}{dt}J_1(t)$ is automatically lying in the contact plane $T^\perp_{X_1(t)}\mathfrak{M}_1$, where $X_1(t)\in\mathfrak{M}_1$ is the point in phase space associated to $(\gamma_1(t), \gamma_1'(t))$. Between collisions, the billiard trajectories are geodesics of the Euclidean metric, hence $J_1''=0$, and

$$\begin{bmatrix} J_1(t) \\ J_1'(t) \end{bmatrix} = \begin{bmatrix} J_1(0)+tJ_1'(0) \\ J_1'(0) \end{bmatrix}.$$

The Jacobi field J_1 can be naturally extended beyond reflections on the boundary of the table. More precisely, at a collision time $t_c \in \mathbb{R}$, if $J_1(t_c^-)$ and $J_1(t_c^+)$ are the Jacobi fields immediately before and after the collision, it holds (see [53, 54])

$$\begin{bmatrix} J_1(t_c^+) \\ J_1'(t_c^+) \end{bmatrix} = \begin{bmatrix} -J_1(t_c^-) \\ \varrho_1 J_1(t_c^-) - J_1'(t_c^-) \end{bmatrix}, \quad \varrho_1 := \frac{2\mathcal{K}_1(s_1)}{\nu_1},$$

where $\mathcal{K}_1(s_1)$ is the curvature at the point $\Upsilon_1(s_1)$ where the collision happens, and $\nu_1 = \cos\varphi_1 \neq 0$, where φ_1 is the angle of collision at $\Upsilon_1(s_1)$.

By the conjugacy map $\widetilde{\Psi}$, to each trajectory $\{\gamma_1(t,u)\}_{t\in\mathbb{R}}$ corresponds a trajectory $\{\gamma_2(t,u)\}_{t\in\mathbb{R}}$ for the flow Φ_2, and as $\widetilde{\Psi}$ is regular, the Jacobi field J_1 is mapped to a Jacobi field J_2 for Φ_2 which undergoes similar transformations as those described above for J_1. Moreover, as $D\widetilde{\Psi}$ preserves the contact planes, perpendicular Jacobi fields for Φ_1 are sent to perpendicular Jacobi fields for Φ_2, hence J_2 are perpendicular.

3.3. **Proof of Theorem D.** Let us now give the proof of Theorem D. Let $\mathcal{D}_1, \mathcal{D}_2$ be two billiards with $\mathcal{C}^k$ boundaries, for some integer $k \geq 3$, and let $\mathcal{F}_1, \mathcal{F}_2$ be the associated billiard maps. Assume that there exists a horseshoe Λ_1, resp. Λ_2 for $\mathcal{F}_1$, resp. $\mathcal{F}_2$, such that $\mathcal{F}_1|_{\Lambda_1}$ and $\mathcal{F}_2|_{\Lambda_2}$ are topologically conjugated and have the same periodic length data. Therefore, there exists a conjugacy $\Psi\colon \Lambda_1 \to \Lambda_2$ between $\mathcal{F}_1|_{\Lambda_1}, \mathcal{F}_2|_{\Lambda_2}$ which is $\mathcal{C}^{k-1}$ in Whitney sense, and such that $\Psi^*(ds_2 \wedge dr_2)|_{\Lambda_1} = ds_1 \wedge dr_1|_{\Lambda_1}$. We further assume that

(1) $\Psi^{-1} \circ \mathcal{I}_2 \circ \Psi \circ \mathcal{I}_1|_{\Lambda_1}$ fixes $\mathcal{F}_1$-orbits, where $\mathcal{I}_i\colon (s_i, r_i) \mapsto (s_i, -r_i)$, for $i=1,2$;
(2) there exists a point $x_1 \in \Lambda_1 \cap \{r_1 = 0\}$ whose orbit is dense in Λ_1, and such that $\Psi(x_1) \in \mathcal{F}_2^{-k}(\{r_2=0\})$ for some $k \in \mathbb{Z}$;
(3) $\mathcal{F}_1^2|_{\Lambda_1}$ is transitive;
(4) $D\Psi$ preserves vertical fibers.

For the billiard flows, the analogue of (4) would be to require that radial Jacobi fields (see Subsection 3.2) are sent to radial Jacobi fields by the conjugacy $\widetilde{\Psi}$.

By (1)-(2) and Lemma 3.3, without loss of generality, we can (and will) assume that the conjugacy map Ψ satisfies $\Psi \circ \mathcal{I}_1 = \mathcal{I}_2 \circ \Psi$ on Λ_1. It follows from assumption (4) that for the functions $a = \frac{\partial s_2}{\partial s_1} = \left(\frac{\partial r_2}{\partial r_1}\right)^{-1}\colon \Lambda_1 \to \mathbb{R}^*$ and $b = \frac{\partial r_2}{\partial s_1}\colon \Lambda_1 \to \mathbb{R}$,

$$D\Psi(s_1,r_1) = \begin{bmatrix} a(s_1,r_1) & 0 \\ b(s_1,r_1) & a^{-1}(s_1,r_1) \end{bmatrix}, \quad \forall\,(s_1,r_1) \in \Lambda_1. \tag{3.10}$$

Besides, as Ψ is $\mathcal{C}^{k-1}$ in Whitney sense, the functions a, b are $\mathcal{C}^{k-2}$ in Whitney sense.

In the following, we derive further consequences on the conjugacy map Ψ. For $(s_2, r_2) \in \Lambda_2$, we see $s_2 = s_2(s_1,r_1)$ and $r_2 = r_2(s_1,r_1)$ as functions of $(s_1,r_1) \in \Lambda_1$.

Claim 3.8. *The functions a, b satisfy $a(s_1,-r_1) = a(s_1,r_1)$, $b(s_1,-r_1) = -b(s_1,r_1)$, for any $(s_1,r_1) \in \Lambda_1$.*

Proof. Fix any $(s_1, r_1) \in \Lambda_1$. Differentiating the relation $\Psi \circ \mathcal{I}_1 = \mathcal{I}_2 \circ \Psi$, we get

$$\begin{bmatrix} a(s_1,-r_1) & 0 \\ b(s_1,-r_1) & a^{-1}(s_1,-r_1) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a(s_1,r_1) & 0 \\ b(s_1,r_1) & a^{-1}(s_1,r_1) \end{bmatrix},$$

which concludes. □

Fix $(s_1, r_1) \in \Lambda_1$, and let $(s_2, r_2) := \Psi(s_1, r_1) \in \Lambda_2$. By (1.3), for $i = 1, 2$, we have

$$D\mathcal{F}_i(s_i, r_i) = -\begin{bmatrix} \frac{1}{\nu_i'}(h_i\mathcal{K}_i + \nu_i) & \frac{h_i}{\nu_i\nu_i'} \\ h_i\mathcal{K}_i\mathcal{K}_i' + \mathcal{K}_i\nu_i' + \mathcal{K}_i'\nu_i & \frac{1}{\nu_i}(h_i\mathcal{K}_i' + \nu_i') \end{bmatrix},$$

where $(s_i', r_i') = \mathcal{F}_i(s_i, r_i)$, $\nu_i := \sqrt{1 - r_i^2}$, $\nu_i' := \sqrt{1 - (r_i')^2}$, $h_i := h_i(s_i, s_i') = \tau_i(s_i, r_i)$ is the length of the associated orbit segment, and $\mathcal{K}_i := \mathcal{K}_i(s_i)$, $\mathcal{K}_i' := \mathcal{K}_i(s_i')$ are the respective curvatures. Let $a := a(s_1, r_1)$, $a' := a(s_1', r_1')$, $b := b(s_1, r_1)$ and $b' := b(s_1', r_1')$, so that

$$D\Psi(s_1, r_1) = \begin{bmatrix} a & 0 \\ b & a^{-1} \end{bmatrix}, \quad D\Psi(s_1', r_1') = \begin{bmatrix} a' & 0 \\ b' & (a')^{-1} \end{bmatrix}.$$

Since $D\Psi(s_1', r_1')D\mathcal{F}_1(s_1, r_1) = D\mathcal{F}_2(s_2, r_2)D\Psi(s_1, r_1)$, it follows that

$$\frac{a'}{\nu_1'}(h_1\mathcal{K}_1 + \nu_1) = \frac{a}{\nu_2'}(h_2\mathcal{K}_2 + \nu_2) + \frac{bh_2}{\nu_2\nu_2'},$$

$$\frac{a'h_1}{\nu_1\nu_1'} = \frac{a^{-1}h_2}{\nu_2\nu_2'},$$

$$\frac{1}{a'\nu_1}(h_1\mathcal{K}_1' + \nu_1') + \frac{b'h_1}{\nu_1\nu_1'} = \frac{1}{a\nu_2}(h_2\mathcal{K}_2' + \nu_2').$$

In particular, we obtain

$$\frac{\nu_1}{h_1}(h_1\mathcal{K}_1 + \nu_1) = \frac{a^2\nu_2}{h_2}(h_2\mathcal{K}_2 + \nu_2) + ab, \tag{3.11}$$

$$\frac{h_2}{h_1} = \frac{a\nu_2}{\nu_1} \cdot \frac{a'\nu_2'}{\nu_1'}. \tag{3.12}$$

Lemma 3.9. *It holds*

$$\Psi^*\tau_2|_{\Lambda_1} = \tau_1|_{\Lambda_1}. \tag{3.13}$$

Proof. Let us show that the function $\frac{\tau_2 \circ \Psi}{\tau_1}$ is $\mathcal{F}_1$-invariant. Fix any $(s_1, r_1) \in \Lambda_1$, and let $(s_2, r_2) := \Psi(s_1, r_1) \in \Lambda_2$. As $\nu\colon r \mapsto \sqrt{1 - (r)^2}$ is even, and by Claim 3.8, the right hand side of (3.12) is unchanged if we replace (s_i, r_i) with $(s_i, -r_i)$, thus,

$$\frac{\tau_2(s_2, r_2)}{\tau_1(s_1, r_1)} = \frac{\tau_2(s_2, -r_2)}{\tau_1(s_1, -r_1)}.$$

Since $\frac{\tau_2(s_2,r_2)}{\tau_1(s_1,r_1)} = \frac{\tau_2\circ\Psi}{\tau_1}(s_1, r_1)$, and by the time-reversal property, we get $\tau_i(s_i, -r_i) = \tau_i(\mathcal{F}_i^{-1}(s_i, r_i))$, for $i = 1, 2$ (see Figure 8), so that $\frac{\tau_2(s_2,-r_2)}{\tau_1(s_1,-r_1)} = \frac{\tau_2\circ\Psi}{\tau_1}\big(\mathcal{F}_1^{-1}(s_1, r_1)\big)$. By the above identity, we thus have

$$\frac{\tau_2 \circ \Psi}{\tau_1}(s_1, r_1) = \frac{\tau_2 \circ \Psi}{\tau_1}\big(\mathcal{F}_1^{-1}(s_1, r_1)\big).$$

In other words, the quantity $\frac{\tau_2\circ\Psi}{\tau_1}$ is $\mathcal{F}_1$-invariant, hence constant on each orbit. Actually, since $\mathcal{F}_1|_{\Lambda_1}$, $\mathcal{F}_2|_{\Lambda_2}$ are transitive, by considering a dense orbit $\mathcal{O}_1$ in Λ_1, and by continuity, we see that the function $\frac{\tau_2\circ\Psi}{\tau_1}$ is constant on Λ_1, i.e., $\tau_2\circ\Psi|_{\Lambda_1} = c\tau_1|_{\Lambda_1}$, for some constant $c\in\mathbb{R}$. In particular, for any periodic orbit $\mathcal{O}_1$ for $\mathcal{F}_1$ and $\mathcal{O}_2 := \Psi(\mathcal{O}_1)$ for $\mathcal{F}_2$, by summing this identity over the different orbit segments, we get that the ratio of the perimeters of $\mathcal{O}_1$ and $\mathcal{O}_2$ is equal to c. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ have the same periodic length data, we conclude that $c = 1$, as wanted. □

Lemma 3.10. *We have $\Psi^* d\tau_2|_{\Lambda_1} = d\tau_1|_{\Lambda_1}$. Moreover, it holds*

$$\Psi^*\lambda_2|_{\Lambda_1} = \lambda_1|_{\Lambda_1}, \quad \text{where } \lambda_i = -r_i ds_i,\ i = 1, 2,$$

and $r_2 a = r_1$ on Λ_1, i.e.,

$$r_2(s_1, r_1)a(s_1, r_1) = r_1, \quad \forall\,(s_1, r_1) \in \Lambda_1. \tag{3.14}$$

Proof. The identity $\Psi^* d\tau_2|_{\Lambda_1} = d\tau_1|_{\Lambda_1}$ is a direct consequence of (3.13), as $(\Psi^* d\tau_2 - d\tau_1)|_{\Lambda_1} = d(\Psi^*\tau_2 - \tau_1)|_{\Lambda_1} = 0$. Recall that $\big(\mathcal{F}_i^*\lambda_i - \lambda_i\big)|_{\Lambda_i} = d\tau_i|_{\Lambda_i}$, for $i = 1, 2$. From $\Psi^* d\tau_2|_{\Lambda_1} = d\tau_1|_{\Lambda_1}$ and $\Psi\circ\mathcal{F}_1|_{\Lambda_1} = \mathcal{F}_2\circ\Psi|_{\Lambda_1}$, we deduce that

$$\big(\mathcal{F}_1^*\Psi^*\lambda_2 - \Psi^*\lambda_2\big)|_{\Lambda_1} = \Psi^*\big(\mathcal{F}_2^*\lambda_2 - \lambda_2\big)|_{\Lambda_1} = \big(\mathcal{F}_1^*\lambda_1 - \lambda_1\big)|_{\Lambda_1},$$

and thus,

$$\mathcal{F}_1^*\big(\Psi^*\lambda_2 - \lambda_1\big)|_{\Lambda_1} = \big(\Psi^*\lambda_2 - \lambda_1\big)|_{\Lambda_1}. \tag{3.15}$$

In other words, the 1-form $\big(\Psi^*\lambda_2 - \lambda_1\big)|_{\Lambda_1}$ is invariant under the dynamics.

Recall that (see (2)) we assume the existence of a point $x_1 \in \Lambda_1 \cap \{r_1 = 0\}$ with a perpendicular bounce and whose orbit is dense in Λ_1. Let $x_2 := \Psi(x_1) \in \Lambda_2$. Since $\Psi\circ\mathcal{I}_1 = \mathcal{I}_2\circ\Psi$ on Λ_1, we have $\mathcal{I}_2(x_2) = \Psi\circ\mathcal{I}_1(x_1) = x_2$, hence $x_2 \in \{r_2 = 0\}$ also has a perpendicular bounce, and clearly, its orbit is dense in Λ_2. Since $\lambda_i = -r_i ds_i$ for $i = 1, 2$, we have $\Psi^*\lambda_2(x_1) = \lambda_1(x_1) = 0$, hence

$$(\Psi^*\lambda_2 - \lambda_1)(x_1) = 0.$$

By (3.15), and as $D\Psi$ is invertible, we deduce that $\Psi^*\lambda_2 - \lambda_1$ vanishes on the orbit of x_1; as the latter is dense, and by continuity, we conclude that $\Psi^*\lambda_2|_{\Lambda_1} = \lambda_1|_{\Lambda_1}$.

Besides, for any $(s_1, r_1) \in \Lambda_1$, $(s_2, r_2) = \Psi(s_1, r_1) \in \Lambda_2$, we have $\Psi^*\lambda_2(s_1, r_1) = -r_2\big(\frac{\partial s_2}{\partial s_1} ds_1 + \frac{\partial s_2}{\partial r_1} dr_1\big) = -r_2 a(s_1, r_1) ds_1$; as $\Psi^*\lambda_2(s_1, r_1) = \lambda_1(s_1, r_1) = -r_1 ds_1$, we deduce that $r_2 a(s_1, r_1) = r_1$. □

Lemma 3.11. *We have $a|_{\Lambda_1} = 1$, $b|_{\Lambda_1} = 0$, and*

$$r_2(s_1, r_1) = r_1, \quad \forall\,(s_1, r_1) \in \Lambda_1.$$

Proof. Let us consider the function $\gamma\colon \Lambda_1 \to \mathbb{R}$, $(s_1, r_1) \mapsto \frac{a(s_1,r_1)\sqrt{1-r_2^2(s_1,r_1)}}{\sqrt{1-r_1^2}}$.

By (3.12) and (3.13), we have

$$\gamma(x_1)\cdot\gamma\circ\mathcal{F}_1(x_1) = 1, \quad \forall\, x_1 \in \Lambda_1. \tag{3.16}$$

We deduce that the function γ is $\mathcal{F}_1^2$-invariant. By our assumption (3) that $\mathcal{F}_1^2|_{\Lambda_1}$ is transitive, and by continuity, it follows that γ is equal to some constant c on Λ_1. By (3.16), we also have $c^2 = 1$, and then $c = 1$ (since $a \geq 0$). Moreover, it follows from (3.14) and the fact that $\gamma = 1$ that

$$\frac{r_1}{\sqrt{1-r_1^2}} = \frac{r_2(s_1, r_1)}{\sqrt{1-r_2^2(s_1, r_1)}}.$$

In particular, r_1 and $r_2(s_1, r_1)$ have the same sign; from the above identity, it follows that $r_2(s_1, r_1) = r_1$, for any $(s_1, r_1) \in \Lambda_1$. Besides, by differentiating the identity $r_2(s_1, r_1) = r_1$, we also deduce that $a = \frac{\partial r_2}{\partial r_1} = 1$ and $b = \frac{\partial r_2}{\partial s_1} = 0$ on Λ_1. □

Fix $(s_1, r_1) \in \Lambda_1$, and let $(s_2, r_2) := \Psi(s_1, r_1) \in \Lambda_2$. For $i = 1, 2$, we let $(s_i', r_i') = \mathcal{F}_i(s_i, r_i)$, $\nu_i := \sqrt{1 - r_i^2}$, $\nu_i' := \sqrt{1 - (r_i')^2}$, and $h_i := h_i(s_i, s_i') = \tau_i(s_i, r_i)$. We also denote by $\mathcal{K}_i \colon s_i \mapsto \mathcal{K}_i(s_i)$ the curvature function.

Lemma 3.12. *The $(k-2)$-jets of $\mathcal{K}_1$ and $\mathcal{K}_2$ at s_1 and s_2 respectively are the same:*

$$\mathcal{K}_1^{(j)}(s_1) = \mathcal{K}_2^{(j)}(s_2), \quad \forall\, j = 0, \dots, k-2.$$

Proof. By Lemma 3.11, we have $r_1 = r_2$, $r_1' = r_2'$, $\nu_1 = \nu_2$, $\nu_1' = \nu_2'$, and $a(s_1, r_1) b(s_1, r_1) = 0$. By Lemma 3.9, and as $h_2 = \tau_2 \circ \Psi(s_1, r_1)$, we also know that $h_1 = h_2$. As $\nu_1 \neq 0$ (we are away from tangential collisions), (3.11) yields

$$\mathcal{K}_1(s_1) = \mathcal{K}_2(s_2) = \mathcal{K}_2(s_2(s_1, r_1)).$$

By differentiating this identity with respect to s_1, we have

$$\mathcal{K}_1'(s_1) = \mathcal{K}_2'(s_2) \frac{\partial s_2}{\partial s_1}(s_1, r_1) = \mathcal{K}_2'(s_2) a(s_1, r_1) = \mathcal{K}_2'(s_2). \tag{3.17}$$

Indeed, by Lemma 3.11, the function a is constant equal to 1 on Λ_1. Differentiating this identity repeatedly, and as $\mathcal{K}_1, \mathcal{K}_2$ are $\mathcal{C}^{k-2}$, we conclude that the $(k-2)$-jets of $\mathcal{K}_1$ and $\mathcal{K}_2$ at s_1 and s_2 respectively are the same. □

Let us now conclude the proof of Theorem D. Point (2) follows from (3.10) and Lemma 3.11. Point (5) was shown in Lemma 3.9. Point (6) follows from Lemma 3.11 (recall that for any $(s_1, r_1) \in \Lambda_1$, we have $r_1 = \sin \varphi_1$ and $r_2 = r_2(s_1, r_1) = \sin \varphi_2$, where φ_1, φ_2 are the angles at the corresponding collisions). By considering a dense orbit in Λ_1, and arguing as in Remark 1.14, we deduce from items (5) and (6) that the traces $\Pi(\Lambda_1), \Pi(\Lambda_2)$ on $\mathcal{D}_1, \mathcal{D}_2$ are isometric, where $\Pi \colon (s_i, r_i) \mapsto s_i$ is the projection on $\partial \mathcal{D}_i$, for $i = 1, 2$. This allows us to define a homeomorphism $\mathcal{Z} = \mathcal{Z}_{\Lambda_1, \Lambda_2} \colon \Pi(\Lambda_1) \to \Pi(\Lambda_2)$ as in point (3). Moreover we can rewrite $\Psi \colon \Lambda_1 \to \Lambda_2$ as $(s_1, r_1) \mapsto (\mathcal{Z}(s_1), r_1)$, hence $\mathcal{Z}$ is $\mathcal{C}^{k-1}$. Finally, point (4) follows from the definition of $\mathcal{Z}$ and from Lemma 3.12. □

3.4. **Proof of Corollary F.** As in Corollary F, fix $\ell \geq 3$, and let $\mathcal{D}_1, \mathcal{D}_2 \in \mathbf{B}_{ne}(\ell)$ with $\mathcal{C}^k$ boundaries, for some $k \geq 3$, such that $\mathcal{D}_1, \mathcal{D}_2$ have the same marked length spectrum. Then, according to Theorem C, the respective billiards maps $\mathcal{F}_1, \mathcal{F}_2$ are conjugated on $\Omega(\mathcal{F}_1), \Omega(\mathcal{F}_2)$ by a map $\Psi \colon \Omega(\mathcal{F}_1) \to \Omega(\mathcal{F}_2)$ that is $\mathcal{C}^{k-1}$ in Whitney sense and such that $\Psi^*(ds_2 \wedge dr_2) = ds_1 \wedge dr_1$ on $\Omega(\mathcal{F}_1)$.

Let us recall that $\mathcal{F}_1|_{\Omega(\mathcal{F}_1)}$, $\mathcal{F}_2|_{\Omega(\mathcal{F}_2)}$ are conjugated to the same subshift of finite type on the alphabet $\mathscr{A} = \{1, \dots, \ell\}$ associated with the transition matrix $(1 - \delta_{i,j})_{1 \leq i,j \leq \ell}$, where $\delta_{i,j} = 1$, when $i = j$, and $\delta_{i,j} = 0$ otherwise. We say that a word $\varsigma = (\varsigma_j)_j \in \mathscr{A}^{\mathbb{Z}}$ is *admissible*, if $\varsigma_{j+1} \neq \varsigma_j$, for all $j \in \mathbb{Z}$. We also let $\mathrm{Adm} \subset \cup_{j \geq 2} \mathscr{A}^j$ be the set of all finite words $\sigma = \sigma_1 \dots \sigma_j$, $j \geq 2$, such that $\sigma^\infty := \cdots \sigma\sigma\sigma \cdots \in \mathrm{Adm}_\infty$. We normalize the conjugacy Ψ by requiring that for each $y_1 \in \Omega(\mathcal{F}_1)$, the points y_1 and $\Psi(y_1) \in \Omega(\mathcal{F}_2)$ are coded by the same admissible word. Symbolically, the actions of $\mathcal{I}_1, \mathcal{I}_2$ amount to switching the symbolic past and future. In particular, by our choice that Ψ preserves the symbolic coding, we have $\Psi \circ \mathcal{I}_1 = \mathcal{I}_2 \circ \Psi$ on $\Omega(\mathcal{F}_1)$, where $\mathcal{I}_i\colon (s_i,r_i) \mapsto (s_i,-r_i)$ is the time-reversal involution, for $i=1,2$ (*a fortiori*, assumption (1) in Theorem D is satisfied).

In the following, we assume (4), i.e., that $D\Psi$ preserves vertical fibers. In order to conclude the proof of Corollary F, it suffices to check the remaining assumptions of Theorem D. Assumption (3) follows from the fact that the billiard map $\mathcal{F}_1|_{\Omega(\mathcal{F}_1)}$, resp. $\mathcal{F}_2|_{\Omega(\mathcal{F}_2)}$ restricted to the basic set $\Omega(\mathcal{F}_1)$, resp. $\Omega(\mathcal{F}_2)$, is topologically mixing. Thus, it remains to show assumption (2) about the existence of a point $x_1 \in \Omega(\mathcal{F}_1)\cap \{r_1=0\}$ whose orbit is dense in $\Omega(\mathcal{F}_1)$, which is done in Lemma 3.13 below. In that case, we necessarily have $\Psi(x_1)\in\{r_2=0\}$, as $\mathcal{I}_2(\Psi(x_1)) = \Psi(\mathcal{I}_1(x_1)) = \Psi(x_1)$.

For $i=1,2$, a point $y_i \in \Omega(\mathcal{F}_i)$ is in $\{r_i=0\}$ if and only if its symbolic coding

$$\varsigma = \dots \varsigma_{-1}\underset{\uparrow}{\varsigma_0}\varsigma_1\dots \in \mathscr{A}^{\mathbb{Z}}$$

is *palindromic* at ς_0, i.e., $\varsigma_j = \varsigma_{-j}$, for all $j\in\mathbb{Z}$. Indeed, $y_i \in \{r_i=0\}$ if and only if its future coincides with its past, which can be seen symbolically.

Lemma 3.13. *There exists a point $x_1 \in \Omega(\mathcal{F}_1)\cap\{r_1=0\}$ whose orbit is dense.*

Proof. Let us choose an enumeration $(\sigma_n)_{n\geq 1}$ of the words in Adm, with $\sigma_n = \sigma_{n,1}\dots\sigma_{n,i_n}\in\mathscr{A}^{i_n}$, for $n\geq 1$. Let us construct an infinite word $\xi=(\xi_j)_{j\geq 1}$ which contains each word in Adm. We proceed by induction, by concatenating the words $(\sigma_n)_{n\geq 1}$ in an appropriate way so as to produce an admissible word; more precisely:

- for each $j\in\{1,\dots,i_1\}$, we let $\xi_j := \sigma_{1,j}$;
- if $\sigma_{1,i_1}\neq\sigma_{2,1}$, we extend ξ by adding the word σ_2 after σ_1;
- otherwise, if $\sigma_{1,i_1}=\sigma_{2,1}$, we shift σ_2 by one, and add the shifted word (i.e., the word $\sigma_{2,2}\sigma_{2,3}\dots\sigma_{2,i_2}\sigma_{2,1}$) to ξ right after σ_1;
- we iterate the procedure to complete the construction of ξ.

Let $\overline{\xi} = (\overline{\xi}_j)_{j\leq -1}$, with $\overline{\xi}_j = \xi_{-j}$, for each $j\leq -1$. Fix a letter $\sigma_0\in\mathscr{A}$, with $\sigma_0\neq\xi_1$. We then let $x_1\in\Omega(\mathcal{F}_1)$ be the point whose symbolic coding is given by

$$x_1 \quad\longleftrightarrow\quad \tau := \overline{\xi}\underset{\uparrow}{\sigma_0}\xi.$$

By construction, the word τ is admissible; besides, it is palindromic at σ_0. By the construction of τ, for any point $y_1\in\Omega(\mathcal{F}_1)$, with symbolic coding $(\rho_i)_{i\in\mathbb{Z}}$, and for any integer $n\geq 0$, there exists $k(y_1,n)\geq 0$ such that the points $\mathcal{F}^k(x_1)$ and y_1 have the same symbolic trajectory for n iterates in the past and in the future, namely,

$$\rho_{-n}\dots\rho_{-1}\underset{\uparrow}{\rho_0}\rho_1\dots\rho_n.$$

By hyperbolicity, by choosing the integer $n\geq 0$ larger and larger, the points $\mathcal{F}^{k(y_1,n)}(x_1)$ and y_1 can be made arbitrarily close to each other, which shows that the orbit (and actually the forward orbit) of x_1 is dense in $\Omega(\mathcal{F}_1)$. ☐

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