ACCESSIBILITY FOR DYNAMICALLY COHERENT PARTIALLY HYPERBOLIC DIFFEOMORPHISMS WITH 2D CENTER

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Abstract. We show that for any integer $r \geq 2$, stable accessibility is C^r-dense among partially hyperbolic diffeomorphisms with two-dimensional center that satisfy some strong bunching and are stably dynamically coherent.

Contents

1. Introduction

In 1871 L. Boltzmann stated his ergodic hypothesis when he was studying the motion of gases and thermodynamics. He wanted a property that could let him "characterize the probability of a state by the average time in which the system *is* in this state". Since then, ergodicity has played a key role in dynamical systems,

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physics and probability. Recall that a dynamical system $f\colon M \to M$ preserving a finite measure m is *ergodic* if every f-invariant set has zero or total measure.

After Birkhoff's ergodic theorem, E. Hopf proved in 1939 the ergodicity of the geodesic flow on a surface of constant negative curvature, introducing an argument to get ergodicity which is now called Hopf's argument [H]. Twenty eight years later, D. Anosov [A] improved Hopf's results by proving the ergodicity of the geodesic flow on surfaces of negative (non necessarily constant) curvature and compact manifolds of constant negative curvature. He also showed the ergodicity of uniformly hyperbolic diffeomorphisms, now called Anosov diffeomorphisms. Since hyperbolicity is a C^1-robust condition, Anosov diffeomorphisms became the first example of *stably ergodic* diffeomorphisms, that is, a C^r ergodic diffeomorphism (preserving a measure m) that remains ergodic after a C^1-small perturbation.

For almost thirty years Anosov diffeomorphisms were the only known examples of stably ergodic systems, until 1995 when M. Grayson, C. Pugh and M. Shub [GPS] proved the C^2 stable ergodicity of the time-one map of the geodesic flow on surfaces of constant negative curvature, hence the first non-Anosov stably ergodic example. Despite being non globally hyperbolic, this example has a weak form of hyperbolicity called *partial hyperbolicity.* With the evidence of this work they formulated in a 1995 conference [PS] the following conjecture:

Conjecture 1.1 (Pugh-Shub's stable ergodicity conjecture [PS, PS1])**.** *On any compact connected Riemannian manifold, stable ergodicity is C^r-dense among the set of volume preserving partially hyperbolic diffeomorphisms, for any integer $r \geq 2$.*

They also proposed a program in order to prove this, and split the conjecture into two conjectures:

Conjecture 1.2 (Accessibility implies ergodicity)**.** *A C^2 partially hyperbolic volume preserving diffeomorphism with the essential accessibility property is ergodic.*

Here, *essential accessibility* is a measure-theoretic version of the accessibility property.

Conjecture 1.3 (Density of accessibility)**.** *For any integer $r \in [2, +\infty]$, stable accessibility is open and dense among the set of C^r partially hyperbolic diffeomorphisms, volume preserving or not.*

There has been a lot of progress on these conjectures, mostly depending on the topology and the dimension of the central bundle.

The main conjecture was proven in [RHRHU] in the case where $\dim E^c = 1$ and for the C^r topology (in fact the authors showed C^∞-density). Recently in [ACW] the conjecture was proved in its full generality (any central dimension) for the C^1 topology. Despite these remarkable results, in the C^r case for $r \geq 2$ the conjecture is far from being solved. Recently, M. Leguil and Z. Zhang [LZ] obtained C^r-density of stable ergodicity for partially hyperbolic diffeomorphisms (for any center dimension) with a strong pinching condition, introducing a new technique based on random perturbations.

With respect to Conjecture 1.2, C. Pugh and M. Shub [PS2] proved that a C^2 volume preserving partially hyperbolic diffeomorphism that is dynamically coherent, center bunched and with the essential accessibility property is ergodic. The state-of-the-art on Conjecture 1.2 is the result of K. Burns and A. Wilkinson [BW1] where the authors improved Pugh-Shub's result by removing the dynamical coherence hypothesis, and weakening the center bunching condition. In other words, by these works, a possible strategy to show that stable ergodicity is typical in the C^r topology would be to go further towards Conjecture 1.3, i.e., that stable accessibility is C^r-dense.

Regarding Conjecture 1.3, in [DW, ACW2] stable accessibility is obtained for a C^1-dense set of • all • volume preserving • symplectic partially hyperbolic diffeomorphisms. The authors strongly use C^1 techniques which seem hard to generalize to other topologies.

For the $\dim E^c = 2$ case, there has been many results lately. The first one is the remarkable result by F. Rodríguez-Hertz [RH] where he classified the central accessibility classes and obtained stable ergodicity of certain automorphisms on the torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. Elaborating on these ideas, in [HS] V. Horita and M. Sambarino proved stable ergodicity for skew-products of surface diffeomorphisms over Anosov diffeomorphisms. Recently, A. Ávila and M. Viana [AV] obtained C^1-openness of accessibility and C^r-density for certain *fibered* partially hyperbolic diffeomorphisms with 2-dimensional center bundle using different techniques.

Our purpose in this article is to contribute to the accessibility conjecture (Conjecture 1.3) by proving the C^r-density of accessibility for (stably) dynamically coherent partially hyperbolic diffeomorphism with 2-dimensional center bundle which satisfy some strong bunching condition, for any integer $r \geq 2$.

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2. Preliminaries

2.1. **Partially hyperbolic diffeomorphisms.**

Let us fix a compact Riemannian manifold M of dimension $m \geq 3$. We denote by Vol the volume form, and we denote by $\|\cdot\|$ the norm on TM associated to the Riemannian metric. We say that a diffeomorphism f of M is *partially hyperbolic* if there exist a nontrivial Df-invariant splitting $TM = E^s_f \oplus E^c_f \oplus E^u_f$ of the tangent bundle and continuous positive functions $\nu, \hat{\nu}, \gamma, \hat{\gamma}$ with

$$\nu, \hat{\nu} < 1, \qquad \nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1}, \tag{2.1}$$

such that for any $(x, v) \in TM$, it holds

$$\begin{aligned} \|D_x f(v)\| &< \nu(x)\|v\|, && \text{if } v \in E^s_f(x) \setminus \{0\},\\ \gamma(x)\|v\| < \|D_x f(v)\| &< \hat{\gamma}(x)^{-1}\|v\|, && \text{if } v \in E^c_f(x) \setminus \{0\},\\ \hat{\nu}(x)^{-1}\|v\| &< \|D_x f(v)\|, && \text{if } v \in E^u_f(x) \setminus \{0\}. \end{aligned}$$

For any integer $r \geq 1$, we denote by $\mathcal{PH}^r(M)$ the set of all partially hyperbolic diffeomorphisms of M of class C^r; we also denote by $\mathcal{PH}^r(M, \mathrm{Vol}) \subset \mathcal{PH}^r(M)$ the subset of volume preserving partially hyperbolic diffeomorphisms.

In the rest of this section, we fix an integer $r \geq 1$ and we consider a partially hyperbolic diffeomorphism $f \in \mathcal{PH}^r(M)$. We will denote $d_s := \dim E^s_f$ and $d_u := \dim E^u_f$. The strong bundles E^u_f and E^s_f are uniquely integrable to continuous foliations $\mathcal{W}^u_f$ and $\mathcal{W}^s_f$ respectively, called the *strong unstable* and *strong stable* foliations. For $* = u, s$, and for any $x \in M$, we denote by $\mathcal{W}^*_f(x)$ the leaf of $\mathcal{W}^*_f$ through x. The foliation $\mathcal{W}^*_f$ is invariant under the dynamics, i.e., $f(\mathcal{W}^*_f(x)) = \mathcal{W}^*_f(f(x))$, for all $x \in M$. Moreover, each leaf $\mathcal{W}^*_f(x)$, $x \in M$, is an immersed C^r manifold.

2.2. Dynamical coherence, plaque expansiveness.

The partially hyperbolic diffeomorphism f is *dynamically coherent* if the *center-unstable* bundle $E^{cu}_f := E^c_f \oplus E^u_f$ and the *center-stable* bundle $E^{cs}_f := E^c_f \oplus E^s_f$ integrate respectively to foliations $\mathcal{W}^{cu}_f$, $\mathcal{W}^{cs}_f$, called the *center-unstable foliation*, resp. the *center-stable foliation*, where $\mathcal{W}^u_f$ subfoliates $\mathcal{W}^{cu}_f$, while $\mathcal{W}^s_f$ subfoliates $\mathcal{W}^{cs}_f$. In this case, the collection $\mathcal{W}^c_f$ obtained by intersecting the leaves of $\mathcal{W}^{cs}_f$ and $\mathcal{W}^{cu}_f$ is a foliation which integrates E^c_f, and subfoliates both $\mathcal{W}^{cs}_f$ and $\mathcal{W}^{cu}_f$; it is called the *center foliation*.

In the following, for any $* \in \{s, c, u, cs, cu\}$, we denote by $d_{\mathcal{W}^*_f}$ the leafwise distance, and for any $x \in M$, for any $\epsilon > 0$, we denote by $\mathcal{W}^*_f(x, \varepsilon) := \{y \in \mathcal{W}^*_f(x) : d_{\mathcal{W}^*_f}(x, y) < \varepsilon\}$ the ε-ball in $\mathcal{W}^*_f$ of center x and radius ε.

It is an open question whether dynamical coherence is a C^1-open condition. A closely related property is *plaque expansiveness*.

Definition 2.1 (Plaque expansiveness)**.** *We say that f is* plaque expansive *(see* [HPS, Section 7]*) if f is dynamically coherent and there exists $\varepsilon > 0$ with the following property: if $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ are ε-pseudo orbits which respect $\mathcal{W}^c_f$ such that $d(p_n, q_n) \leq \varepsilon$ for all $n \geq 0$, then $q_n \in \mathcal{W}^c_f(p_n)$. It is known that plaque expansiveness is a C^1-open condition (see Theorem 7.4 in* [HPS]*).*

The following result is due to Hirsch-Pugh-Shub.

Theorem 2.2 (Theorem 7.1, [HPS], see also Theorem 1 in [PSW2])**.** *Let us assume that f is dynamically coherent and plaque expansive. Then any $g \in \mathcal{PH}^1(M)$ which is sufficiently C^1-close to f is also dynamically coherent and plaque expansive. Moreover, there exists a homeomorphism $\mathfrak{h} = \mathfrak{h}_g : M \to M$, called a* leaf conjugacy*, such that $\mathfrak{h}$ maps a f-center leaf to a g-center leaf, and $\mathfrak{h} \circ f(\mathcal{W}^c_f(\cdot)) = g \circ \mathfrak{h}(\mathcal{W}^c_f(\cdot))$.*

2.3. Holonomies.

Let us assume that the diffeomorphism f is dynamically coherent. Let $x_1 \in M$ and let $x_2 \in M$ be sufficiently close to x_1.[1] By transversality, there exist a neighbourhood $\mathcal{U}_1$ of x_1 within $\mathcal{W}^{cu}_f(x_1)$ and a neighbourhood $\mathcal{U}_2$ of x_2 within $\mathcal{W}^{cu}_f(x_2)$ such that for any $z \in \mathcal{U}_1$, the local stable leaf through z intersects $\mathcal{U}_2$ at a unique point, denoted by $H^s_{f,x_1,x_2}(z) \in \mathcal{U}^{cu}_{x_2}$. We thus get a well defined local homeomorphism

$$H^s_{f,x_1,x_2} : \mathcal{U}_1 \to \mathcal{U}_2 \subset \mathcal{W}^{cu}_f(x_2),$$

called the *stable holonomy map*. Note that as a consequence of dynamical coherence, if $x_2 \in \mathcal{W}^s_{f,\mathrm{loc}}(x_1)$, then the image of the restriction $H^s_{f,x_1,x_2}|_{\mathcal{U}_1 \cap \mathcal{W}^c_f(x_1)}$ to the center leaf $\mathcal{W}^c_f(x_1)$ is contained in the center leaf $\mathcal{W}^c_f(x_2)$. Unstable holonomies are defined in a similar way, following local unstable leaves.

[1] In the rest of the paper, all the constructions will be local.

Definition 2.3 (Center bunching)**.** *We say that f is* center bunched *if the functions $\nu, \hat{\nu}, \gamma, \hat{\gamma}$ in (2.1) can be chosen such that*

$$\max(\nu, \hat{\nu}) < \gamma\hat{\gamma}.$$

Theorem 2.4 (see [HPS] and Theorem B in [PSW1])**.** *If $f \in \mathcal{PH}^2(M)$ is dynamically coherent and center bunched, then local stable/unstable holonomy maps between center leaves are C^1 when restricted to some center-stable/center-unstable leaf and have uniformly continuous derivatives.*

Indeed, the authors prove that the strong stable/unstable foliation is C^1 when restricted to a center-stable/unstable leaf. However, from their proof, it is not clear how the holonomies $H^s_{f,x_1,x_2}|_{\mathcal{W}^c_{f,\mathrm{loc}}(x_1)}$, resp. $H^u_{f,x_1,x_2}|_{\mathcal{W}^c_{f,\mathrm{loc}}(x_1)}$ vary in the C^1-topology with the choices of the points x_1 and $x_2 \in \mathcal{W}^s_{f,\mathrm{loc}}(x_1)$, resp. $x_2 \in \mathcal{W}^u_{f,\mathrm{loc}}(x_1)$. This question is investigated in Obata's work [O], where it is shown that under some stronger bunching condition, these holonomy maps vary continuously with the choices of the base points x_1, x_2.

Definition 2.5 (see [O])**.** *For any integer $r \geq 1$, we denote by $\mathcal{PH}^r_*(M)$ the set of all partially hyperbolic diffeomorphisms $f \in \mathcal{PH}^r(M)$ such that, for some $\theta \in (0,1)$,*

$$\|D_x f|_{E^s_f}\|^\theta < \frac{m(D_x f|_{E^c_f})}{\|D_x f|_{E^c_f}\|}, \quad \frac{\|D_x f|_{E^c_f}\|}{m(D_x f|_{E^c_f})} < m(D_x f|_{E^u_f})^\theta,$$
$$\|D_x f|_{E^s_f}\| < m(D_x f|_{E^c_f}) m(D_x f|_{E^s_f})^\theta,$$
$$\|D_x f|_{E^c_f}\| \cdot \|D_x f|_{E^u_f}\|^\theta < m(D_x f|_{E^u_f}).$$

Note that any diffeomorphism $f \in \mathcal{PH}^r_*(M)$ is automatically center bunched.

Theorem 2.6 (Theorem 0.3 in [O])**.** *Assume that $f \in \mathcal{PH}^2_*(M)$. Then, for $* = s, u$, the family $\left\{H^*_{f,x_1,x_2}|_{\mathcal{W}^c_{f,\mathrm{loc}}(x_1)}\right\}_{x_1 \in M,\, x_2 \in \mathcal{W}^*_{f,\mathrm{loc}}(x_1)}$ is a family of C^1 maps depending continuously in the C^1-topology with the choices of the points x_1 and $x_2 \in \mathcal{W}^*_{f,\mathrm{loc}}(x_1)$.*

2.4. **Accessibility classes.** A f-*accessibility sequence* is a sequence $[x_1, \ldots, x_k]$ of $k \geq 1$ points in M such that for any $i \in \{1, \ldots, k-1\}$, the points x_i and x_{i+1} belong to the same stable or unstable leaf of f. In particular, the points x_1 and x_k can be connected by some f-*path*, i.e., a continuous path in M obtained by concatenating finitely many arcs in $\mathcal{W}^s_f$ or $\mathcal{W}^u_f$. We will refer to the points $x_1, \ldots, x_k$ as the *corners* of the accessibility sequence $[x_1, \ldots, x_k]$.

For any point $x \in M$, we denote by $\mathrm{Acc}_f(x)$ the *accessibility class* of x. By definition, it is the set of all points $y \in M$ which can be connected to x by some f-path. We also let

$$\mathrm{C}_f(x) := \mathrm{cc}(\mathrm{Acc}_f(x) \cap \mathcal{W}^c_f(x,1), x)$$

be the connected component containing x of the intersection of the accessibility class of x and the local center leaf through x. Similarly, for any $\varepsilon > 0$, we let $\mathrm{C}_f(x,\varepsilon) := \mathrm{cc}(\mathrm{Acc}_f(x) \cap \mathcal{W}^c_f(x,\varepsilon), x)$. By definition, accessibility classes form a partition of M. We say that the diffeomorphism f is *accessible* if this partition is trivial, i.e., the whole manifold M is a single accessibility class; we say that f is *stably accessible* if the diffeomorphisms which are sufficiently C^1-close to f are accessible.

Moreover, given any f-accessibility sequence $\gamma = [x_1, \ldots, x_k]$, we let $H_{f,\gamma}\colon \mathcal{W}^c_{f,\mathrm{loc}}(x_1) \to \mathcal{W}^c_{f,\mathrm{loc}}(x_k)$ be the holonomy map obtained by concatenating the local holonomy maps along the arcs of γ, i.e.,

$$H_{f,\gamma} := H^{*_{k-1}}_{f,x_{k-1},x_k} \circ \cdots \circ H^{*_1}_{f,x_1,x_2}, \tag{2.2}$$

where for $j \in \{1, \ldots, k-1\}$, $*_j \in \{s, u\}$ is such that $x_{j+1} \in \mathcal{W}^{*_j}_f(x_j)$.

The next lemma is elementary; it follows from the local product structure and the continuous dependence of the invariant foliations with respect to the diffeomorphism.

Lemma 2.7 (Continuation of accessibility sequences)**.** *Let $\gamma = [x_0, x_1, \ldots, x_k]$ be a f-accessibility sequence, for some integer $k \geq 0$. Then there exist a neigbourhood $\mathcal{O}$ of x_0 and a C^1-neighbourhood $\mathcal{U}$ of f such that for any point $x \in \mathcal{O}$, and for any diffeomorphism $g \in \mathcal{U}$, there exists a natural continuation $\gamma^{x,g} = [x, x_1^{x,g}, \ldots, x_k^{x,g}]$ of γ for x and g. Indeed, the g-accessibility sequence $\gamma^{x,g}$ is defined as*

$$\begin{aligned} x_1^{x,g} &:= H^{*_0}_{g,x,x_1}(x);\\ x_2^{x,g} &:= H^{*_1}_{g,x_1^{x,g},x_2}(x_1^{x,g});\\ &\cdots\\ x_k^{x,g} &:= H^{*_{k-1}}_{g,x_{k-1}^{x,g},x_k}(x_{k-1}^{x,g}); \end{aligned}$$

here, for each $j \in \{0, \ldots, k-1\}$, we let $*_j \in \{s, u\}$ be such that $x_{j+1} \in \mathcal{W}^{*_j}_f(x_j)$. Moreover, $\gamma^{x,g}$ depends continuously on the pair (x, g).

Definition 2.8. *Given a point $x \in M$ and an integer $n \geq 2$, a $2n$* us-loop *at (f, x_0) is a f-accessibility sequence $\gamma = [x_0, x_1, x_2, \ldots, x_{2n}] \in M^{2n+1}$ with $2n$ legs such that*

$$\begin{aligned} x_1 &\in \mathcal{W}^u_{f,\mathrm{loc}}(x_0),\\ x_2 &\in \mathcal{W}^s_{f,\mathrm{loc}}(x_1), \ldots\\ \ldots\ x_{2n-1} &\in \mathcal{W}^u_{f,\mathrm{loc}}(x_{2n-2}) \cap \mathcal{W}^{cs}_{f,\mathrm{loc}}(x_0),\\ x_{2n} &:= H^s_{f,x_{2n-1},x}(x_{2n-1}) \in \mathcal{W}^c_{f,\mathrm{loc}}(x_0). \end{aligned}$$

We define $2n$ su-loops accordingly (with $x_1 \in \mathcal{W}^s_{f,\mathrm{loc}}(x_0)$ etc.).

The length *of a $2n$ us-loop $\gamma = [x_0, x_1, x_2, \ldots, x_{2n}] \in M^{2n+1}$ is defined as*

$$\ell(\gamma) := d_{\mathcal{W}^u_f}(x_0, x_1) + \sum_{i=1}^{n-1} \Big[d_{\mathcal{W}^s_f}(x_{2i-1}, x_{2i}) + d_{\mathcal{W}^u_f}(x_{2i}, x_{2i+1}) \Big] + d_{\mathcal{W}^{cs}_f}(x_{2n-1}, x_0).$$

Moreover, we say that the us-loop γ is

- closed, *if $x_{2n} = x_0$;*
- trivial, *if $x_0 = x_1 = x_2 = \cdots = x_{2n}$;*
- non-degenerate, *if x_1 is distinct from the other corners $x_0, x_2, \ldots, x_{2n}$.*

We also denote by $\overline{\gamma}$ the $2n$ su-loop $\overline{\gamma} := [x_{2n}, x_{2n-1}, \ldots, x_2, x_1, x_0] \in M^{2n+1}$ at (f, x_{2n}). Finally, given an integer $m \geq 2$ and a $2m$ us-loop $\gamma' = [x_{2n}, x'_1, \ldots, x'_{2m}]$ at (f, x_{2n}), the concatenation $\gamma\gamma'$ of γ and γ' is the $2(m+n)$ us-loop $\gamma\gamma' := [x_0, x_1, \ldots, x_{2n}, x'_1, \ldots, x'_{2m}]$ at (f, x_0).

Definition 2.9. *Given $x \in M$ and $n \geq 2$, a one-parameter family $\gamma = \{\gamma(t) = [x, x_1(t), \ldots, x_{2n}(t)]\}_{t \in [0,1]}$ of $2n$ us-loops at (f, x) is said to be* continuous *if for any $i = 1, \ldots, 2n$, the map $t \mapsto x_i(t)$ is continuous. We define $\ell(\gamma) := \sup_{t \in [0,1]} \ell(\gamma(t))$.*

2.5. **Structure of center accessibility classes.** Let M be a compact Riemannian manifold of dimension $d \geq 4$. Let $r \geq 2$ be some integer, and let $f \in \mathcal{PH}^r(M)$ be a partially hyperbolic diffeomorphism with $\dim E_f^c = 2$ that is center bunched and dynamically coherent.

By Theorem 2.4, for $* = s, u$, the $*$-holonomy maps are C^1 when restricted to a $\mathcal{W}_f^{c*}$ leaf; by C^1-homogeneity arguments, this allowed [RH, RHV] to obtain a classification of center accessibility classes.

Theorem 2.10 ([RH, RHV])**.** *For any point $x \in M$, and for any sufficiently small $\varepsilon > 0$, the local center accessibility class $\mathrm{C}_f(x, \varepsilon)$ can be either*

- *trivial, i.e., reduced to a point;*
- *a one-dimensional submanifold of $\mathcal{W}_f^c(x)$;*
- *open; in this case, $\mathrm{Acc}_f(x)$ is also open.*

In the following, for any subset $\mathscr{S} \subset M$, we let

- $\Gamma_f^0(\mathscr{S}) := \{x \in \mathscr{S} : \mathrm{C}_f(x) \text{ is trivial}\}$;
- $\Gamma_f^1(\mathscr{S}) := \{x \in \mathscr{S} : \mathrm{C}_f(x) \text{ is one-dimensional}\}$;
- $\Gamma_f(\mathscr{S}) := \Gamma_f^0(\mathscr{S}) \cup \Gamma_f^1(\mathscr{S})$.

In particular, $\mathscr{S} \setminus \Gamma_f(\mathscr{S})$ is the set of points $x \in \mathscr{S}$ whose accessibility class $\mathrm{Acc}_f(x)$ is open. When $\mathscr{S} = M$, we abbreviate $\Gamma_f^0(\mathscr{S}), \Gamma_f^1(\mathscr{S}), \Gamma_f(\mathscr{S})$ respectively as $\Gamma_f^0, \Gamma_f^1, \Gamma_f$.

3. Main results

We fix a compact Riemannian manifold M of dimension $d \geq 4$ and an integer $r \geq 2$. Our main result is about the C^r-density of the accessibility property for partially hyperbolic diffeomorphisms with two-dimensional center which are stably dynamically coherent and satisfy some strong bunching condition as in Definition 2.5, i.e., $f \in \mathcal{PH}_*^r(M)$.

Theorem A. *For any partially hyperbolic diffeomorphism $f \in \mathcal{PH}_*^r(M)$, resp. $f \in \mathcal{PH}_*^r(M, \mathrm{Vol})$, with $\dim E_f^c = 2$, that is dynamically coherent and plaque expansive, and for any $\delta > 0$, there exists a partially hyperbolic diffeomorphism $g \in \mathcal{PH}^r(M)$, resp. $g \in \mathcal{PH}^r(M, \mathrm{Vol})$, with $d_{C^r}(f, g) < \delta$, such that g is stably accessible.*

In particular, by the work of Burns-Wilkinson [BW1], this implies that for any partially hyperbolic diffeomorphism $f \in \mathcal{PH}_*^r(M, \mathrm{Vol})$, with $\dim E_f^c = 2$, that is dynamically coherent and plaque expansive, and for any $\delta > 0$, there exists $g \in \mathcal{PH}^r(M, \mathrm{Vol})$, with $d_{C^r}(f, g) < \delta$, such that g is stably ergodic.

One intermediate step is to show that trivial accessibility classes can be broken by C^r-small perturbations. This part of the proof also holds when the center is higher dimensional and only requires center bunching.

Theorem B. *For any partially hyperbolic diffeomorphism $f \in \mathcal{PH}^r(M)$, resp. $f \in \mathcal{PH}^r(M, \mathrm{Vol})$, with $\dim E_f^c \geq 2$, that is center bunched, dynamically coherent, and plaque expansive, and for any $\delta > 0$, there exists a partially hyperbolic diffeomorphism $g \in \mathcal{PH}^r(M)$, resp. $g \in \mathcal{PH}^r(M, \mathrm{Vol})$, with $d_{C^r}(f, g) < \delta$, such that $\mathrm{C}_g(x)$ is non-trivial, for all $x \in M$.*

Let us briefly summarize the main steps of the proof:

(1) we study the structure of local center accessibility classes, i.e., the set of points which can be attained within some small center disk around a given point, following accessibility sequences with a given number of legs of prescribed size; in particular, we identify which are the configurations to break in order to make each accessibility class open;
(2) given a small center disk $\mathcal{D}$, we construct continuous families of local accessibility sequences at points in $\mathcal{D}$; these families depend on the nature of the center accessibility class of the base point (which can be zero, one or two-dimensonal), and allow us to have sufficiently many "degrees of freedom" to create local accessibility after perturbation;
(3) once these families are constructed, we design families of perturbations, localized near one of the corners of the accessibility sequences, and which depend in a differentiable way on the perturbation parameter;
(4) we study the variation of the endpoint of these accessibility sequences once the perturbation parameter is turned on, and show that for suitable perturbations, we obtain a submersion from the space of perturbations to the phase space; in particular, bad configurations in phase space (non-open accessibility classes) correspond to special configurations in the space of perturbations, which can be broken to create local accessibility;
(5) we globalize the argument using spanning families.

Let us say a few more words about the previous points.

The details about point (1) are given in Section 4. For partially hyperbolic diffeomorphisms with two-dimensional center that are center bunched, it is known (by the works of Rodriguez-Hertz [RH], Rodriguez-Hertz and Vásquez [RHV] etc.) that center accessibility classes are zero, one or two-dimensional submanifolds. Moreover, Horita-Sambarino [HS] have studied the organization of center accessibility classes within a small center disk all of whose points have non-trivial center accessibility classes; in particular, they have shown that the set of one-dimensional center accessibility classes of points in the disk forms a C^1 lamination. In Section 4, we go further in this direction, and investigate the variation of center accessibility classes for perturbations of a given partially hyperbolic diffeomorphism. In particular, we show that if the center accessibility class of a point x remains one-dimensional after perturbation, it stays in a certain "cone" around x.

The construction of loops mentioned in point (2) is outlined in Section 5. Indeed, in the subsequent argument, given a point x whose accessibility class is not open, we need to construct (non-trivial) *closed* accessibility sequences at x; moreover, we show that it is possible to construct these loops in such a way that they depend nicely on x.

The details about point (3) are in Section 6, and follow the arguments of [LZ]. Given a point $x \in M$ that is non-periodic, we construct a family $\{\gamma(t)\}_{t\in[0,1]}$ of contractible *us-loops* at (f, x), and we define a family of perturbations such that the support of the perturbations is contained in some small neighbourhood of the first corner of $\gamma(1)$. By taking the loop sufficiently small, the first return time to the support of the perturbation can be made arbitrarily large, and we show that it induces a change of the holonomy along the continuation of $\gamma(1)$ for the perturbed diffeomorphisms. More precisely, by the results of [LZ], we get a submersion from the space of perturbations to the phase space – here, the local center leaf of x.

The submersion property is sufficient to show that after perturbation, the center accessibility class of x can be made non-trivial. This part of the proof is explained in Subsection 7.1 and holds in a more general setting, as it does not require the center to be two-dimensional. When the center accessibility class of x is one-dimensional, by point (1), it varies continuously with respect to the diffeomorphism in the C^1 topology. In particular, if the center accessibility class of x were one-dimensional for every diffeomorphism in a C^r-neighbourhood of f, then all those classes would stay in some cone around the point x; but this is in contradiction with the submersion property for the family of perturbations we construct. The details of this part are given in Subsection 7.2.

The details about point (5) are given in Section 8, where we explain how to globalize the arguments in order to verify the accessibility property, through the notion of spanning family of center disks, as in [DW] (see Subsection 8.1). In Subsection 8.2, given some small center disk in the family, we explain how by a C^r-small perturbation, it is possible to make the center accessibility class of each point in the disk non-trivial. One difficulty is that the perturbation used to break trivial center accessibility classes may create new trivial classes in other places (at points with non-trivial, but very small center accessibility classes). The idea to bypass this difficulty is to take two families of us-loops which we can perturb "independently", in order to increase the codimension of "bad" situations for which the center accessibility class of some point in the disk would be trivial. Once all classes in the disk are non-trivial, we have to make a further perturbation to make all these classes simultaneously open. One important step in the argument is the aforementioned result (inspired by the work of Horita-Sambarino [HS]) that within the center disk, one-dimensional center accessibility classes vary C^1-continuously both in perturbation space and phase space. In particular, if the center disk is chosen sufficiently small, then the set of tangent directions associated to one-dimensional classes (even for small perturbations of the diffeomorphism f) stay in a small cone that is uniform in the points of the disk. Thanks to the submersion property, we can then choose a perturbation for which each point x in the center disk will have a point y in its center accessibility class lying outside this cone, which forces the accessibility class of x to be open. There again, one difficulty is to check that the perturbations which we make preserve the accessibility classes which were already open. Repeating the same argument for each center disk in the spanning family, we thus construct a C^r-small perturbation of f that is accessible.

4. Variation of one-dimensional center accessibility classes

In this section, given an integer $r \geq 2$, we prove that the set of one-dimensional center accessibility classes varies continuously in the C^1 topology with respect to $f \in \mathcal{PH}^r_*(M)$. The idea of the proof is similar to Proposition 2.19 from [HS] where it is proved that for a fixed partially hyperbolic diffeomorphism, and for a given center disk, the one-dimensional accessibility classes form a C^1-lamination. To prove this, we have to see that for a given $x \in M$, the direction $T_x\mathrm{C}_f(x)$ varies continuously with respect to f in the C^1 topology.

Let us fix an integer $r \geq 2$. We denote by $\mathscr{F} \subset \mathcal{PH}^r_*(M)$ the set of C^r dynamically coherent, plaque expansive, partially hyperbolic diffeomorphisms with two-dimensional center which satisfy the bunching condition in Definition 2.5. Let

$f \in \mathscr{F}$. By center bunching, for $* = s, u$, for any $x \in M$, $y \in \mathcal{W}^*_{f,\mathrm{loc}}(x)$, the holonomy map $H^*_{f,x,y}$ is C^1 when restricted to the leaf $\mathcal{W}^{c*}_{f,\mathrm{loc}}(x)$. For any C^1 neighbourhood $\mathcal{U}$ of f, we will denote by $\mathcal{U}^{\mathscr{F}}$ the set $\mathcal{U}^{\mathscr{F}} := \mathcal{U} \cap \mathscr{F}$.

In the following, we will need to have uniform control of the differential of the holonomies $H^*_{f,x,y}$ in two ways:

- with respect to the points x, y (in the same stable/unstable manifold);
- when the diffeomorphism f is replaced with another C^r partially hyperbolic diffeomorphism in a C^1-neighbourhood of f.

This is the content of the next lemma.

Lemma 4.1 (See [O], and also [B, BW2])**.** *Let $f \in \mathscr{F}$. Then there exists a C^1 neighbourhood $\mathcal{U}$ of f such that for $* = s, u$ and $\mathcal{U}^{\mathscr{F}} = \mathcal{U} \cap \mathscr{F}$, the family of C^1 maps $\{H^*_{g,x,y}|_{\mathcal{W}^c_g(x)}\}_{g \in \mathcal{U}^{\mathscr{F}},\, x \in M,\, y \in \mathcal{W}^*_f(x)}$ depends continuously in the C^1 topology with the choices of the points x, y and of the map $g \in \mathcal{U}^{\mathscr{F}}$.*

Remark 4.2. *In fact, Obata [O] shows that for $* = s, u$, the family of holonomy maps $\{H^*_{f,x,y}|_{\mathcal{W}^c_f(x)}\}_{x \in M,\, y \in \mathcal{W}^*_f(x)}$ depends continuously in the C^1 topology with the choices of the points x, y, when f is dynamically coherent and satisfies a strong bunching condition. For our purpose, we also need to have a uniform control with respect to the diffeomorphism g in a C^1-small neighbourhood of f. It is indeed possible as the estimates in [O] are written in terms of the functions as in (2.1) controlling the growth rates along the different invariant bundles, which depend continuously on the map g in the C^1 topology.*

The holonomy map associated to some accessibility sequence is obtained by composing the holonomy maps between two consecutive corners (recall (2.2)). By the previous lemma, we thus have:

Corollary 4.3. *Let $f \in \mathscr{F}$, and let $\gamma = [x_1, x_2, \dots, x_k] \in M^k$ be a f-accessibility sequence for some integer $k \geq 1$. We take a small neighbourhood $\mathcal{O} \subset M$ of x_1 and a C^1 neighbourhood $\mathcal{U}$ of f such that for any $x \in \mathcal{O}$ and for any $g \in \mathcal{U}$, the continuation $\gamma^{x,g} = [x, x_2^{x,g}, \dots, x_k^{x,g}]$ of γ starting at x given by Lemma 2.7 is well-defined.*

Then, the family of C^1 maps $\{H^*_{g,\gamma^{x,g}}|_{\mathcal{W}^c_g(x)}\}_{x \in \mathcal{O},\, g \in \mathcal{U}^{\mathscr{F}}}$ depends continuously in the C^1 topology with the choices of the point $x \in \mathcal{O}$ and the map $g \in \mathcal{U}^{\mathscr{F}}$.

For any point $x \in M$ and any subset $\mathscr{G} \subset \mathscr{F}$, we let $\mathscr{G}_1(x) \subset \mathscr{G}$ be the subset of maps f for which the center accessibility class $\mathrm{C}_f(x)$ is one-dimensional. For any $f \in \mathscr{G}_1(x)$, and for any sufficiently small $\theta, \varepsilon > 0$, we let $\mathscr{C}_f(x, \theta, \varepsilon) \subset \mathcal{W}^c_f(x) \cap B(x, \varepsilon)$ be the set of points in the ε-ball $B(x, \varepsilon)$ centered at x which belong to (the image by the exponential map of) the cone of angle θ around $T_x\mathrm{C}_f(x)$, i.e.,

$$\mathscr{C}_f(x, \theta, \varepsilon) := \exp_x\{y \in T_xM : \angle(y, T_x\mathrm{C}_f(x)) < \theta\} \cap B(x, \varepsilon). \tag{4.1}$$

The main result of this section is the following:

Proposition 4.4. *Take $f \in \mathscr{F}$ and $x \in M$ such that $\mathrm{C}_f(x)$ is one-dimensional, i.e., $f \in \mathscr{F}_1(x)$. Then, for every $\theta > 0$ there exists a C^1 neighbourhood $\mathcal{U}$ of f such that for every $g \in \mathcal{U}_1^{\mathscr{F}}(x)$, the angle at x between $\mathrm{C}_f(x)$ and $\mathrm{C}_g(x)$ satisfies*

$$\angle(T_x\mathrm{C}_f(x), T_x\mathrm{C}_g(x)) < \theta.$$

Moreover, there exists $\varepsilon_0 > 0$ *such that for any* $g \in \mathcal{U}_1^{\mathscr{F}}(x)$ *and* $\varepsilon \in (0, \varepsilon_0)$*, it holds*

$$\mathrm{C}_g(x, \varepsilon) \subset \mathscr{C}_f(x, \theta, \varepsilon).$$

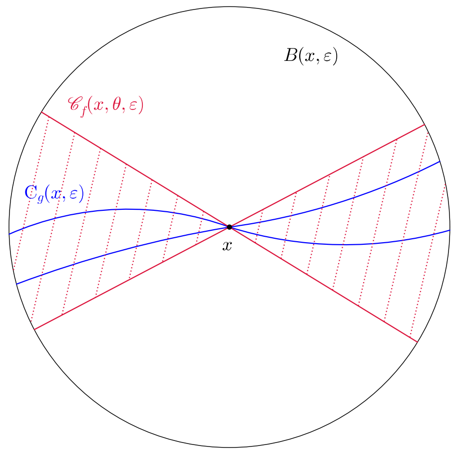

FIGURE 1. Variation of 1-dimensional center accessibility classes.

The idea of the proof consists in showing some "uniform" homogeneity of one-dimensional center accessibility classes $\mathrm{C}_g(x)$ of all maps $g \in \mathcal{U}^{\mathscr{F}}$, for a sufficiently small C^1 neighbourhood $\mathcal{U}$ of a fixed $f \in \mathscr{F}$. Indeed, the tangent spaces at two points x, y in the same center accessibility class are naturally related through the differential of the holonomy map along an accessibility sequence connecting x to y. Since everything we are doing here is local, we are able to compare angles and norms of vectors in different tangent spaces, using trivialization charts as follows. Recall that $d := \dim M$, and that we denote $d_s := \dim E_f^s$, $d_u := \dim E_f^u$.

Lemma 4.5 (see Construction 9.1, [LZ]). *There exist* C^2*-uniform constants* $\overline{h} = \overline{h}(f) > 0$ *and* $\overline{C} = \overline{C}(f) > 1$ *such that for any* $x \in M$*, there exists a* C^r *volume preserving map* $\phi = \phi_x : (-\overline{h}, \overline{h})^d \to M$ *such that* $\phi(0_{\mathbb{R}^d}) = x$ *and*

1. $\mathcal{W}_f^c(x, \frac{\overline{h}}{5}) \subset \phi\big((-\frac{\overline{h}}{4}, \frac{\overline{h}}{4})^2 \times \{0\}^{d_u+d_s}\big) \subset \phi\big((-\frac{2\overline{h}}{3}, \frac{2\overline{h}}{3})^2 \times \{0\}^{d_u+d_s}\big) \subset \mathcal{W}_f^c(x, \overline{h})$;
2. $\|\phi\|_{C^2} < \overline{C}$;
3. $D\phi(0, \mathbb{R}^2 \times \{0_{\mathbb{R}^{d_u+d_s}}\})$, $D\phi(0, \{0_{\mathbb{R}^2}\} \times \mathbb{R}^{d_u} \times \{0_{\mathbb{R}^{d_s}}\})$, $D\phi(0, \{0_{\mathbb{R}^{2+d_u}}\} \times \mathbb{R}^{d_s})$ *are respectively equal to* $E_f^c(x), E_f^u(x), E_f^s(x)$;
4. *for any* $y \in \phi((-\overline{h}, \overline{h})^d)$, $\Pi^c D(\phi^{-1})_y : E_f^c(y) \to \mathbb{R}^2$ *has determinant in* $\big(\overline{C}^{-1}, \overline{C}\big)$*, where* $\Pi^c : \mathbb{R}^d \simeq \mathbb{R}^2 \times \mathbb{R}^{d_u+d_s} \to \mathbb{R}^2$ *is the canonical projection;*
5. *for any* $\zeta > 0$*, there exists a* C^1*-uniform constant* $\overline{h}_\zeta = \overline{h}_\zeta(f) \in (0, \overline{h})$ *so that if* $h \in (0, \overline{h}_\zeta)$*, then for any* $y \in \phi((-h, h)^d)$*, the norm of* $\Pi^c (D\phi^{-1})_y : E_f^{su}(y) \to \mathbb{R}^2$ *is smaller than* ζ.

In the following, we will denote by Π_x^c *the map* $\Pi_x^c := \Pi^c \circ \phi_x^{-1} : M \to \mathbb{R}^2$.

Before giving the proof of Proposition 4.4, let us state an elementary lemma and introduce a notation. Let $\alpha\colon [0,1] \to M$ be a C^1 arc of M and given $\epsilon > 0$, consider an ϵ tubular neighbourhood $\mathscr{N}_{\alpha,\epsilon}$ of α. This tubular neighbourhood is diffeomorphic to $[0,1] \times [-\epsilon,\epsilon]^{d-1}$. We identify points in $\mathscr{N}_{\alpha,\epsilon}$ with pairs (t,s), where $t \in [0,1]$ and $s \in [-\epsilon,\epsilon]^{d-1}$. We call the boundary $\{0\} \times [-\epsilon,\epsilon]^{d-1}$ the *left side* of $\mathscr{N}_{\alpha,\epsilon}$, and we call the boundary $\{1\} \times [-\epsilon,\epsilon]^{d-1}$ its *right side*. We denote by $\xi\colon \mathscr{N}_{\alpha,\epsilon} \to \alpha$ the projection $\xi\colon (t,s) \mapsto \alpha(t)$.

Lemma 4.6. *With the notation above, given $\delta > 0$, there exists $\epsilon > 0$ such that if $\beta\colon [0,1] \to \mathscr{N}_{\alpha,\epsilon}$ is a C^1 curve in $\mathscr{N}_{\alpha,\epsilon}$ from the left to the right side, then there exists some $(t,s) = \beta(\tilde{t})$ with $\tilde{t} \in [0,1]$ such that the angle between α and β satisfies*

$$\angle(\dot{\alpha}(t), \dot{\beta}(\tilde{t})) < \delta.$$

Proof of Proposition 4.4. Let us show the first part. Suppose by contradiction that for some $\eta > 0$, there exists a sequence $(g_n)_{n\geq 0} \in \mathscr{F}^{\mathbb{N}}$ of maps such that $g_n \to f$ in the C^1 topology, with $g_n \in \mathscr{F}_1(x)$ and

$$\angle(T_x\mathrm{C}_f(x), T_x\mathrm{C}_{g_n}(x)) > \eta, \quad \text{for all } n \geq 0. \tag{4.2}$$

Since $\mathrm{C}_f(x)$ is one-dimensional, for some integer $n \geq 2$, there exists a $2n$ us-loop $\gamma = [x, x_1, \ldots, x_{2n}]$ at (f,x) such that $x_{2n} \neq x$. By shrinking the size of the legs, we get a one-parameter family $\{\gamma(t) = [x, x_1(t), \ldots, x_{2n}(t)]\}_{t\in[0,1]}$ of $2n$ us-loops at (f,x), where $\gamma(1) = \gamma$ and $\gamma(0)$ is trivial. By Lemma 2.7, there exists a C^1 neighbourhood $\widetilde{\mathcal{U}}$ of f such that for any $g \in \widetilde{\mathcal{U}}$ and for any $t \in [0,1]$, there exists a one-parameter family $\{\gamma^{x,g}(t) = [x, x_1^{x,g}(t), \ldots, x_{2n}^{x,g}(t)]\}_{t\in[0,1]}$ of $2n$ us-loops at (g,x) such that $\gamma^{x,g}(0)$ is the trivial loop. We also denote $\alpha_{g,x}\colon t \mapsto x_{2n}^{x,g}(t) \in \mathrm{C}_g(x)$.

For each pair $(g,t) \in \widetilde{\mathcal{U}} \times [0,1]$ we have the corresponding holonomy map $H_g^t := H_{g,\gamma^{x,g}(t)}|_{\mathcal{W}^c_{g,\mathrm{loc}}(x)}\colon \mathcal{W}^c_{g,\mathrm{loc}}(x) \to \mathcal{W}^c_{g,\mathrm{loc}}(x)$. Given some small $h > 0$, and assuming that $\widetilde{\mathcal{U}}$ is sufficiently small, then for every map $g \in \widetilde{\mathcal{U}}^{\mathscr{F}}$, we take a C^1 chart $\phi_{x,g}\colon (-h,h)^2 \to \mathcal{W}^c_{g,\mathrm{loc}}(x)$ as in Lemma 4.5; as center leaves vary continuously with respect to g in the C^1 topology, the map $g \mapsto \phi_{x,g}$ depends continuously on g in the C^1 topology. After replacing H_g^t with $\phi_{x,g}^{-1} \circ H_g^t \circ \phi_{x,g}$, we can compare angles and norms of vectors for diffeomorphisms in a neighbourhood of f; by a slight abuse of notation, we will still denote this map by H_g^t for simplicity. By Corollary 4.3, and by compactness of $[0,1]$, we deduce that the family of holonomy maps $\{H_g^t\}_{(t,g)\in[0,1]\times\widetilde{\mathcal{U}}^{\mathscr{F}}}$ is uniformly C^1. In particular, for any $\delta > 0$, there exists a C^1 neighbourhood $\mathcal{U}_\delta$ of f such that for $\mathcal{U}_\delta^{\mathscr{F}} := \mathcal{U}_\delta \cap \mathscr{F}$, it holds

$$\sup_{(t,g)\in[0,1]\times\mathcal{U}_\delta^{\mathscr{F}}} \|DH_g^t - DH_f^t\| < \delta. \tag{4.3}$$

Therefore, for every $\theta > 0$, there exist $\delta_0 > 0$, $\rho_0 > 0$ such that for $g \in \mathcal{U}_{\delta_0}^{\mathscr{F}}$ and for any $t \in [0,1]$, if $y \in \mathcal{W}^c_{g,\mathrm{loc}}(x)$ is such that $d(x,y) < \rho_0$ and if the vectors $v, w \in \mathbb{R}^2$ satisfy $\angle(v,w) > \theta$, then we have

$$\angle(D_xH_f^t(v), D_yH_g^t(w)) > \delta_0. \tag{4.4}$$

As invariant manifolds depend continuously on the diffeomorphism $g \in \mathcal{U}_{\delta_0}^{\mathscr{F}}$, for any $\epsilon_0 > 0$, there exists $\rho(\epsilon_0) > 0$ such that for any $y \in B(x, \rho(\epsilon_0))$, for any $t \in [0,1]$

and for any $g \in \mathcal{U}^{\mathscr{F}}_{\delta_0}$ (taking a smaller δ_0 if necessary), it holds

$$d(H^t_f(x), \xi(H^t_g(y))) < \epsilon_0. \tag{4.5}$$

Since the center accessibility class $\mathrm{C}_f(x)$ is C^1, the map $\mathrm{C}_f(x) \ni z \mapsto T_z\mathrm{C}_f(x)$ is continuous, hence, if $\epsilon_0 > 0$ is chosen sufficiently small, then for any $t \in [0,1]$, $g \in \mathcal{U}^{\mathscr{F}}_{\delta_0}$, and $y \in B(x, \rho(\epsilon_0))$, we have

$$\angle(T_{H^t_f(x)}\mathrm{C}_f(x), T_{\xi(H^t_g)(y)}\mathrm{C}_f(x)) < \frac{\delta_0}{2}. \tag{4.6}$$

Now we argue as in Proposition 2.19 of [HS]. For $\theta = \eta$ (recall (4.2)) we take $\delta_0 = \delta_0(\theta) > 0$ as in (4.4) and we set $\delta := \frac{\delta_0}{2} > 0$.

Since $g_n \to f$, we can take n large enough so that $g_n \in \mathcal{U}^{\mathscr{F}}_{\delta_0}$ and such that the arc $\{\alpha_{g_n,x}(t)\}_{t\in[0,1]}$ is a curve that crosses $\mathscr{N}_{\alpha_{f,x},\epsilon_0}$ from the left to the right side. Set $\beta := \alpha_{g_n,x}$. Note that if $t \in [0,1]$ is such that $\beta(t) \in \mathscr{N}_{\alpha_{f,x},\epsilon_0}$, then

$$\mathrm{Span}(\dot\beta(t)) = D_xH^t_{g_n}(T_x\mathrm{C}_{g_n}(x)). \tag{4.7}$$

Then, by (4.2), (4.4), (4.6), (4.7), we deduce that for any $t \in [0,1]$,

$$\begin{aligned}
&\angle(\dot\beta(t), T_{\xi(\beta(t))}\mathrm{C}_f(x)) = \angle(D_xH^t_{g_n}(T_x\mathrm{C}_{g_n}(x)), T_{\xi(H^t_{g_n}(x))}\mathrm{C}_f(x))\\
&\geq \angle(D_xH^t_{g_n}(T_x\mathrm{C}_{g_n}(x)), T_{H^t_f(x)}\mathrm{C}_f(x)) - \angle(T_{H^t_f(x)}\mathrm{C}_f(x), T_{\xi(H^t_{g_n}(x))}\mathrm{C}_f(x))\\
&= \angle(D_xH^t_{g_n}(T_x\mathrm{C}_{g_n}(x)), D_xH^t_f(T_x\mathrm{C}_f(x))) - \angle(T_{H^t_f(x)}\mathrm{C}_f(x), T_{\xi(H^t_{g_n}(x))}\mathrm{C}_f(x))\\
&> \delta_0 - \frac{\delta_0}{2} = \delta,
\end{aligned}$$

which contradicts Lemma 4.6.

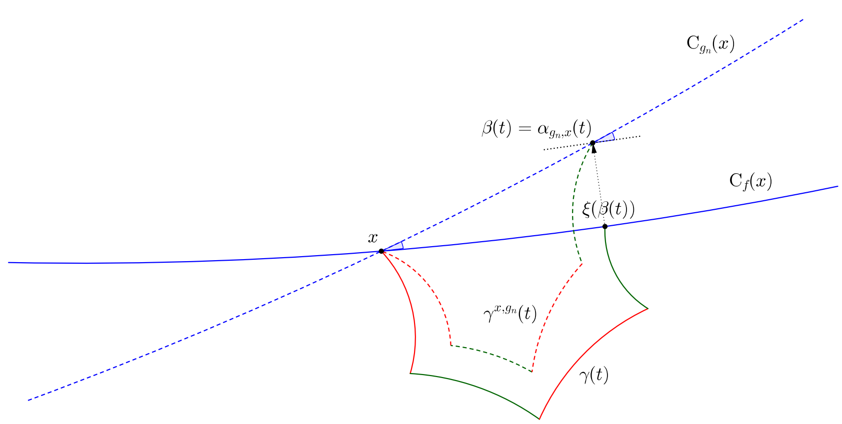

FIGURE 2. Tangent spaces to $\mathrm{C}_f(x)$, resp. $\mathrm{C}_{g_n}(x)$ at $\xi(\beta(t))$, resp. $\beta(t)$.

The proof of the second part of the proposition is similar. We know from the previous part that given $\theta > 0$ there is a C^1 neighbourhood $\mathcal{U}$ of f such that for every $g \in \mathcal{U}_1^{\mathscr{F}}(x)$, it holds $\angle(T_x \mathrm{C}_f(x), T_x \mathrm{C}_g(x)) < \theta$. Now we want to see the variation of the leaves at uniform (small) scale. Let us then suppose by contradiction that there are sequences $g_n \to f$ and $x_n \to x$ such that $x_n \in \mathrm{C}_{g_n}(x, \frac{1}{n}) \setminus \mathscr{C}_f(x, \theta, \frac{1}{n})$. By Lagrange Mean Value Theorem, this implies that there is $y_n \in \mathrm{C}_{g_n}(x, \frac{1}{n})$ such that $\angle(T_{y_n} \mathrm{C}_{g_n}(x), T_x \mathrm{C}_f(x)) > \theta$. Take $\delta_0 = \delta_0(\theta) > 0$, $\epsilon_0 > 0$ sufficiently small, and $n > 0$ sufficiently large such that $g_n \in \mathcal{U}_{\delta_0}^{\mathscr{F}}$, $y_n \in B(x, \rho(\epsilon_0))$, and such that the curve $\beta_1 \colon t \mapsto H_{g_n}^t(y_n)$ crosses $\mathscr{N}_{\alpha_{f,x,\epsilon_0}}$ from the left to the right side. Now we argue as above, the only difference being that the role of the "big angle" is played by $\angle(T_{y_n} \mathrm{C}_{g_n}(x), T_x \mathrm{C}_f(x))$ instead of $\angle(T_x \mathrm{C}_{g_n}(x), T_x \mathrm{C}_f(x))$: for any $t \in [0,1]$, it holds

$$\begin{aligned}
&\angle(\dot{\beta}_1(t), T_{\xi(\beta_1(t))} \mathrm{C}_f(x)) = \angle(D_{y_n} H_{g_n}^t(T_{y_n} \mathrm{C}_{g_n}(x)), T_{\xi(H_{g_n}^t(y_n))} \mathrm{C}_f(x)) \\
&> \angle(D_{y_n} H_{g_n}^t(T_{y_n} \mathrm{C}_{g_n}(x)), T_{H_f^t(x)} \mathrm{C}_f(x)) - \angle(T_{H_f^t(x)} \mathrm{C}_f(x), T_{\xi(H_{g_n}^t(y_n))} \mathrm{C}_f(x)) \\
&= \angle(D_{y_n} H_{g_n}^t(T_{y_n} \mathrm{C}_{g_n}(x)), D_x H_f^t(T_x \mathrm{C}_f(x))) - \angle(T_{H_f^t(x)} \mathrm{C}_f(x), T_{\xi(H_{g_n}^t(x))} \mathrm{C}_f(x)) \\
&> \delta_0 - \frac{\delta_0}{2} = \delta,
\end{aligned}$$

which again contradicts Lemma 4.6. This concludes the proof. □

As it will be used in the proof, let us recall the following result of [HS]:

Proposition 4.7 (Corollary 2.20, [HS]). *Let $\mathcal{C}$ be a center disk of f such that $\mathcal{C} \cap \Gamma_f^0 = \emptyset$. Then the set $\Gamma_f^1(\mathcal{C})$ of points with 1-dimensional center accessibility classes in $\mathcal{C}$ admits a C^1 lamination whose leaves are the manifolds $\mathrm{C}_f(y) \cap \mathcal{C}$, $y \in \Gamma_f^1(\mathcal{C})$.*

5. Construction of adapted accessibility loops

Let $r \geq 2$, and let us consider a partially hyperbolic diffeomorphism $f \in \mathcal{PH}^r(M)$ with $\dim E_f^c \geq 2$ that is center bunched, dynamically coherent, and plaque expansive. In this section, given a point $x \in M$, we build suitable loops starting at x which will later be used to construct perturbations to break non-open accessibility classes. The loops which we construct will depend on whether the accessibility class of the point x is already open or not. In fact, although the accessibility class of x is a homogeneous set, when working with specific families of loops with a prescribed number of legs of a certain size, the set of points which we can reach from x moving along these loops may not exhibit the global structure of the accessibility class (for example, if the class of x is open, to be able to reach any point in a neighborhood of x, we may need to consider very long accessibility paths instead of local ones), which leads us to the following definitions.

Fix a subset $\mathscr{S} \subset M$. For any $\sigma > 0$, we let $\tilde{\Gamma}_f^0(\mathscr{S}, \sigma)$ be the set of all points $x \in \mathscr{S}$ whose center accessibility class is *locally trivial* in the following sense: for any 4 us-loop $\gamma = [x, x_1, x_2, x_3, x_4]$ at (f, x) such that $\ell(\gamma) < 10^{-2}\sigma$, we have $x_4 = x$. We also set $\tilde{\Gamma}_f^0(\mathscr{S}) := \cup_{\sigma \in (0,1)} \tilde{\Gamma}_f^0(\mathscr{S}, \sigma)$. When $\mathscr{S} = M$, we abbreviate $\tilde{\Gamma}_f^0(\mathscr{S}, \sigma), \tilde{\Gamma}_f^0(\mathscr{S})$ respectively as $\tilde{\Gamma}_f^0(\sigma), \tilde{\Gamma}_f^0$.

The next lemma explains how to construct *closed* us-loops at points x whose center accessibility class is (locally) one-dimensional; it will be useful later to show that after a C^r-small perturbation, the accessibility class of x can be made open.

Lemma 5.1. *There exist C^2-uniform constants $\sigma_0 = \sigma_0(f) > 0$, $K_0 = K_0(f) > 0$ such that for any $\sigma \in (0, \sigma_0)$, for any point $x_0 \in \Gamma^1_f \setminus \tilde{\Gamma}^0_f(\sigma)$, if $\phi = \phi_{x_0}$ is the chart given by Lemma 4.5, then for any point $x \in \Gamma^1_f \cap \phi(B(0_{\mathbb{R}^d}, \frac{K_0}{10}\sigma))$, there exists a non-degenerate closed us-loop $\gamma_x = [x, x_1, \ldots, x_9, x]$ at (f, x) such that*

(1) *$\ell(\gamma_x) < \sigma$;*
(2) *$B(z_1, K_0\sigma) \cap \{z, z_2, \ldots, z_9\} = \emptyset$, where $z = \phi^{-1}(x)$, and $z_i = \phi^{-1}(x_i)$, for each integer $i = 2, \ldots, 9$;*
(3) *the map $\Gamma_f \cap \phi(B(0_{\mathbb{R}^d}, \frac{K_0}{10}\sigma)) \ni x \mapsto \gamma_x$ is continuous.*

Proof. Fix some small $\sigma > 0$, let $x_0 \in \Gamma^1_f \setminus \tilde{\Gamma}^0_f(\sigma)$, and let $\sigma' \in (\frac{\sigma}{30}, \frac{\sigma}{20})$. By definition, there exists a non-degenerate 4 us-loop $\gamma = [x_0, x_1, x_2, x_3, x_4]$ such that

- $x_1 \in \mathcal{W}^u_f(x_0)$, with $\frac{\sigma'}{2} < d_{\mathcal{W}^u_f}(x_0, x_1) < \sigma'$;
- $x_2 \in \mathcal{W}^s_f(x_1)$, with $\frac{\sigma'}{2} < d_{\mathcal{W}^s_f}(x_1, x_2) < \sigma'$;
- $x_3 := H^u_{f,x_2,x_0}(x_2) \in \mathcal{W}^u_f(x_2, \sigma') \cap \mathcal{W}^{cs}_f(x_0, \sigma')$;
- $x_4 := H^s_{f,x_3,x_0}(x_3) \in \mathcal{W}^s_f(x_3, \sigma') \cap \mathcal{W}^c_f(x_0, \sigma')$, with $x_4 \in \mathrm{C}_f(x) \setminus \{x_0\}$.

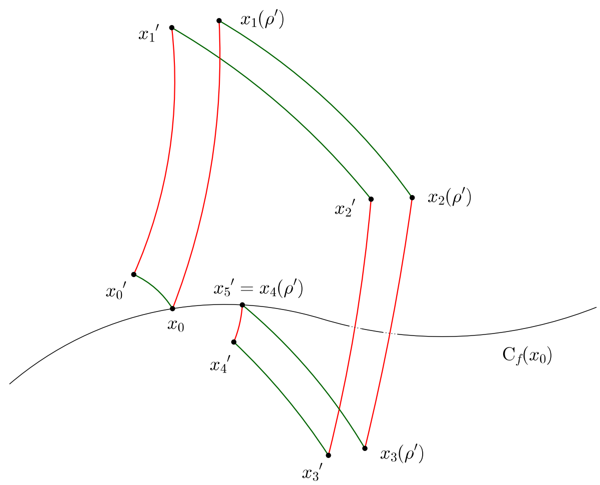

FIGURE 3. Construction of a non-degenerate closed us-loop.

As $\mathrm{C}_f(x_0)$ is one-dimensional, for the chart $\phi = \phi_{x_0}$ given by Lemma 4.5, we can assume that $\phi^{-1}(\mathrm{C}_f(x_0, \sigma)) = (-\rho_1, \rho_2) \times \{0_{\mathbb{R}^{d-1}}\} \simeq (-\rho_1, \rho_2)$, with $\rho_1, \rho_2 > 0$, $x_0 \simeq 0$, and $x_4 \simeq \rho \in (0, \rho_2)$. By varying the size of the legs, we can construct a continuous family $\{\gamma(t) = [x_0, x_1(t), x_2(t), x_3(t), x_4(t)]\}_{t \in [\frac{\rho}{2}, \rho]}$ of non-degenerate 4 us-loops at (f, x_0) such that $x_4(t) \simeq t \in [\frac{\rho}{2}, \rho]$.

Let us take $x_0' \in \mathcal{W}^s_f(x_0, \frac{\sigma'}{10}) \setminus \mathcal{W}^s_f(x_0, \frac{\sigma'}{20})$ and $t_0 \in [\frac{\rho}{2}, \rho)$ close to ρ. As in Lemma 2.7, we let $\gamma^{x_0', f}(t_0) = [x_0', x_1', x_2', x_3', x_4']$ be the natural continuation of $\gamma(t_0)$ starting at x_0' in place of x_0. Since $\mathcal{W}^{cu}_f(x_4') = \mathcal{W}^{cu}_f(x_4) = \mathcal{W}^{cu}_f(x_0)$, we can also define $\{x_5'\} := H^u_{f,x_4',x_0}(x_4') \in \mathcal{W}^u_{f,\mathrm{loc}}(x_4') \cap \mathcal{W}^c_{f,\mathrm{loc}}(x_0)$, and we set $\gamma' := [x_0, x_0', x_1', \ldots, x_5']$. In particular, $x_5' \in \mathrm{C}_f(x, \sigma)$, and $x_5' \simeq \rho'$ for some $\rho' \in (0, \rho)$. As $x_5' = x_4(\rho')$, we can concatenate the 4 us-loop $\gamma(\rho')$ at (f,x) with the 6 us-loop $\overline{\gamma'}$ at (f,x_5') to produce a closed 10 us-loop $\gamma_{x_0} := \overline{\gamma'}\gamma(\rho')$ at (f,x_0). By construction, γ_{x_0} is non-degenerate, and we have $\ell(\gamma_{x_0}) < \sigma$.

Let us check that $d(x_1', x_1(\rho')) > \frac{\sigma}{800}$ provided that σ is taken sufficiently small. By definition, we have $d_{\mathcal{W}_f^s}(x_0,x_0') \in [\frac{\sigma}{600}, \frac{\sigma}{200}]$. Since we work in a σ-neighbourhood of x_0, and as the map $z \mapsto E_f^u(z)$ is Hölder continuous (see [PSW1]), we deduce that the distance between the unstable bundles at any two points $z_1 \in \mathcal{W}_f^u(x_0',\sigma)$, $z_2 \in \mathcal{W}_f^u(x_0,\sigma)$ is at most $\tilde{c}_1\sigma^\theta$, for two C^2-uniform constants $\theta = \theta(f) > 0$, $\tilde{c}_1 = \tilde{c}_1(f) > 0$. Integrating the discrepancy along the unstable arcs from x_0' to x_1' and from x_0 to $x_1(\rho')$ yields

$$d(x_1', x_1(\rho')) \geq d(x_0', x_0) - \tilde{c}_2\sigma^\theta \times \sigma \geq \frac{\sigma}{600} - \tilde{c}_2\sigma^{1+\theta},$$

for some constant $\tilde{c}_2 > 0$. We conclude that $d(x_1', x_1(\rho')) \geq \frac{\sigma}{800}$ provided that σ is chosen sufficiently small, i.e., $\sigma \in (0,\sigma_0)$, for some C^2-uniform constant $\sigma_0 = \sigma_0(f) > 0$. Moreover, by construction, $B(x_1, \frac{\sigma}{100}) \cap \{x_0,x_2,x_3,x_4\} = \emptyset$. Similarly, we have $B(x_1', \frac{\sigma}{200}) \cap \{x_0,x_0',x_2',x_3',x_4',x_5'\} = \emptyset$ and $B(x_1(\rho'), \frac{\sigma}{200}) \cap \{x_0, x_2(\rho'), x_3(\rho'), x_4(\rho')\} = \emptyset$.

Let us now explain how this construction can be performed for points x near x_0 whose center accessibility class is also one-dimensional. By Lemma 2.7, for any point $x \in M$ which is sufficiently close to x_0, and for any $t \in [\frac{\rho}{2}, \rho]$, the us-loop $\gamma(t)$ admits a natural continuation $(\gamma(t))^{x,f} =: \check{\gamma}^x(t)$ that is a 4 us-loop at (f,x). Moreover, the map $t \mapsto \check{\gamma}^x(t)$ is continuous. Similarly, the 6 su-loop γ' has a natural continuation $(\gamma')^{x,f} = [x, (x_0')^{x,f}, \dots, (x_5')^{x,f}]$. The point $(x_5')^{x,f}$ depends continuously on x, hence we can choose a continuous map $\rho'(\cdot)$ such that $\rho'(x_0) = \rho'$ and such that the endpoint of $\check{\gamma}^x(\rho'(x))$ coincides with the endpoint $(x_5')^{x,f}$ of $(\gamma')^{x,f}$. In particular, the continuations $(\gamma')^{x,f}$, $\check{\gamma}^x(\rho'(x))$ depend continuously on x. We conclude that the closed 10 us-loop $\gamma_x := \overline{(\gamma')^{x,f}}\check{\gamma}^x(\rho'(x))$ at (f,x) depends continuously on the point x in a small neighbourhood of x_0. In particular, for x sufficiently close to x_0, we have $\ell(\gamma_x) < \sigma$. ☐

Actually, given a small center disk $\mathcal{D}$, we will need to construct *closed* us-loops at points $x \in \mathcal{D}$ whose center accessibility class is not open, i.e., either zero or one-dimensional. Let us introduce some notation. Fix some small $\sigma > 0$. For any $x \in \Gamma_f = \Gamma_f^0 \cup \Gamma_f^1$, we let

- let $\overline{\Gamma}_f(x) := \tilde{\Gamma}_f^0(\sigma)$, and $n(x) := 2$, if $x \in \tilde{\Gamma}_f^0(\sigma)$;
- otherwise, let $\overline{\Gamma}_f(x) := \Gamma_f^1 \setminus \tilde{\Gamma}_f^0(\sigma)$, and $n(x) := 5$, if $x \in \Gamma_f^1 \setminus \tilde{\Gamma}_f^0(\sigma)$.

Lemma 5.2. *There exist C^2-uniform constant $\tilde{K} = \tilde{K}(f) \in (0,1)$, $\tilde{\sigma} = \tilde{\sigma}(f) > 0$ such that for any $R_0 > 0$, for any integer $k_0 \geq 1$,*[2] *for any $\sigma \in (0,\tilde{\sigma})$, and for any point $x_0 \in \Gamma_f$ there exists a continuous map $\overline{\Gamma}_f(x_0) \cap \mathcal{W}_f^c(x_0, \tilde{K}\sigma) \ni x \mapsto \gamma^x$ such that $\gamma^x = \{\gamma^x(t) = [x, x_1^x(t), \dots, x_{2n}^x(t)]\}_{t\in[0,1]}$ is a continuous family of $2n$ us-loops at (f,x), with $n := n(x_0) \in \{2,5\}$, $\ell(\gamma^x) < \sigma$, such that $\gamma^x(0)$ is trivial, and for any integer $k \in \{1,\dots,k_0\}$, $\gamma^x(\frac{k}{k_0})$ is a non-degenerate closed us-loop.*

Proof. Let $R_0 > 0$, and let $k_0 \geq 1$ be some integer. Let $\sigma_0 = \sigma_0(f) > 0$, $K_0 = K_0(f) > 0$ be as in Lemma 5.1, and take some small $\sigma \in (0, \min(\overline{h}, \sigma_0))$.

[2]We will apply this lemma with $k_0 = 1$ or 2 in the following.

We consider a point $x_0 \in \Gamma_f$ and set $n := n(x_0) \in \{2,5\}$. Let $\overline{h} = \overline{h}(f) > 0$ and $\phi = \phi_{x_0} \colon (-\overline{h}, \overline{h})^d \to M$ be given by Lemma 4.5. We distinguish between two cases.

(1) If $x_0 \in \tilde{\Gamma}^0_f(\sigma)$, then there exists a non-degenerate closed $2n$ us-loop $\tilde{\gamma} = [x_0, x_1, x_2, x_3, x_0]$ at (f, x_0) with $n = 2$, $\ell(\tilde{\gamma}) < \frac{\sigma}{2}$ and $B(z_1, K_0\sigma) \cap \{0_{\mathbb{R}^d}, z_2, z_3\} = \emptyset$, where $z_i := \phi^{-1}(x_i)$, for $i = 1,2,3$. By decreasing continuously the size of the legs of $\tilde{\gamma}$, we obtain a family of $2n$ us-loops $\{\gamma(t) = [x_0, x_1(t), x_2(t), x_3(t), x_0]\}_{t\in[0,1]}$ at (f, x_0) such that $\gamma(0)$ is trivial and $\gamma(1) = \tilde{\gamma}$. Moreover, by choosing the map $t \mapsto \gamma(t)$ carefully, we can ensure that for any $k \in \{1, \dots, k_0\}$, it holds $B(z_1(\frac{k}{k_0}), \frac{K_0}{2}\sigma) \cap \{0_{\mathbb{R}^d}, z_2(\frac{k}{k_0}), z_3(\frac{k}{k_0})\} = \emptyset$, where $z_i(\frac{k}{k_0}) := \phi^{-1}\big((x_i)(\frac{k}{k_0})\big)$, for $i = 1,2,3$, and $d\big(z_1(\frac{k}{k_0}), z_1(\frac{k'}{k_0})\big) \geq \frac{K_0}{2k_0}\sigma$,[3] for all $k' \in \{1, \dots, k_0\} \setminus \{k\}$.

For any point $x \in \tilde{\Gamma}^0_f(\sigma) \cap \mathcal{W}^c_{f,\mathrm{loc}}(x_0)$ with $d(0_{\mathbb{R}^d}, \phi^{-1}(x)) \leq \frac{K_0}{10}\sigma$, and for $t \in [0,1]$, let $\gamma^x(t) = [x, x^x_1(t), x^x_2(t), x^x_3(t), x]$ be the closed $2n$ us-loop whose corners are:

- $x^x_1(t) := H^u_{f,x,x_1(t)}(x) \in \mathcal{W}^u_{f,\mathrm{loc}}(x) \cap \mathcal{W}^{cs}_{f,\mathrm{loc}}(x_1(t))$;
- $x^x_2(t) := H^s_{f,x^x_1(t),x_2(t)}(x^x_1(t)) \in \mathcal{W}^s_{f,\mathrm{loc}}(x^x_1(t)) \cap \mathcal{W}^{cu}_{f,\mathrm{loc}}(x_2(t))$;
- $x^x_3(t) := H^u_{f,x^x_2(t),x}(x^x_2(t)) \in \mathcal{W}^u_{f,\mathrm{loc}}(x^x_2(t)) \cap \mathcal{W}^{cs}_{f,\mathrm{loc}}(x)$.

We let γ^x be the continuous family $\gamma^x := \{\gamma^x(t)\}_{t\in[0,1]}$. If σ is sufficiently small, then $\ell(\gamma^x) < \sigma$, and for any $k \in \{1, \dots, k_0\}$, $\gamma^x(\frac{k}{k_0})$ is a non-degenerate closed us-loop at (f, x). Let $z^x_0 := \phi^{-1}(x)$, and $z^x_i(\frac{k}{k_0}) := \phi^{-1}\big((x^x_i)(\frac{k}{k_0})\big)$, for $i = 1,2,3$. Arguing as in the proof of Lemma 5.1, we have $B\big(z^x_1(\frac{k}{k_0}), \frac{K_0}{5}\sigma\big) \cap \{z^x_0, z^x_2(\frac{k}{k_0}), z^x_3(\frac{k}{k_0})\} = \emptyset$, and $d\big(z^x_1(\frac{k}{k_0}), z^x_1(\frac{k'}{k_0})\big) \geq \frac{K_0}{5k_0}\sigma$, for all $k' \in \{1, \dots, k_0\} \setminus \{k\}$, provided that σ is sufficiently small.

(2) Otherwise, we have $x_0 \in \Gamma^1_f \setminus \tilde{\Gamma}^0_f(\sigma)$. By Lemma 5.1, after possibly taking K_0 smaller, then for any point $x \in \Gamma_f \cap \mathcal{W}^c_{f,\mathrm{loc}}(x_0)$ such that $d(0_{\mathbb{R}^d}, \phi^{-1}(x)) \leq \frac{K_0}{10}\sigma$, there exists a non-degenerate closed $2n$ us-loop $\gamma_x = [x, x_1, \dots, x_{2n-1}, x]$ at (f, x) with $n = 5$, $\ell(\gamma_x) < \frac{\sigma}{2}$, such that the map $\Gamma_f \cap \phi(B(0_{\mathbb{R}^d}, \frac{K_0}{10}\sigma)) \ni x \mapsto \gamma_x$ is continuous, and such that $B(z^x_1, K_0\sigma) \cap \{z^x_0, z^x_2, \dots, z^x_{2n-1}\} = \emptyset$, where $z^x_0 := \phi^{-1}(x)$, and $z^x_i := \phi^{-1}\big(x^x_i\big)$, for each integer $i = 1, \dots, 2n-1$.

By decreasing continuously the size of the legs of γ_x, keeping $x_{2n-1}(t) \in \mathcal{W}^{cs}_f(x)$ and letting $x_{2n}(t) := H^s_{f,x_{2n-1}(t),x}(x_{2n-1}(t))$, we obtain a continuous family $\gamma^x = \{\gamma^x(t) = [x, x^x_1(t), \dots, x^x_{2n}(t)]\}_{t\in[0,1]}$ of $2n$ us-loops at (f, x) such that $\gamma^x(0)$ is trivial, $\gamma^x(1) = \gamma_x$, and $\ell(\gamma^x) < \sigma$.

Moreover, by choosing carefully the map $t \mapsto \gamma^x(t)$, we can ensure that for any integer $k \in \{1, \dots, k_0\}$, $\gamma^x(\frac{k}{k_0})$ is a non-degenerate closed us-loop at (f, x). Indeed, as in the proof of Lemma 5.1, we consider a one-parameter family $(\check{\gamma}^x(t))_{t\in[0,1]}$ of 4 us-loops at (f, x) such that $\check{\gamma}^x(0)$ is the trivial loop and such that the first corners of $\check{\gamma}^x(t)$ and $\check{\gamma}^x(t')$ are distinct for $t \neq t' \in [0,1]$. We can also perform the same construction as in Lemma 5.1 in order to obtain a closed 10-us loop $\gamma^x(t)$ at the times $t = 1, \frac{k_0-1}{k_0}, \frac{k_0-2}{k_0}, \dots, \frac{1}{k_0}$, and such that $B(z^x_1(\frac{k}{k_0}), \frac{K_0}{5}\sigma) \cap \{z^x_0, z^x_2(\frac{k}{k_0}), \dots, z^x_{2n-1}(\frac{k}{k_0})\} = \emptyset$,

[3]For instance, we choose the map $t \mapsto x_1(t) \in \mathcal{W}^u_{f,\mathrm{loc}}(x_0)$ in such a way that $d(z_1(t), z_1(t')) = d(0_{\mathbb{R}^d}, z_1) \cdot |t - t'|$, for all $t, t' \in [0,1]$.

where we let $z_i^x(\frac{k}{k_0}) := \phi^{-1}\big((x_i^x)(\frac{k}{k_0})\big)$, for $i = 1, \dots, 2n-1$, and such that $d\big(z_1^x(\frac{k}{k_0}), z_1^x(\frac{k'}{k_0})\big) \geq \frac{K_0}{5k_0}\sigma$, for all $k' \in \{1, \dots, k_0\} \setminus \{k\}$. □

We will also need to construct certain us/su-paths for all points in a small center disk. Take $f \in \mathscr{F}$ and let $\sigma > 0$ be small. We assume that for some point $x_0 \in M$, and some constant $K > 0$, it holds $x \notin \tilde{\Gamma}_f^0(\sigma)$, for all $x \in \mathcal{W}_f^c(x_0, K\sigma)$. Fix $\theta > 0$ small. By Proposition 4.7 and Proposition 4.4, there exists a C^1 neighbourhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}^{\mathscr{F}}$ and for any $x \in \Gamma_g^1 \cap \mathcal{W}_f^c(x_0, K\sigma)$, it holds

$$\Pi_x^c \mathrm{C}_g(x, 10\sigma) \subset \mathscr{C}_1, \tag{5.1}$$

where $\mathscr{C}_1 \subset \mathbb{R}^2$ is the cone of angle θ centered at $0_{\mathbb{R}^2}$, and $\Pi_x^c \colon M \to \mathbb{R}^2$ is the map in Lemma 4.5 for f. In the following, we let $\mathscr{C} := \big(\mathbb{R}^2 \setminus \mathscr{C}_1\big) \cup \{0_{\mathbb{R}^2}\}$, we denote by $\mathscr{C}_*^+, \mathscr{C}_*^-$ the two components of the set $\mathscr{C} \setminus \{0_{\mathbb{R}^2}\}$, and let $\mathscr{C}^+ := \mathscr{C}_*^+ \cup \{0_{\mathbb{R}^2}\}$, $\mathscr{C}^- := \mathscr{C}_*^- \cup \{0_{\mathbb{R}^2}\}$. Assume that $\mathscr{C}^+$, resp. $\mathscr{C}^-$ is the top, resp. bottom component in Figure 4.

Lemma 5.3. *Take f, x_0, σ, θ, $\mathcal{U}$ as above, and let $\mathscr{C}$, $\mathscr{C}^+$, and $\mathscr{C}^-$ as defined above. After possibly taking K smaller, there exist continuous maps $\mathcal{W}_f^c(x_0, K\sigma) \ni x \mapsto \gamma_1^x, \gamma_2^x$ such that for any $x \in \mathcal{W}_f^c(x_0, K\sigma)$, $\gamma_1^x = [x, \alpha_1^x, \dots, \omega_1^x]$, resp. $\gamma_2^x = [x, \alpha_2^x, \dots, \omega_2^x]$, is a non-degenerate closed 10 us-loop, resp. 10 su-loop at (f, x) such that $\ell(\gamma_1^x), \ell(\gamma_2^x) < \sigma$, such that the endpoints $\omega_1^x = H_{\gamma_1^x}(x)$, $\omega_2^x = H_{\gamma_2^x}(x)$ satisfy*

$$\big(\Pi_x^c \omega_1^x, \Pi_x^c \omega_2^x\big) \in \big(\mathscr{C}^+ \times \mathscr{C}^-\big) \cup \big(\mathscr{C}^- \times \mathscr{C}^+\big),$$

and such that for $\star = 1, 2$, for some C^2-uniform constant $\hat{K}_\star > 0$, we have

$$B(\alpha_\star^x, \hat{K}_\star \sigma) \cap \{z\} = \emptyset, \text{ for any corner } z \neq \alpha_\star^x \text{ of } \gamma_\star^x. \tag{5.2}$$

Proof. As $x_0 \in \Gamma_f^1 \setminus \tilde{\Gamma}_f^0(\sigma)$, there exists a non-degenerate 4 us-loop $\gamma = [x_0, x_1, x_2, x_3, x_4]$ with $\ell(\gamma) < \sigma$ and $x_4 \in \mathrm{C}_f(x_0) \setminus \{x_0\}$. By shrinking the size of the legs, we construct a continuous family $\{\gamma(t) = [x_0, x_1(t), x_2(t), x_3(t), x_4(t)]\}_{t \in [0,1]}$ of non-degenerate 4 us-loops at (f, x_0) such that $\gamma(0)$ is trivial and $\gamma(1) = \gamma$.

Assuming that $K > 0$ is sufficiently small, the family $\{\gamma(t)\}_{t \in [0,1]}$ extends to a continuous map $\mathcal{W}_f^c(x_0, K\sigma) \ni x \mapsto \gamma^x = \{\gamma^x(t)\}_{t \in [0,1]}$ such that for each $x \in \mathcal{W}_f^c(x_0, K\sigma)$, and for each $t \in [0,1]$, $\gamma^x(t) = [x, x_1^x(t), x_2^x(t), x_3^x(t), x_4^x(t)]$ is a 4 us-loop at (f, x), and $\gamma^x(0)$ is trivial. Moreover, up to reparametrization, there exists $\vartheta > 0$ such that for each $x \in \mathcal{W}_f^c(x_0, K\sigma)$, it holds

$$\begin{cases} \big\{\Pi_x^c\big(x_4^x(t)\big)\big\}_{t \in [\frac{1}{4}, \frac{1}{3}]} \subset B\Big(\Pi_x^c\Big(x_4\Big(\frac{1}{3}\Big)\Big), \frac{1}{2}\vartheta\Big) \subset B\big(0_{\mathbb{R}^2}, 3\vartheta\big), \\ \big\{\Pi_x^c\big(x_4^x(t)\big)\big\}_{t \in [\frac{1}{2}, \frac{2}{3}]} \subset B\Big(\Pi_x^c\Big(x_4\Big(\frac{2}{3}\Big)\Big), \frac{1}{2}\vartheta\Big) \subset \mathscr{C}_1^r \cap \Big(B\big(0_{\mathbb{R}^2}, 7\vartheta\big) \setminus B\big(0_{\mathbb{R}^2}, 4\vartheta\big)\Big), \\ \big\{\Pi_x^c\big(x_4^x(t)\big)\big\}_{t \in [\frac{3}{4}, 1]} \subset B\big(\Pi_x^c\big(x_4(1)\big), \frac{1}{2}\vartheta\big) \subset \mathscr{C}_1^r \cap \Big(B\big(0_{\mathbb{R}^2}, 10\vartheta\big) \setminus B\big(0_{\mathbb{R}^2}, 8\vartheta\big)\Big), \end{cases} \tag{5.3}$$

denoting by $\mathscr{C}_1^r$ the connected component of $\mathscr{C}_1 \setminus \{0_{\mathbb{R}^2}\}$ containing $\Pi_{x_0}^c(x_4)$. Moreover, after possibly changing the parametrization by t, we can also assume that for all $x \in \mathcal{W}_f^c(x_0, K\sigma)$, we have

$$d_{\mathcal{W}_f^u}(x_1^x(t), x) \geq \frac{1}{200}\sigma, \quad \text{for all } t \in \Big[\frac{1}{4}, 1\Big]. \tag{5.4}$$

Now, as in Lemma 5.1, we take $x_0' \in \mathcal{W}^s_f(x_0)$ such that $\frac{1}{200}\sigma \leq d_{\mathcal{W}^s_f}(x_0, x_0') \leq \frac{1}{100}\sigma$. For any $t \in [0,1]$, let $\tilde{\gamma}(t) = [y_0(t), \ldots, y_4(t)]$ be the natural continuation of $\gamma(t)$ starting at $y_0(t) = x_0'$ in place of x_0. As $\mathcal{W}^{cu}_f(y_4(t)) = \mathcal{W}^{cu}_f(x_4) = \mathcal{W}^{cu}_f(x_0)$, we may also define $\{y_5(t)\} := \mathcal{W}^u_{f,\mathrm{loc}}(y_4(t)) \cap \mathcal{W}^c_{f,\mathrm{loc}}(x_0)$, and set $\gamma_*(t) := [x_0, y_0(t), \ldots, y_4(t), y_5(t)]$. In the same way, for each point $x \in \mathcal{W}^c_f(x_0, K\sigma)$, we let $\gamma^x_*(t) = [x, y^x_0(t), \ldots, y^x_5(t)]$ be the continuation of $\gamma_*(t)$ starting at x given by Lemma 2.7.

For each $(x,t) \in \mathcal{W}^c_f(x_0, K\sigma) \times [0,1]$, we denote by $\check{\gamma}^x(t)$ the continuation of $\overline{\gamma^x_*(\frac{2}{3})}$ starting at $x^x_4(t)$ as in Lemma 2.7, and by concatenation, we obtain the 10 us-loop $\gamma^x_1(t) := \gamma^x(t)\check{\gamma}^x(t) = [x, \alpha^x_1(t), \ldots, \omega^x_1(t)]$. If σ is sufficiently small, x_0' is very close to x_0, and by (5.3), for any $x \in \mathcal{W}^c_f(x_0, K\sigma)$, it holds

$$\Pi^c_x\Big(\omega^x_1\Big(\frac{1}{4}\Big)\Big) \notin \mathscr{C}^r_1, \qquad \Pi^c_x(\omega^x_1(1)) \in \mathscr{C}^r_1.$$

Since the set $\{\omega^x_1(t)\}_{t\in[0,1]}$ of endpoints is connected, its image under Π^c_x has to cross the cone $\mathscr{C} = \mathscr{C}^+ \cup \mathscr{C}^-$. We then let $t^x \in [0,1]$ be the smallest $t \in [0,1]$ such that $\Pi^c_x(\omega^x_1(t)) \in \mathscr{C}^r_1$; we also denote by $\gamma^x_1 = [x, \alpha^x_1, \ldots, \omega^x_1]$ the 10 us-loop $\gamma^x_1(t^x) = \gamma^x(t^x)\check{\gamma}^x(t^x)$, with $\alpha^x_1 := \alpha^x_1(t^x)$ and $\omega^x_1 := \omega^x_1(t^x)$. In particular, we have $\Pi^c_x(\omega^x_1) \in \mathscr{C}$; without loss of generality, we assume that $\Pi^c_x(\omega^x_1) \in \mathscr{C}^+$.

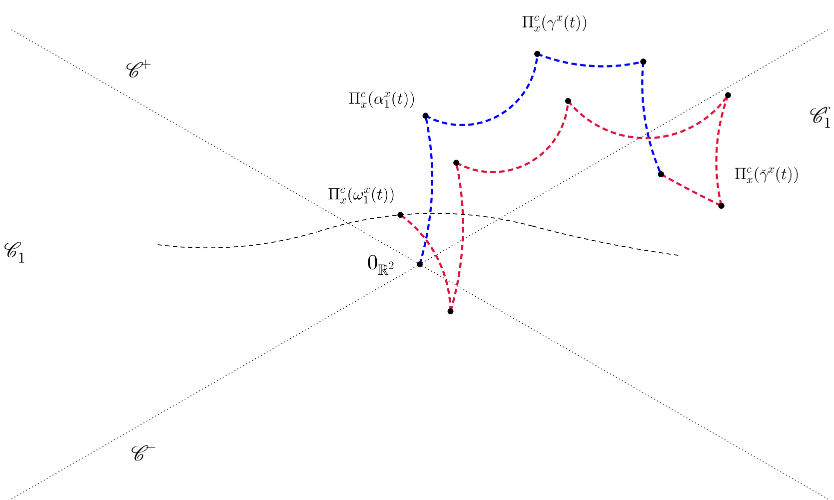

FIGURE 4. Construction of the loop γ^x_1.

For each $s \in [0,1]$, we also denote by $\gamma^x_2(s) = [x, \alpha^x_2(s), \ldots, \omega^x_2(s)]$ the 10 su-loop obtained by taking the continuation of $\overline{\gamma^x_1(s)}$ starting at x in place of $\omega^x_1(s)$. In this case, arguing as above, we see that for certain values $s \in [0,1]$, it holds $\Pi^c_x(\omega^x_2(s)) \in \mathscr{C}^-$; we then let $s^x \in [0,1]$ be the largest $s \in [0,1]$ with that property, and we define the 10 su-loop $\gamma^x_2 := \gamma^x_2(s^x)$, with $\gamma^x_2 = [x, \alpha^x_2, \ldots, \omega^x_2]$, and $\Pi^c_x(\omega^x_2) \in \mathscr{C}^-$.

Besides, (5.2) follows from arguments similar to those in Lemma 5.1, using (5.4), and since x_0' was chosen such that $d_{\mathcal{W}^s_f}(x_0, x_0') \geq \frac{1}{200}\sigma$. □

6. A submersion from the space of perturbations to the phase space

As above, we consider a partially hyperbolic diffeomorphism $f \in \mathcal{PH}^r(M)$, $r \geq 2$, with $\dim E^c_f \geq 2$ that is center bunched, dynamically coherent, and plaque expansive. In Subsection 6.1, we recall some general results from [LZ] about random perturbations and the changes those perturbations induce on certain holonomy maps. In Subsection 6.2, we construct a family of perturbations and show how the results of the previous part can be applied to the particular setting we are interested in.

6.1. **Random perturbations.** As in [LZ], we will use the following suspension construction to show that certain holonomy maps are differentiable with respect to the perturbation parameter. The idea is to incorporate the perturbation parameter into a higher dimensional partially hyperbolic diffeomorphism, which, under some assumptions, is still dynamically coherent and center bunched.

Definition 6.1 (C^r deformation). *Let $I \geq 1$ be some integer, and let $\mathcal{U}$ be an open neighbourhood of $\{0\}$ in $\mathbb{R}^I$. A C^r map $\hat{f} \colon \mathcal{U} \times M \to M$ satisfying $\hat{f}(0, \cdot) = f$ and $\hat{f}(b, \cdot) \in \mathcal{PH}^r(M)$ for all $b \in \mathcal{U}$ is called a C^r* deformation at f with I-parameters. *We associate with $\hat{f}$ the suspension map $T(\hat{f})$ defined by*

$$T = T(\hat{f}) \colon \mathcal{U} \times M \to \mathcal{U} \times M, \quad (b, x) \mapsto (b, \hat{f}(b, x)), \tag{6.1}$$

and we denote $f_b := \hat{f}(b, \cdot)$. If in addition $f_b \in \mathcal{PH}^r(M, \mathrm{Vol})$ for all $b \in \mathcal{U}$, then $\hat{f}$ is said to be volume preserving.

Definition 6.2 (Infinitesimal C^r deformation). *Let $I \geq 1$ be an integer. A C^r map $V \colon \mathbb{R}^I \times M \to TM$ is called an* infinitesimal C^r deformation with I-parameters *if*

1. *for each $B \in \mathbb{R}^I$, $V(B, \cdot)$ is a C^r vector field on M;*
2. *for each $x \in M$, $B \mapsto V(B, x)$ is a linear map from $\mathbb{R}^I$ to T_xM.*

Remark 6.3. *Given $I \geq 1$, an infinitesimal C^r deformation V with I-parameters, and some small $\epsilon > 0$, we associate with V a C^r deformation at f with I-parameters, denoted by $\hat{f}$, which is defined by*

$$\hat{f}(b, x) := \mathcal{F}_{V(b,\cdot)}(1, f(x)), \quad \forall\, (b, x) \in \mathcal{U} \times M,$$

where $\mathcal{U} = B(0, \epsilon) \subset \mathbb{R}^I$ and for any $B \in \mathbb{R}^I$, $\mathcal{F}_{V(B,\cdot)} \colon \mathbb{R} \times M \to M$ denotes the C^r flow generated by the vector field $V(B, \cdot)$. In this case, we say that $\hat{f}$ is generated by V. *If in addition $V(B, \cdot)$ is divergence-free for each $B \in \mathbb{R}^I$, then $\hat{f}$ is volume preserving as in Definition 6.1, and we say that V is* volume preserving.

Lemma 6.4 (Lemma 4.11 in [LZ]). *Let $I \geq 1$ be some integer, let $\mathcal{U} \subset \mathbb{R}^I$ be an open neighbourhood of $\{0\}$, and let $\hat{f} \colon \mathcal{U} \times M \to M$ be a C^r deformation at f with I-parameters. If $\mathcal{U}$ is chosen sufficiently small, then the map $T = T(\hat{f})$ is a C^r dynamically coherent partially hyperbolic system for some T-invariant splitting*

$$T_{(b,x)}(\mathcal{U} \times M) \simeq T_b\mathcal{U} \oplus T_xM = E^s_T(b, x) \oplus E^c_T(b, x) \oplus E^u_T(b, x),$$

for all $(b, x) \in \mathcal{U} \times M$. Moreover, for any $(b, x) \in \mathcal{U} \times M$, we have

$$E^*_T(b, x) = \{0\} \oplus E^*_{f_b}(x), \qquad \mathcal{W}^*_T(b, x) = \{b\} \times \mathcal{W}^*_{f_b}(x), \qquad \text{for } * = u, s,$$

and

$$E^c_T(b,x) = Graph(\nu_b(x,\cdot)) \oplus E^c_{f_b}(x), \tag{6.2}$$

for a unique linear map $\nu_b(x,\cdot)\colon T_b\mathcal{U} \to E^{su}_{f_b}(x) := E^s_{f_b}(x) \oplus E^u_{f_b}(x)$.

If in addition f *is center bunched, then, after reducing the size of* $\mathcal{U}$, *u/s-holonomy maps between local center leaves of* T *(within distance* 1*) are* C^1 *when restricted to some cu/cs-leaf, with uniformly continuous, uniformly bounded derivatives.*

Let $I \geq 1$ be some integer, let $\mathcal{U} \subset \mathbb{R}^I$ be some small neighbourhood of $\{0\}$ in $\mathbb{R}^I$, let $\hat{f}\colon \mathcal{U}\times M \to M$ be a C^1 deformation at f with I-parameters, and let $T = T(\hat{f})$.

Definition 6.5 (Lift of a us/su-loop)**.** *For any point* $x \in M$, *for any integer* $n \geq 2$, *and for any* $2n$ *us/su-loop* $\gamma = [x, x_1, \ldots, x_{2n}]$ *at* (f,x), *we define the* lift *of* γ *as*

$$\hat{\gamma} := [(0,x), (0,x_1), \ldots, (0,x_{2n})].$$

In particular, by Lemma 6.4, $\hat{\gamma}$ *is a* $2n$ *us/su-loop at* $(T,(0,x))$.

Remark 6.6. *In the following, we will mostly consider us-loops; for that reason, we will state the technical lemmas needed for the proof only for us-loops, but similar results hold for su-loops as well.*

Similarly to Lemma 2.7, given a point $x \in M$ and a us-loop at (f,x), we can define a natural continuation for the C^1 deformation $\hat{f}$ with I-parameters we consider:

Definition 6.7. *Let* $x \in M$, *let* $n \geq 2$, *we say that* $\gamma = \{\gamma(t) = [x, x_1(t), \ldots, x_{2n}(t)]\}_{t\in[0,1]}$ *is a continuous family of* $2n$ *us-loops at* (f,x) *if for each* $t \in [0,1]$, $\gamma(t)$ *is a* $2n$ *us-loop, and for each* $i = 1,\ldots,2n$, *the map* $t \mapsto x_i(t)$ *is continuous. Given such a family, for any* $t \in [0,1]$, *we let* $\hat{\gamma}(t)$ *be the lift of* $\gamma(t)$ *as above. Then by continuity, there exists a* C^2*-uniform constant* $\hat{\delta} = \hat{\delta}(T,\gamma) > 0$ *such that* $B(0,\hat{\delta}) \subset \mathcal{U}$, *and for any* $(b,y,t) \in \mathcal{W}^c_T((0,x),\hat{\delta})\times[0,1]$, *for some constant* $\hat{h} = \hat{h}(T,\gamma) > 0$, *the following intersections exist and are unique:*

- $\{(b,\hat{x}^{b,y}_1(t))\} := \mathcal{W}^u_{T,\mathrm{loc}}((b,y),\hat{h}) \cap \mathcal{W}^{cs}_{T,\mathrm{loc}}((0,x_1(t)),\hat{h})$;
- $\{(b,\hat{x}^{b,y}_2(t))\} := \mathcal{W}^s_{T,\mathrm{loc}}((b,\hat{x}^{b,y}_1(t)),\hat{h}) \cap \mathcal{W}^{cu}_{T,\mathrm{loc}}((0,x_2(t)),\hat{h})\ldots$
- $\ldots\{(b,\hat{x}^{b,y}_{2n-1}(t))\} := \mathcal{W}^u_{T,\mathrm{loc}}((b,\hat{x}^{b,y}_{2n-2}(t)),\hat{h}) \cap \mathcal{W}^{cs}_{T,\mathrm{loc}}((0,x),\hat{h})$;
- $\{(b,\hat{x}^{b,y}_{2n}(t))\} := \mathcal{W}^s_{T,\mathrm{loc}}((b,\hat{x}^{b,y}_{2n-1}(t)),\hat{h}) \cap \mathcal{W}^c_{T,\mathrm{loc}}((0,x),\hat{h})$.

We thus have a continuous family of $2n$ *us-loops at* (f_b,y), *denoted by* $\{\hat{\gamma}^{b,y}(t)\}_{t\in[0,1]}$*:*

$$\hat{\gamma}^{b,y}(t) := [y, \hat{x}^{b,y}_1(t), \ldots, \hat{x}^{b,y}_{2n}(t)], \quad \forall\, t \in [0,1].$$

We define the map

$$\widehat{\psi} = \widehat{\psi}(T,x,\gamma)\colon \left\{ \begin{array}{rcl} \mathcal{W}^c_T((0,x),\hat{\delta})\times[0,1] & \to & \mathcal{W}^c_T(0,x), \\ (b,y,t) & \mapsto & H_{T,\hat{\gamma}(t)}(b,y) = (b,\hat{x}^{b,y}_{2n}(t)). \end{array}\right. \tag{6.3}$$

For any $(b,y) \in \mathcal{W}^c_T((0,x),\hat{\delta})$, *we thus get a map* $\psi = \psi(T,x,\gamma)$*:*

$$\psi(b,y,\cdot) := \pi_M\widehat{\psi}(b,y,\cdot)\colon [0,1] \to \mathcal{W}^c_{f_b}(y), \tag{6.4}$$

where $\pi_M\colon \mathcal{U}\times M \to M$ *denotes the canonical projection.*

Definition 6.8. *Let $I \geq 1$ be some integer. For any infinitesimal C^r deformation with I-parameters $V : \mathbb{R}^I \times M \to TM$, we define*

$$\mathrm{supp}(V) := \{x \in M \mid \exists\, B \in \mathbb{R}^I \text{ such that } V(B,x) \neq 0\}.$$

Given an open neighbourhood $\mathcal{U}$ of $\{0\}$ in $\mathbb{R}^I$, and a C^r deformation at f with I-parameters $\hat{f} : \mathcal{U} \times M \to M$, we define

$$\mathrm{supp}(\hat{f}) := \{x \in M \mid \exists\, b \in \mathcal{U} \text{ such that } \hat{f}(b,x) \neq f(x)\}.$$

We introduce the following definitions in order to control return times of a map to the support of a deformation; they are motivated by the fact that for very large return times, it is possible to achieve a good control on how certain holonomies change after perturbation.

Definition 6.9. *For any subsets $A, B \subset M$, and for $* \in \{+,-\}$, we define*

$$R(f,A,B) := \inf\{n \geq 0 \mid f^n(A) \cap B \neq \emptyset \text{ or } f^{-n}(A) \cap B \neq \emptyset\};$$
$$R_*(f,A,B) := \inf\{n \geq 1 \mid f^{*n}(A) \cap B \neq \emptyset\}.$$

We abbreviate $R(f,A,A)$, $R_*(f,A,A)$ respectively as $R(f,A)$, $R_*(f,A)$. Similarly, for a C^1 deformation $\hat{f} : \mathcal{U} \times M \to M$ of f, and for $* \in \{+,-\}$, we set

$$R(\hat{f},A,B) := \inf\{n \geq 0 \mid \exists\, b \in \mathcal{U} \text{ s.t. } \hat{f}(b,\cdot)^n(A) \cap B \neq \emptyset \text{ or } \hat{f}(b,\cdot)^{-n}(A) \cap B \neq \emptyset\},$$
$$R_*(\hat{f},A,B) := \inf\{n \geq 1 \mid \exists\, b \in \mathcal{U} \text{ s.t. } \hat{f}(b,\cdot)^{*n}(A) \cap B \neq \emptyset\},$$

and we abbreviate $R(\hat{f},A,A)$, $R_*(\hat{f},A,A)$ respectively as $R(\hat{f},A)$, $R_*(\hat{f},A)$.

In the following, most of the time[4], we restrict ourselves to the case of deformations with 2-parameters, i.e., we take a small neighbourhood $\mathcal{U} \subset \mathbb{R}^2$ of $\{0_{\mathbb{R}^2}\}$, we let $\hat{f} : \mathcal{U} \times M \to M$ be a C^1 deformation at f with 2-parameters generated by an infinitesimal C^1 deformation with 2-parameters $V : \mathbb{R}^2 \times M \to TM$, and we set $T = T(\hat{f})$.

Definition 6.10 (Adapted deformation)**.** *Let $x \in M$, let $n \geq 2$ be some integer, and let $\gamma = [x, x_1, \ldots, x_{2n}]$ be a $2n$ us-loop or su-loop at (f,x) with $\ell(\gamma) < \sigma$ for some small $\sigma > 0$. Given two constants $C, R_0 > 0$, we say that an infinitesimal C^r deformation V is* adapted to (γ, σ, C, R_0) *if*

1. $\sigma\|\partial_b \partial_x V\|_M + \|\partial_b V\|_M < C$;
2. $R(f, \{z\}, \mathrm{supp}(V)) > R_0$ *for* $z = x, x_2, \ldots, x_{2n}$;
3. $R_\pm(f, \{x_1\}, \mathrm{supp}(V)) > R_0$.

Proposition 6.11 (see Proposition 5.6, [LZ])**.** *For any $C, \kappa > 0$, there exist C^2-uniform constants $R_0 = R_0(f,C,\kappa) > 0$ and $\kappa_0 = \kappa_0(f,C,\kappa) > 0$ such that the following is true.*

Let $x \in M$, let $n \geq 2$ be some integer, and let $\gamma = [x, x_1, \ldots, x_{2n}]$ be a $2n$ us-loop at (f,x) of length $\sigma > 0$ such that there exists an infinitesimal C^r deformation V that is adapted to (γ, σ, C, R_0). In the following, we denote by $B = (B_1, B_2)$ an element of $T_0\mathcal{U} \simeq \mathbb{R}^2$. Assume that for all $z \in \{x, x_2, \ldots, x_{2n}\}$, we have

$$D_B(\pi_c V(B,z)) = 0, \tag{6.5}$$

[4] Except in Subsection 8.2 where deformations with 4-parameters are needed.

while

$$\left|\det\ (B \mapsto D_B(\pi_c V(B, x_1)))\right| > \kappa, \tag{6.6}$$

where $\pi_c \colon TM \to E^c_f$ *denotes the canonical projection.*

Then, the map

$$\Xi\colon \begin{cases} T_0\mathcal{U} & \to & E^c_f(x_{2n}), \\ B & \mapsto & \hat{\pi}_c DH_{T,\hat{\gamma}}(B + \nu_0(x, B)), \end{cases}$$

satisfies

$$\det\ \Xi \geq \kappa_0,$$

where $\hat{\gamma}$ *is the lift of* γ *for* T*, and* $\hat{\pi}_c \colon E^c_T(0, x_{2n}) = Graph(\nu_0(x_{2n}, \cdot)) \oplus E^c_f(x_{2n}) \to E^c_f(x_{2n})$ *denotes the canonical projection.*

6.2. **Construction of C^r deformations at f.** In the following, we assume that $\dim E^c_f = 2$. Recall that $\Pi^c \colon \mathbb{R}^d \simeq \mathbb{R}^2 \times \mathbb{R}^{d_u+d_s} \to \mathbb{R}^2$ is the canonical projection, and that Π^c_x is the map $\Pi^c_x := \Pi^c \circ \phi_x^{-1} \colon M \to \mathbb{R}^2$.

Lemma 6.12. *Let* $\tilde{K} = \tilde{K}(f) \in (0, 1)$*,* $\tilde{\sigma} = \tilde{\sigma}(f) > 0$ *be as in Lemma 5.2. Then, for any* $R_0 > 0$*, for any integer* $k_0 \geq 1$*, for any* $\sigma \in (0, \tilde{\sigma})$*, and for any point* $x_0 \in \Gamma_f$ *satisfying* $R_\pm(f, B(x_0, 10\sigma)) > R_0$*,*[5] *there exists an infinitesimal* C^r *deformation at* f *with* $2k_0$*-parameters* $V \colon \mathbb{R}^{2k_0} \times M \to TM$ *such that* $\mathrm{supp}(V) \subset B(x_0, 10\sigma)$*,*[5] *and there exists a continuous map* $\overline{\Gamma}_f(x_0) \cap \mathcal{W}^c_f(x_0, \tilde{K}\sigma) \ni x \mapsto \gamma^x$ *such that* $\gamma^x = \{\gamma^x(t) = [x, x_1^x(t), \ldots, x_{2n}^x(t)]\}_{t\in[0,1]}$ *is a continuous family of* $2n$ *us-loops at* (f, x)*, with* $n := n(x_0) \in \{2, 5\}$*,* $\ell(\gamma^x) < \sigma$*, such that* $\gamma^x(0)$ *is trivial, and for any integer* $k \in \{1, \ldots, k_0\}$*, we have:*

1. $\gamma^x(\frac{k}{k_0})$ *is a non-degenerate closed us-loop;*
2. V *is adapted to* $(\gamma^x(\frac{k}{k_0}), \sigma, \widetilde{C}, R_0)$*, for some* C^2*-uniform constant* $\widetilde{C} = \widetilde{C}(f, k_0) > 0$*;*
3. *for any* $z \in \left\{x, x_2^x(\frac{k}{k_0}), \ldots, x_{2n-1}^x(\frac{k}{k_0})\right\}$*, it holds*

$$D_B(\pi_c V(B, z)) = 0,$$

and there exists a 2*-dimensional vector space* $E_k \subset \mathbb{R}^{2k_0}$ *such that*

$$\left|\det\ \Big(E_k \ni B \mapsto D_B\big(\pi_c V\big(B, x_1^x(\frac{k}{k_0})\big)\big)\Big)\right| > \tilde{\kappa},$$

for some C^2*-uniform constant* $\tilde{\kappa} = \tilde{\kappa}(f) > 0$*, where* $\pi_c \colon TM \to E^c_f$ *denotes the canonical projection.*

[5]Recall Definition 6.8 and Definition 6.9.

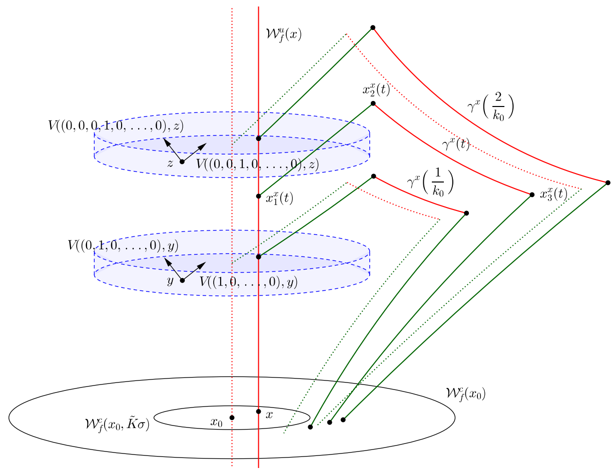

FIGURE 5. Localization of the perturbations.

Actually, we will apply Lemma 6.12 with $k_0 = 1$ or 2 in the following. The construction in the proof of Lemma 6.12 is adapted from [LZ].

Proof of Lemma 6.12. Let $R_0 > 0$, and let $k_0 \geq 1$ be some integer. Let $\tilde{K} = \tilde{K}(f) \in (0,1)$, $\tilde{\sigma} = \tilde{\sigma}(f) > 0$ be as in Lemma 5.2, and take some small $\sigma \in (0, \tilde{\sigma})$.

We consider a point $x_0 \in \Gamma_f$ such that $R_\pm(f, B(x_0, 10\sigma)) > R_0$, and set $n := n(x_0) \in \{2,5\}$. We let $\overline{h} = \overline{h}(f) > 0$ and $\phi = \phi_{x_0} : (-\overline{h}, \overline{h})^d \to M$ be given by Lemma 4.5.

For each $x \in \overline{\Gamma}_f(x_0) \cap \mathcal{W}^c_f(x_0, \tilde{K}\sigma)$ we let $\gamma^x = \{\gamma^x(t) = [x, x^x_1(t), \dots, x^x_{2n}(t)]\}_{t \in [0,1]}$ be the continuous family of $2n$ us-loops at (f,x) constructed in Lemma 5.2.

For each integer $k \in \{1, \dots, k_0\}$, we let $z_k = (z^c_k, z^u_k, z^s_k) := z^{x_0}_1(\frac{k}{k_0}) \in (-\overline{h}, \overline{h})^d = (-\overline{h}, \overline{h})^2 \times (-\overline{h}, \overline{h})^{d_u} \times (-\overline{h}, \overline{h})^{d_s}$. We define a collection of functions and vector fields as follows.

For each $1 \leq j \leq 2$, let $U_j : (-\frac{1}{5}, \frac{1}{5})^2 \times (-\frac{1}{5}, \frac{1}{5})^{d_u} \times (-\frac{1}{3}, \frac{1}{3})^{d_s} \to \mathbb{R}^d$ be a compactly supported C^∞ divergence-free vector field such that U_j restricted to $(-\frac{1}{10}, \frac{1}{10})^2 \times (-\frac{1}{10}, \frac{1}{10})^{d_u} \times (-\frac{1}{5}, \frac{1}{5})^{d_s}$ is equal to the constant vector e_j, where $e_1 := (1, 0, 0, \dots, 0) \in \mathbb{R}^d$ and $e_2 := (0, 1, 0, \dots, 0) \in \mathbb{R}^d$. Moreover, we can assume that U_j satisfies $\|U_j\|_{C^1} < C_*$ for some constant $C_* = C_*(d) > 0$.

For any $x_c \in \mathbb{R}^2$, $x_u \in \mathbb{R}^{d_u}$, $x_s \in \mathbb{R}^{d_s}$, for any $\lambda_c, \lambda_u, \lambda_s > 0$, and for any $z_c \in \mathbb{R}^2$, $z_u \in \mathbb{R}^{d_u}$, $z_s \in \mathbb{R}^{d_s}$, we set:

$$\Lambda^{\lambda_c, \lambda_u, \lambda_s}_{x_c, x_u, x_s}(z_c, z_u, z_s) = (x_c + \lambda_c z_c, x_u + \lambda_u z_u, x_s + \lambda_s z_s).$$

For any $j \in \{1,2\}$, any $x_u \in \mathbb{R}^{d_u}$, we let $U^\sigma_{j,x_u} : (-\overline{h}, \overline{h})^d \to \mathbb{R}^d$ be the vector field

$$U^\sigma_{j,x_u} = U_j \big(\Lambda^{\tilde{K}\sigma, \frac{\tilde{K}}{k_0}\sigma, \tilde{K}\sigma}_{0_{\mathbb{R}^2}, x_u, 0_{\mathbb{R}^{d_s}}}\big)^{-1}.$$

The support of U^σ_{j,x_u} is contained in

$$\Big(-\frac{\tilde K}{5}\sigma,\frac{\tilde K}{5}\sigma\Big)^2\times\Big(x_u+\big(-\frac{\tilde K}{5k_0}\sigma,\frac{\tilde K}{5k_0}\sigma\big)^{d_u}\Big)\times\Big(-\frac{\tilde K}{3}\sigma,\frac{\tilde K}{3}\sigma\Big)^{d_s}.$$

Moreover, for any $z_c\in(-\frac{\tilde K}{10}\sigma,\frac{\tilde K}{10}\sigma)^2$, for any $z_u\in x_u+(-\frac{\tilde K}{10k_0}\sigma,\frac{\tilde K}{10k_0}\sigma)^{d_u}$ and for any $z_s\in\big(-\frac{\tilde K}{5}\sigma,\frac{\tilde K}{5}\sigma\big)^{d_s}$, it holds

$$U^\sigma_{j,x_u}(z_c,z_u,z_s)=e_j.$$

We set

$$V^\sigma_{j,x_u}:=D\phi(U^\sigma_{j,x_u}).$$

The vector field V^σ_{j,x_u} is divergence-free and satisfies:

$$\sigma\|\partial_x V^\sigma_{j,x_u}\|_M+\|V^\sigma_{j,x_u}\|_M<\widetilde C_0, \tag{6.7}$$

for some C^2-uniform constant $\widetilde C_0=\widetilde C_0(f,k_0)>0$.

Let $V\colon\mathbb R^{2k_0}\times M\to TM$ be the infinitesimal C^r deformation defined as

$$\begin{aligned}V(B,\cdot):=&\big(B_{1,1}V^\sigma_{1,z_1^u}+B_{2,1}V^\sigma_{2,z_1^u}\big)+\big(B_{1,2}V^\sigma_{1,z_2^u}+B_{2,2}V^\sigma_{2,z_2^u}\big)+\cdots+\\&+\big(B_{1,k_0-1}V^\sigma_{1,z^u_{k_0-1}}+B_{2,k_0-1}V^\sigma_{2,z^u_{k_0-1}}\big)+\big(B_{1,k_0}V^\sigma_{1,z^u_{k_0}}+B_{2,k_0}V^\sigma_{2,z^u_{k_0}}\big),\end{aligned}\tag{6.8}$$

for all $B=\sum_{k=1}^{k_0}B_{1,k}u_{2k-1}+B_{2,k}u_{2k}\in\mathbb R^{2k_0}$, where $(u_i)_{i=1}^{2k_0}$ denotes the canonical basis of $\mathbb R^{2k_0}$.

By definition, the map V is linear in B. Moreover, by (6.7), (6.8), it holds

$$\sigma\|\partial_b\partial_x V\|_M+\|\partial_b V\|_M<\widetilde C, \tag{6.9}$$

with $\widetilde C:=2k_0\widetilde C_0>0$.

As $\ell(\gamma^x)<\sigma$, we have $\operatorname{supp}(V)\subset B(x_0,10\sigma)$, and for any integer $k\in\{1,\dots,k_0\}$, it holds $x_2^x(\frac{k}{k_0}),x_3^x(\frac{k}{k_0}),\dots,x^x_{2n-1}(\frac{k}{k_0})\in B(x_0,10\sigma)$, for all $x\in\overline\Gamma_f(x_0)\cap\mathcal W^c_f(x_0,\tilde K\sigma)$. Recall that by assumption, we have

$$R_\pm(f,B(x_0,10\sigma))>R_0. \tag{6.10}$$

By (6.9) and (6.10), we conclude that V is adapted to $(\gamma^x(\frac{k}{k_0}),\sigma,\widetilde C,R_0)$.

By construction, for any $z\in\big\{x,x_2^x(\frac{k}{k_0}),\dots,x^x_{2n-1}(\frac{k}{k_0})\big\}$, it holds

$$D_B(\pi_c V(B,z))=0,$$

and

$$\Big|\det\Big(E_k\ni B\mapsto D_B\big(\pi_c V\big(B,x_1^x(\frac{k}{k_0})\big)\big)\Big)\Big|>\tilde\kappa,$$

for some C^2-uniform constant $\tilde\kappa=\tilde\kappa(f)>0$, where $E_k:=\operatorname{Span}(u_{2k-1},u_{2k})\subset\mathbb R^{2k_0}$, and $\pi_c\colon TM\to E^c_f$ denotes the canonical projection. □

Corollary 6.13. *For any integers $k_0\ge1$, $r\ge2$, for any $\delta>0$, there exist C^2-uniform constants $\tilde K_0=\tilde K_0(f)\in(0,1)$, $\tilde\sigma_0=\tilde\sigma_0(f,k_0)>0$, $\tilde R_0=\tilde R_0(f,k_0)>0$ and $\tilde\delta_0=\tilde\delta_0(f,r,\delta)>0$ such that for any $\sigma\in(0,\tilde\sigma_0)$, for any point $x_0\in\Gamma_f$ satisfying $R_\pm(f,B(x_0,10\sigma))>\tilde R_0$, there exists a C^r deformation $\hat f\colon B(0_{\mathbb R^{2k_0}},\tilde\delta_0)\times M\to M$ at f with $2k_0$-parameters generated by an infinitesimal C^r deformation $V\colon\mathbb R^{2k_0}\times$*

$M \to TM$, such that $\mathrm{supp}(\hat{f}) \subset B(x_0, 10\sigma)$,[6] *and there exists a continuous map* $\overline{\Gamma}_f(x_0) \cap \mathcal{W}^c_f(x_0, \tilde{K}_0\sigma) \ni x \mapsto \gamma^x$*, such that*

(1) $\gamma^x = \{\gamma^x(t) = [x, x_1^x(t), \dots, x_{2n}^x(t)]\}_{t\in[0,1]}$ *is a continuous family of* $2n$ *us-loops at* (f, x) *as in Lemma 6.12, with* $n := n(x_0) \in \{2, 5\}$, $\ell(\gamma^x) < \sigma$*, such that* $\gamma^x(0)$ *is trivial, and for any integer* $k \in \{1, \dots, k_0\}$, $\gamma^x(\frac{k}{k_0})$ *is a non-degenerate closed us-loop;*

(2) *let* $T = T(\hat{f})$*, let* $\psi_x := \psi(T, x, \gamma^x)$ *be the map defined in (6.4), let* $\Pi^c_x \colon M \to \mathbb{R}^2$ *be the map given by Lemma 4.5, and for* $k \in \{1, \dots, k_0\}$*, let* $\Phi^{(k)} \colon (b, x) \mapsto \Pi^c_x \psi_x(b, x, \frac{k}{k_0})$*; then, the map*

$$\Phi \colon \begin{cases} B(0_{\mathbb{R}^{2k_0}}, \tilde{\delta}_0) \times \Big(\overline{\Gamma}_f(x_0) \cap \mathcal{W}^c_f(x_0, \tilde{K}_0\sigma)\Big) & \to \quad \mathbb{R}^{2k_0} \\ \qquad\qquad\qquad\qquad\qquad\qquad (b, x) & \mapsto \quad \big(\Phi^{(k)}(b, x)\big)_{k=1,\dots,k_0} \end{cases}$$

is continuous; besides, for any $x \in \overline{\Gamma}_f(x_0) \cap \mathcal{W}^c_f(x_0, \tilde{K}_0\sigma)$, $\Phi(\cdot, x)$ *is* C^1*, and*

$$|\det D_b|_{b=0}\big(\Phi(\cdot, x)\big)| > \tilde{\kappa}_0,$$

for some C^2*-uniform constant* $\tilde{\kappa}_0 = \tilde{\kappa}_0(f, k_0) > 0$*;*

(3) $d_{C^r}(f, f_b) < \delta$*, for all* $b \in B(0_{\mathbb{R}^{2k_0}}, \tilde{\delta}_0)$*, where* $f_b := \hat{f}(b, \cdot) \in \mathcal{PH}^r(M)$*.*

Proof. Fix two integers $k_0 \geq 1$ and $r \geq 2$. Let $\tilde{K}_0 := \tilde{K}(f) > 0$, $\tilde{\sigma} := \tilde{\sigma}(f) > 0$, $\widetilde{C} = \widetilde{C}(f, k_0) > 0$ and $\tilde{\kappa} = \tilde{\kappa}(f) > 0$ be the constants given by Lemma 6.12, and let $\tilde{R}_0 := R_0(f, \widetilde{C}, \tilde{\kappa}) > 0$, $\kappa_0 := \kappa_0(f, \widetilde{C}, \tilde{\kappa}) > 0$ be the constants given by Proposition 6.11. Given $\sigma \in (0, \tilde{\sigma})$, we consider a point $x_0 \in \Gamma_f$ such that $R_\pm(f, B(x_0, 10\sigma), B(x_0, 10\sigma)) > \tilde{R}_0$, and set $n := n(x_0) \in \{2, 5\}$.

We let $V \colon \mathbb{R}^{2k_0} \times M \to TM$ be the infinitesimal C^r deformation at f with $2k_0$-parameters given by Lemma 6.12. Take a point $x \in \overline{\Gamma}_f(x_0) \cap \mathcal{W}^c_f(x_0, \tilde{K}_0\sigma)$ and let $\gamma^x = \{\gamma^x(t) = [x, x_1^x(t), \dots, x_{2n}^x(t)]\}_{t\in[0,1]}$ be the continuous family of $2n$ us-loops at (f, x) given by Lemma 6.12. Recall that the map $x \mapsto \gamma^x$ is continuous. Let $\hat{f}$ be the C^r deformation at f with 2-parameters generated by V, and let $T = T(\hat{f})$. By the properties of V in Lemma 6.12, we have $\mathrm{supp}(\hat{f}) \subset B(x_0, 10\sigma)$.

For any $t \in [0, 1]$, let $\hat{\gamma}^x(t)$ be the lift of $\gamma^x(t)$ for T, and let us denote by $\hat{\pi}_c \colon E^c_T(0, x) = Graph(\nu_0(x, \cdot)) \oplus E^c_f(x) \to E^c_f(x)$ the canonical projection. Fix an integer $k \in \{1, \dots, k_0\}$. We let $\Xi^{(k)}_x$ be the map defined as

$$\Xi^{(k)}_x \colon \begin{cases} \mathbb{R}^2 \simeq E_k & \to \quad E^c_f(x), \\ \qquad\quad B & \mapsto \quad \hat{\pi}_c DH_{T, \hat{\gamma}^x(\frac{k}{k_0})}(B + \nu_0(x, B)). \end{cases}$$

By points (2)-(2) of Lemma 6.12, and by Proposition 6.11, it holds

$$|\det \Xi^{(k)}_x| \geq \kappa_0. \tag{6.11}$$

Let $\widehat{\psi}_x := \widehat{\psi}(T, x, \gamma^x)$ and $\psi_x := \pi_M \widehat{\psi}_x$ be the maps defined in (6.3)-(6.4), and let $\Pi^c_x \colon M \to \mathbb{R}^2$ be the map given by Lemma 4.5. Let $\hat{\delta} > 0$ be such that $\hat{\delta} < \hat{\delta}(T, \gamma^x)$ for all $x \in \overline{\Gamma}_f(x_0) \cap \mathcal{W}^c_f(x_0, \tilde{K}_0\sigma)$, with $\hat{\delta}(T, \gamma^x) > 0$ as in Definition 6.7. Let

$$\Phi^{(k)} \colon \begin{cases} \mathcal{W}^c_T((0, x_0), \hat{\delta}) & \to \quad \mathbb{R}^2, \\ \qquad\qquad (b, x) & \mapsto \quad \Pi^c_x \psi_x(b, x, \frac{k}{k_0}). \end{cases}$$

[6] Recall Definition 6.8 .

As $x \mapsto \gamma^x$ is continuous, the maps $x \mapsto \psi_x$ and $\Phi^{(k)}$ are continuous as well.

For each $x \in \overline{\Gamma}_f(x_0) \cap \mathcal{W}_f^c(x_0, \tilde{K}_0\sigma)$, and for each $B \in \mathbb{R}^2 \simeq E_k$, we have

$$\begin{aligned} D\Phi_x^{(k)}(0, B + \nu_0(x,B)) &= D\Pi_x^c \pi_M DH_{T,\hat{\gamma}^x(\frac{k}{k_0})}(B + \nu_0(x,B)) \\ &= D\Pi_x^c \Big[\hat{\pi}_c DH_{T,\hat{\gamma}^x(\frac{k}{k_0})}(B + \nu_0(x,B)) + \nu_0(x,B)\Big], \end{aligned} \tag{6.12}$$

where $\Phi_x^{(k)} := \Phi^{(k)}(\cdot, x)$, and $\pi_M \colon \mathbb{R}^2 \times M \to M$ denotes the projection onto the second coordinate.

By Lemma 4.5 there exists a constant $D > 0$ such that for $\zeta > 0$ small, if $\sigma \in (0, \overline{h}_\zeta(f))$, then for any $x \in \overline{\Gamma}_f(x_0) \cap \mathcal{W}_f^c(x_0, \tilde{K}_0\sigma)$ and for any $B \in \mathbb{R}^2$, it holds

$$\|D\Pi_x^c \nu_0(x,B)\| \leq D\zeta\|B\|. \tag{6.13}$$

If $\zeta > 0$ is sufficiently small (depending only on κ_0), then for any $\sigma \in (0, \tilde{\sigma}_0)$, with $\tilde{\sigma}_0 := \min(\tilde{\sigma}, \overline{h}_\zeta(f)) > 0$, and for any $x \in \overline{\Gamma}_f(x_0) \cap \mathcal{W}_f^c(x_0, \tilde{K}_0\sigma)$, by (6.11)-(6.12)-(6.13), we deduce that

$$\left|\det D_b|_{b=0}\big(\Phi_x^{(k)}|_{E_k}\big)\right| > \frac{1}{2}\kappa_0,$$

which concludes the proof of point (2), for $\tilde{\kappa}_0 := \left(\frac{1}{2}\kappa_0\right)^{k_0} > 0$.

Finally, point (3) is a direct observation. □

7. Local accessibility

Let us fix an integer $r \geq 2$, and let us consider $f \in \mathscr{F}$, where as before, $\mathscr{F} \subset \mathcal{PH}_*^r(M)$ is the set of C^r dynamically coherent, plaque expansive, partially hyperbolic diffeomorphisms with two-dimensional center, which satisfy some strong bunching condition as in Definition 2.5.

In this part, we show that it is possible to make the accessibility class of any non-periodic point open by a C^r-small perturbation. First, we explain how to break trivial accessibility classes, and then, we show how to open one-dimensional accessibility classes, based on some transversality arguments.

7.1. Breaking trivial accessibility classes.

Proposition 7.1. *For any non-periodic point $x_0 \in M$, for any $\delta > 0$, and for any $\sigma > 0$, there exists a partially hyperbolic diffeomorphism $g \in \mathscr{F}$ such that $d_{C^r}(f,g) < \delta$ and such that $x_0 \notin \tilde{\Gamma}_g^0(\sigma)$; in particular, the center accessibility class $\mathrm{C}_g(x_0)$ is at least one-dimensional.*

Proof. Take a non-periodic point $x_0 \in M$. Fix some small $\delta > 0$, let $k_0 := 1$, and let $\tilde{\sigma}_0 = \tilde{\sigma}_0(f,1) > 0$, $\tilde{R}_0 = \tilde{R}_0(f,1) > 0$ and $\tilde{\delta}_0 = \tilde{\delta}_0(f,r,\delta) > 0$ be the constants given by Corollary 6.13. As x_0 is non-periodic, then for $\sigma \in (0, \tilde{\sigma}_0)$ sufficiently small, it holds $R_\pm(f, B(x_0, 10\sigma)) > \tilde{R}_0$. Assume that $x_0 \in \tilde{\Gamma}_f^0(\sigma)$ (otherwise there is nothing to prove).

By Corollary 6.13, for $n := n(x_0) = 2$, there exist a continuous family

$$\gamma = \gamma^{x_0} = \{\gamma(t) = [x_0, x_1(t), x_2(t), x_3(t), x_0]\}_{t \in [0,1]} \tag{7.1}$$

of 4 us-loops at (f, x_0) such that $\ell(\gamma) < \sigma$, $\gamma(0)$ is trivial, $\gamma(1)$ is a non-degenerate closed 4 us-loop, a C^r deformation $\hat{f} \colon B(0_{\mathbb{R}^2}, \tilde{\delta}_0) \times M \to M$ at f with 2-parameters, so that $\mathrm{supp}(\hat{f}) \subset B(x_0, 10\sigma)$, and such that the map

$$\Phi_{x_0}\colon B(0_{\mathbb{R}^2}, \tilde{\delta}_0) \ni b \mapsto \Pi_x^c \psi(b, x_0, 1) \tag{7.2}$$

is C^1 and satisfies

$$|\det D_b|_{b=0}\Phi_{x_0}| > \tilde{\kappa}_0, \tag{7.3}$$

for some C^2-uniform constant $\tilde{\kappa}_0 = \tilde{\kappa}_0(f, 1) > 0$. Recall that in (7.2), $\Pi_x^c\colon M \to \mathbb{R}^2$ is the map defined in Lemma 4.5, $T = T(\hat{f})$, and $\psi = \psi(T, x_0, \gamma^{x_0})$.

Moreover, by Definition 6.7, for all $b \in B(0_{\mathbb{R}^2}, \tilde{\delta}_0)$ and all $t \in [0, 1]$, we have $\psi(b, x_0, t) \in \mathcal{W}^c_{f_b}(x_0) \cap \mathrm{Acc}_{f_b}(x_0) = \mathrm{C}_{f_b}(x_0)$, where $f_b := \hat{f}(b, \cdot)$. Besides, (7.3) implies that the map $\psi(\cdot, x_0, 1)$ is a submersion in a neighbourhood of $0_{\mathbb{R}^2}$, hence

$$\psi(\cdot, x_0, 1)^{-1}\{x_0\} \cap B(0_{\mathbb{R}^2}, \delta_1) = \{0_{\mathbb{R}_2}\},$$

for some sufficiently small $\delta_1 \in (0, \tilde{\delta}_0)$. Fix $b \in B(0_{\mathbb{R}^2}, \delta_1) \setminus \{0_{\mathbb{R}^2}\}$ such that $g := f_b \in \mathscr{F}$; we have $\psi(b, x_0, 1) \in \mathrm{C}_g(x_0) \setminus \{x_0\}$, and $[0, 1] \ni t \mapsto \psi(b, x_0, t) \in \mathrm{C}_g(x_0)$ is a non-trivial g-path connecting x_0 to $\psi(b, x_0, 1) \neq x_0$ within $\mathrm{C}_g(x_0)$. In particular, $x_0 \notin \tilde{\Gamma}^0_g(\sigma)$; in fact, by Theorem 2.10, $\mathrm{C}_g(x_0)$ is at least one-dimensional. Moreover, by point (3) of Corollary 6.13, we have $d_{C^r}(f, g) < \delta$, which concludes the proof. □

7.2. **Opening one-dimensional accessibility classes.** The following result shows that local accessibility can be achieved near non-periodic points after a C^r-small perturbation.

Proposition 7.2. *For any non-periodic point $x_0 \in M$, and for any $\delta > 0$, there exists a partially hyperbolic diffeomorphism $g \in \mathscr{F}$ such that $d_{C^r}(f, g) < \delta$ and such that the accessibility class $\mathrm{Acc}_g(x_0)$ is open.*

Proof. Let us consider a non-periodic point $x_0 \in M$. Let $k_0 := 1$, and let $\tilde{\sigma}_0 = \tilde{\sigma}_0(f, 1) > 0$, $\tilde{R}_0 = \tilde{R}_0(f, 1) > 0$ be the constants given by Corollary 6.13. As x_0 is non-periodic, we can fix $\sigma \in (0, \tilde{\sigma}_0)$ such that $R_\pm(f, B(x_0, 10\sigma)) > \tilde{R}_0$.

Assume by contradiction that $\mathrm{Acc}_f(x_0)$ is stably not open in the C^r topology. In other words, by Theorem 2.10, for some $\delta_1 > 0$, and for all diffeomorphism $g \in \mathcal{PH}^r_*(M)$ such that $d_{C^r}(f, g) < \delta_1$, we have $x_0 \in \Gamma_g$. Fix some small $\delta \in (0, \delta_1)$. By Proposition 7.1, there exists a C^r diffeomorphism $f_0 \in \mathscr{F}$ with $d_{C^r}(f, f_0) < \frac{\delta}{2}$ such that $x_0 \notin \tilde{\Gamma}^0_{f_0}(\frac{\sigma}{2})$. In particular, by our choice of δ, we have $x_0 \in \Gamma^1_{f_0} \setminus \tilde{\Gamma}^0_{f_0}(\frac{\sigma}{2})$. Besides, there exists $\delta_2 \in (0, \frac{\delta}{2})$ such that for any diffeomorphism g satisfying $d_{C^r}(f_0, g) < \delta_2$, we have $g \in \mathscr{F}$, and $x_0 \in \Gamma^1_g \setminus \tilde{\Gamma}^0_g(\sigma)$. In particular, with the notations in Section 4, $\mathcal{U}^{\mathscr{F}} = \mathcal{U}^{\mathscr{F}}_1(x_0)$, where $\mathcal{U}$ is a δ_2-neighbourhood of f_0 in the C^r topology.

By Corollary 6.13, for $n := n(x_0) = 5$, there exist a continuous family

$$\gamma = \gamma^{x_0} = \{\gamma(t) = [x_0, x_1(t), x_2(t), \ldots, x_{10}(t)]\}_{t \in [0,1]} \tag{7.4}$$

of 10 us-loops at (f_0, x_0) such that $\ell(\gamma) < \sigma$, $\gamma(0)$ is trivial, $\gamma(1)$ is a non-degenerate closed 10 us-loop, a C^r deformation $\hat{f}\colon B(0_{\mathbb{R}^2}, \tilde{\delta}_0) \times M \to M$ at f_0 with 2-parameters, with $\tilde{\delta}_0 = \tilde{\delta}_0(f_0, r, \delta) > 0$, so that for the map $\Pi^c_{x_0}\colon M \to \mathbb{R}^2$ defined in Lemma 4.5, $T = T(\hat{f})$, and $\psi = \psi(T, x_0, \gamma^{x_0})$, the map $\Phi_{x_0}\colon B(0_{\mathbb{R}^2}, \tilde{\delta}_0) \ni b \mapsto \Pi^c_{x_0}\psi(b, x_0, 1)$ is C^1, and for some constant $\tilde{\kappa}_0 = \tilde{\kappa}_0(f, 1) > 0$, it holds

$$|\det D_b|_{b=0}\Phi_{x_0}| > \tilde{\kappa}_0. \tag{7.5}$$

Fix some small $\theta > 0$. It follows from the previous discussion and Proposition 4.4 that for $\delta_0 \in (0, \tilde{\delta}_0)$, $\varepsilon_0 > 0$ sufficiently small, then for all $b \in B(0_{\mathbb{R}^2}, \delta_0)$, the diffeomorphism $f_b := \hat{f}(b, \cdot)$ satisfies $f_b \in \mathcal{U}_1^{\mathscr{F}}(x_0)$, and

$$\mathrm{C}_{f_b}(x_0, \varepsilon_0) \subset \mathscr{C}_{f_0}(x_0, \theta, \varepsilon_0), \tag{7.6}$$

where $\mathscr{C}_{f_0}(x_0, \theta, \varepsilon_0)$ is as in (4.1). Let us set

$$\mathscr{C}(x_0, \theta) := \Pi^c_{x_0}\big(\mathscr{C}_{f_0}(x_0, \theta, \varepsilon_0)\big).$$

By Definition 6.7, and since $\psi(0, x_0, 1) = x_0$,[7] for $\tilde{\delta}_1 \in (0, \delta_0)$ sufficiently small, we have $\psi(b, x_0, 1) \in \mathrm{C}_{f_b}(x_0, \varepsilon_0)$, for all $b \in B(0_{\mathbb{R}^2}, \tilde{\delta}_1)$, and by (7.6), we deduce that

$$\Phi_{x_0}\big(B(0_{\mathbb{R}^2}, \tilde{\delta}_1)\big) \subset \Pi^c_x\big(\mathrm{C}_{f_b}(x_0, \varepsilon_0)\big) \subset \mathscr{C}(x_0, \theta).$$

On the one hand, by the definition of the cone $\mathscr{C}(x_0, \theta)$, we have $\mathbb{R}^2 \setminus \Phi_{x_0}\big(B(0_{\mathbb{R}^2}, \tilde{\delta}_1)\big) \supset \Delta_0$ for some straight line Δ_0 through the origin $0_{\mathbb{R}^2}$. But on the other hand, it follows from (7.5) that $\Phi_{x_0}\big(B(0_{\mathbb{R}^2}, \tilde{\delta}_1)\big)$ contains an open neighbourhood of $0_{\mathbb{R}^2}$, a contradiction. By Theorem 2.10, we conclude that for some $b \in B(0_{\mathbb{R}^2}, \tilde{\delta}_0)$, $\mathrm{Acc}_{f_b}(x_0)$ is open; moreover, by construction, $g := f_b \in \mathscr{F}$ satisfies

$$d_{C^r}(f, g) \le d_{C^r}(f, f_0) + d_{C^r}(f_0, f_b) < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

which concludes the proof. □

8. C^r-density of accessibility

In this part, we conclude the proof of our main results stated in Section 3. As above, we fix an integer $r \ge 2$, and let $f \in \mathscr{F}$, where $\mathscr{F} \subset \mathcal{PH}^r_*(M)$ is the set of C^r dynamically coherent, plaque expansive, partially hyperbolic diffeomorphisms with two-dimensional center, which satisfy some strong bunching condition as in Definition 2.5. Our goal is to conclude the proof of our main result (Theorem A):

Proposition 8.1. *For any $\delta > 0$, there exists a partially hyperbolic diffeomorphism $g \in \mathscr{F}$ with $d_{C^r}(f, g) < \delta$ such that g is stably accessible.*

8.1. Spanning c-families.

For the proof of Proposition 8.1, we combine ideas from the last section with some global argument; this is done by means of spanning families of center-disks; this notion was already present in the work of Dolgopyat-Wilkinson [DW] and is also used in [LZ].

Definition 8.2 (c-disk)**.** *For each $x \in M$ and $\sigma > 0$, $\mathcal{C} = \mathcal{W}^c_f(x, \sigma)$ is called the* center disk *of f (or c-disk of f for short) centered at x with radius σ. We set $\varrho(\mathcal{C}) := \sigma$, and for any $\theta \in (0, 1]$, we also define $\theta\mathcal{C} := \mathcal{W}^c_f(x, \theta\sigma)$.*

Definition 8.3. *A collection of disjoint center disks $\mathcal{D} = \{\mathcal{C}_1, \dots, \mathcal{C}_J\}$ is called a* family of center disks for f *(or* c-family for f *for short). In addition, we set*

$$\underline{r}(\mathcal{D}) := \inf_{\mathcal{C} \in \mathcal{D}} \{\varrho(\mathcal{C})\}, \quad \overline{r}(\mathcal{D}) := \sup_{\mathcal{C} \in \mathcal{D}} \{\varrho(\mathcal{C})\}.$$

Given $\theta \in (0, 1)$ and $k \ge 1$, we say that $\mathcal{D}$ is a (θ, k)-spanning *c-family for f if*

$$M = \bigcup_{\mathcal{C} \in \mathcal{D}} \bigcup_{x \in \theta\mathcal{C}} Acc_f(x, k),$$

[7] Recall that $\gamma(1)$ is a closed 10 us-loop at (f_0, x_0).

where $Acc_f(x,k)$ denotes the set of all points $y \in M$ which can connected to x by a f-accessibility sequence with at most k legs of length less than one.

Given any subset $\mathcal{C} \subset M$, and $\sigma \geq 0$, we set $(\mathcal{C},\sigma) := \{x \in M \,|\, d(x,\mathcal{C}) \leq \sigma\}$. Given $\sigma \geq 0$ and a c-family $\mathcal{D} = \{\mathcal{C}_1, \dots, \mathcal{C}_J\}$ for f, we set

$$(\mathcal{D},\sigma) := \bigcup_{j=1}^{J} (\mathcal{C}_j,\sigma).$$

We say that $\mathcal{D}$ is σ-sparse *if for any two distinct $\mathcal{C},\mathcal{C}' \in \mathcal{D}$, $(\mathcal{C},\sigma),(\mathcal{C}',\sigma)$ are disjoint. Any c-family for f is σ-sparse for some $\sigma > 0$.*

Proposition 8.4 (Corollary 6.2, [LZ])**.** *Assume that $f \in \mathcal{PH}^1(M)$ is dynamically coherent, plaque expansive, and that the fixed points of f^k are isolated for all $k \geq 1$. Then for every $\overline{R} > 1$, there exist C^1-uniform constants $\overline{N} = \overline{N}(f,\overline{R}) > 0$, $\overline{\rho} = \overline{\rho}(f,\overline{R}) \in (0,\overline{R}^{-1})$ and $\overline{\sigma} = \overline{\sigma}(f,\overline{R}) > 0$ such that the following is true. For all diffeomorphism g sufficiently C^1-close to f, there exists a $(\frac{1}{40},4)$-spanning c-family $\mathcal{D}_g$ for g with at most $\overline{N}$ elements such that*

1. $\overline{\rho} < \underline{r}(\mathcal{D}_g) \leq \overline{r}(\mathcal{D}_g) < \overline{R}^{-1}$;
2. $\mathcal{D}_g$ *is $\overline{\sigma}$-sparse;*
3. $R_\pm(g,(\mathcal{D}_g,\overline{\sigma})) > \overline{R}$.

Moreover, the map $g \mapsto \mathcal{D}_g$ can be chosen to be continuous.

8.2. **Density of diffeomorphisms with no trivial accessibility class.** The following result strengthens Proposition 7.1.

Proposition 8.5. *There exist C^2-uniform constants $\tilde{\sigma}_1 = \tilde{\sigma}_1(f) > 0$, $\tilde{K}_1 = \tilde{K}_1(f) \in (0,1)$ and $\tilde{R}_1 = \tilde{R}_1(f) > 0$ such that for any $\delta > 0$, for any $\sigma \in (0,\tilde{\sigma}_1)$, for any point $x_0 \in M$ satisfying $R_\pm(f,B(x_0,10\sigma)) > \tilde{R}_1$, there exists a partially hyperbolic diffeomorphism $g \in \mathscr{F}$ such that $d_{C^r}(f,g) < \delta$ and such that for some $\delta' = \delta'(x_0,g) > 0$, we have $x \notin \tilde{\Gamma}_h^0(\sigma)$, for all $x \in \mathcal{W}_f^c(x_0,\tilde{K}_1\sigma)$ and for all $h \in \mathscr{F}$ with $d_{C^1}(g,h) < \delta'$. In particular, the center accessibility class $\mathrm{C}_h(x)$ of each point $x \in \mathcal{W}_f^c(x_0,\tilde{K}_1\sigma)$ is at least one-dimensional.*

Remark 8.6. *In order to deal with all the points in a given center disk, the idea is to increase the codimension of "bad" configurations; this is done by considering two 4 us-loops at each point in the center disk, and show that we can construct a perturbation in such a way that for each of those points, at least one of the endpoints of the 4 us-loops is not the original point.*

Proof. Fix some small $\delta > 0$, let $k_0 := 2$, and let $\tilde{K}_1 := \tilde{K}_0(f) \in (0,1)$, $\tilde{\sigma}_1 := \tilde{\sigma}_0(f,2) > 0$, $\tilde{R}_1 := \tilde{R}_0(f,2) > 0$ and $\tilde{\delta}_1 := \tilde{\delta}_0(f,r,\delta) > 0$ be the constants given by Corollary 6.13. Let us take $\sigma \in (0,\tilde{\sigma}_1)$, and let us consider a point $x_0 \in \tilde{\Gamma}_f^0(\sigma)$ satisfying $R_\pm(f,B(x_0,10\sigma)) > \tilde{R}_1$.

By Corollary 6.13, for $n := n(x_0) = 2$, there exists a continuous map $\tilde{\Gamma}_f^0(\sigma) \cap \mathcal{W}_f^c(x_0,\tilde{K}_1\sigma) \ni x \mapsto \gamma^x$ such that $\gamma^x = \{\gamma^x(t) = [x,x_1^x(t),x_2^x(t),x_3^x(t),x]\}_{t\in[0,1]}$ is a continuous family of 4 us-loops at (f,x), with $\ell(\gamma^x) < \sigma$, such that $\gamma^x(0)$ is trivial, for $k = 1,2$, $\gamma^x(\frac{k}{2})$ is a non-degenerate closed us-loop, and there exists a C^r deformation $\hat{f}\colon B(0_{\mathbb{R}^4},\tilde{\delta}_1) \times M \to M$ at f with 4-parameters, so that $\mathrm{supp}(\hat{f}) \subset B(x_0,10\sigma)$, and such that the map

$$\Phi\colon \begin{cases} B(0_{\mathbb{R}^4},\tilde{\delta}_1)\times\Big(\tilde{\Gamma}_f^0(\sigma)\cap\mathcal{W}_f^c(x_0,\tilde{K}_1\sigma)\Big) & \to \quad \mathbb{R}^4=\mathbb{R}^2\times\mathbb{R}^2\\ \qquad\qquad\qquad\qquad\qquad (b,x) & \mapsto \quad (\Phi^{(1)}(b,x),\Phi^{(2)}(b,x)) \end{cases}$$

is continuous and satisfies

$$|\det D_b|_{b=0}\big(\Phi(\cdot,x)\big)|>\tilde{\kappa}_0,$$

for some C^2-uniform constant $\tilde{\kappa}_0=\tilde{\kappa}_0(f,2)>0$. Recall that $\Pi_x^c\colon M\to\mathbb{R}^2$ is the map given by Lemma 4.5, $T=T(\hat{f})$, $\psi_x=\psi(T,x,\gamma^x)$, and $\Phi^{(k)}(\cdot,x):=\Pi_x^c\psi_x(\cdot,x,\frac{k}{2})$, for $k=1,2$.

By Lemma 2.7, we can extend the map $x\mapsto\gamma^x=\{\gamma^x(t)\}_{t\in[0,1]}$ to all the points $x\in\mathcal{W}_f^c(x_0,\tilde{K}_1\sigma)$ (note that for $x\in\mathcal{W}_f^c(x_0,\tilde{K}_1\sigma)\setminus\tilde{\Gamma}_f^0(\sigma)$, the us-loops $\gamma^x(\frac{1}{2}),\gamma^x(1)$ may not be closed). Considering the associated maps $\psi_x=\psi(T,x,\gamma^x)$ and $\Phi^{(k)}(\cdot,x)=\Pi_x^c\psi_x(\cdot,x,\frac{k}{2})$, for $k=1,2$, we can thus extend Φ to a map

$$\Phi\colon \begin{cases} B(0_{\mathbb{R}^4},\tilde{\delta}_1)\times\mathcal{W}_f^c(x_0,\tilde{K}_1\sigma) & \to \quad \mathbb{R}^4=\mathbb{R}^2\times\mathbb{R}^2\\ \qquad\qquad\qquad (b,x) & \mapsto \quad (\Phi^{(1)}(b,x),\Phi^{(2)}(b,x)) \end{cases}$$

such that

$$|\det D_b|_{b=0}\big(\Phi(\cdot,x)\big)|>\frac{1}{2}\tilde{\kappa}_0.$$

Take $\hat{\delta}>0$ suitably small, and let

$$\Psi\colon \begin{cases} \mathcal{W}_T^c((0,x_0),\hat{\delta}) & \to \quad \mathbb{R}^6=\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2,\\ \qquad (b,x) & \mapsto \quad \big(\Pi_x^c(x),\Phi^{(1)}(b,x),\Phi^{(2)}(b,x)\big). \end{cases}$$

For any point $x\in\mathcal{W}_f^c(x_0,\tilde{K}_1\sigma)$, the map $\Phi(\cdot,x)$ is a submersion, and thus, the map Ψ is uniformly transverse to the diagonal

$$\Sigma_0:=\{(z,z,z):z\in\mathbb{R}^2\}\subset\mathbb{R}^6.$$

Therefore, $\Psi^{-1}(\Sigma_0)$ is a submanifold of codimension 4. Let $\pi_B\colon\mathcal{W}_T^c((0,x_0),\hat{\delta})\to\mathbb{R}^4$, $(b,x)\mapsto b$. Let $b\in B(0_{\mathbb{R}^4},\tilde{\delta}_1)\setminus\pi_B\big(\Psi^{-1}(\Sigma_0)\big)$, and let $g:=f_b:=\hat{f}(b,\cdot)\in\mathscr{F}$. Then, for any $x\in\mathcal{W}_f^c(x_0,\tilde{K}_1\sigma)$, we have $\Psi((b,x))\notin\Sigma_0$, i.e., $\psi_x(b,x,\frac{1}{2})\in\mathrm{C}_g(x)\setminus\{x\}$ or $\psi_x(b,x,1)\in\mathrm{C}_g(x)\setminus\{x\}$. We conclude that $x\notin\tilde{\Gamma}_g^0(\sigma)$.

Actually, the same holds for any diffeomorphism h that is sufficiently C^1-close to g. Indeed, for any $x\in\mathcal{W}_f^c(x_0,\tilde{K}_1\sigma)$, let γ_1^x, γ_2^x be the 4 us-loops at (g,x) coming from $\gamma^x(\frac{1}{2})$, $\gamma^x(1)$, with respective endpoints $\psi_x(b,x,\frac{1}{2})$, $\psi_x(b,x,1)$. For any diffeomorphism h which is C^1-close to g, we let $\gamma_1^{x,h}$, $\gamma_2^{x,h}$ be the respective continuations of γ_1^x, γ_2^x given by Lemma 2.7, and we set

$$\widetilde{\Psi}(h,x):=\big(\Pi_x^c(x),\Pi_x^cH_{h,\gamma_1^{x,h}}(x),\Pi_x^cH_{h,\gamma_2^{x,h}}(x)\big).$$

By our choice of b, and by compactness, there exists $\varepsilon_0>0$ such that

$$d(\widetilde{\Psi}(g,x),\Sigma_0)=d(\Psi(b,x),\Sigma_0)>\varepsilon_0,$$

for all $x\in\mathcal{W}_f^c(x_0,\tilde{K}_1\sigma)$. Thus, there exists $\delta'>0$ such that for any diffeomorphism h with $d_{C^1}(g,h)<\delta'$, and for any $x\in\mathcal{W}_f^c(x_0,\tilde{K}_1\sigma)$, it holds

$$d(\widetilde{\Psi}(h,x),\Sigma_0)>\frac{\varepsilon_0}{2}>0.$$

Therefore, $H_{h,\gamma_1^{x,h}}(x) \in \mathrm{C}_h(x) \setminus \{x\}$ or $H_{h,\gamma_2^{x,h}}(x) \in \mathrm{C}_h(x) \setminus \{x\}$, so that $x \notin \tilde{\Gamma}_h^0(\sigma)$, which concludes the proof. □

We can now give the proof of Theorem B.

Corollary 8.7. *There exists a C^2-uniform constant $\hat{\sigma}_1 = \hat{\sigma}_1(f) > 0$ such that for any $\sigma \in (0, \hat{\sigma}_1)$, and for any $\delta > 0$, there exists a partially hyperbolic diffeomorphism $g \in \mathscr{F}$ such that $d_{C^r}(f, g) < \delta$ and such that for some $(\frac{1}{40}, 4)$-spanning c-family $\mathcal{D}_g$ for g, it holds $x \notin \tilde{\Gamma}_g^0(\sigma)$, for all $\mathcal{C} \in \mathcal{D}_g$, and for all $x \in \frac{1}{20}\mathcal{C}$. In particular, the center accessibility class $\mathrm{C}_g(x)$ of each point $x \in M$ is non-trivial.*

Proof. Fix some small $\delta > 0$. By Kupka-Smale's Theorem (see for instance [K]), C^r-generically, periodic points are hyperbolic. Therefore, without loss of generality, we can assume that the fixed points of f^k are isolated, for all $k \geq 1$.

Let $\tilde{\sigma}_1 = \tilde{\sigma}_1(f) > 0$, $\tilde{K}_1 = \tilde{K}_1(f) \in (0,1)$ and $\tilde{R}_1 = \tilde{R}_1(f) > 0$ be the constants given by Proposition 8.5. For $\overline{R} > \max(\tilde{R}_1, 1)$, let $\overline{N} = \overline{N}(f, \overline{R}) > 0$, $\overline{\rho} = \overline{\rho}(f, \overline{R}) \in (0, \overline{R}^{-1})$ and $\overline{\sigma} = \overline{\sigma}(f, \overline{R}) > 0$ be the constants given by Proposition 8.4. Then, there exists a constant $\delta_0' \in (0, \delta)$ such that for any diffeomorphism g with $d_{C^1}(f, g) < \delta_0'$, there exists a $(\frac{1}{40}, 4)$-spanning c-family $\mathcal{D}_g$ for g with at most $\overline{N}$ elements such that the map $g \mapsto \mathcal{D}_g$ is continuous, and

$$\overline{\rho} < \underline{r}(\mathcal{D}_g) \leq \overline{r}(\mathcal{D}_g) < \overline{R}^{-1}; \qquad \mathcal{D}_g \text{ is } \overline{\sigma}\text{-sparse}; \qquad R_\pm(g, (\mathcal{D}_g, \overline{\sigma})) > \overline{R}.$$

Take $\sigma \in \left(0, \min\left(\tilde{\sigma}_1, \frac{\overline{\sigma}}{10}\right)\right)$, and let $z_1, z_2, \ldots, z_\ell$, $\ell \geq 1$, be a finite collection of points such that for any diffeomorphism g with $d_{C^1}(f, g) < \delta_0'$, we have $g \in \mathscr{F}$, and

$$\bigcup_{\mathcal{C} \in \mathcal{D}_g} \frac{1}{20}\mathcal{C} \subset \bigcup_{i=1}^{\ell} \mathcal{W}_f^c(z_i, \tilde{K}_1\sigma) \subset (\mathcal{D}_g, 10\sigma). \tag{8.1}$$

As $\sigma \in (0, \tilde{\sigma}_1)$ and $R_\pm(f, B(z_1, 10\sigma)) > \tilde{R}_1$, we can apply Proposition 8.5 to get a diffeomorphism $f_1 \in \mathscr{F}$ such that for some $\delta_1' \in (0, \delta'(z_1, f_1))$, it holds $B_{C^r}(f_1, \delta_1') \subset B_{C^r}(f, \delta_0')$, and $x \notin \tilde{\Gamma}_h^0(\sigma)$, for all $x \in \mathcal{W}_f^c(z_1, \tilde{K}_1\sigma)$ and for all $h \in B_{C^r}(f_1, \delta_1')$.

Similarly, as $R_\pm(f_1, B(z_2, 10\sigma)) > \tilde{R}_1$, we can apply Proposition 8.5 to get a diffeomorphism $f_2 \in \mathscr{F}$ such that for some $\delta_2' > 0$, it holds $B_{C^r}(f_2, \delta_2') \subset B_{C^r}(f_1, \delta_1') \subset B_{C^r}(f, \delta_0')$, and $x \notin \tilde{\Gamma}_h^0(\sigma)$, for all $x \in \mathcal{W}_f^c(z_2, \tilde{K}_1\sigma)$ and for all $h \in B_{C^r}(f_2, \delta_2')$; in fact, as $B_{C^r}(f_2, \delta_2') \subset B_{C^r}(f_1, \delta_1')$, we have $x \notin \tilde{\Gamma}_h^0(\sigma)$, for all $x \in \mathcal{W}_f^c(z_1, \tilde{K}_1\sigma) \cup \mathcal{W}_f^c(z_2, \tilde{K}_1\sigma)$.

Recursively, we thus obtain a diffeomorphism $g = f_\ell \in \mathscr{F}$ such that $d_{C^r}(f, g) < \delta_0' < \delta$ and such that $x \notin \tilde{\Gamma}_g^0(\sigma)$, for all $x \in \mathcal{W}_f^c(z_1, \tilde{K}_1\sigma) \cup \cdots \cup \mathcal{W}_f^c(z_\ell, \tilde{K}_1\sigma)$. By (8.1), we conclude that for each $\mathcal{C} \in \mathcal{D}_g$, and for each $x \in \frac{1}{20}\mathcal{C}$, we have $x \notin \tilde{\Gamma}_g^0(\sigma)$. In particular, as $\mathcal{D}_g$ is a $(\frac{1}{40}, 4)$-spanning c-family for g, the center accessibility class $\mathrm{C}_g(x)$ of each point $x \in M$ is non-trivial. □

Remark 8.8. *In fact, Corollary 8.7 also holds when the center dimension $\dim E_f^c$ is larger than 2. Indeed, the proof relies on the submersion from the space of perturbations to the phase space – here, some center leaf – constructed in Lemma 6.12 and Corollary 6.13, which can be carried out also when $\dim E_f^c > 2$.*

8.3. **Density of accessibility.** In this part, we conclude the proof of Proposition 8.1. Let us start with the following result, which strengthens Proposition 7.2.

Proposition 8.9. *There exist C^2-uniform constants $\tilde{\sigma}_2 = \tilde{\sigma}_2(f) > 0$, $\tilde{K}_2 = \tilde{K}_2(f) \in (0,1)$ and $\tilde{R}_2 = \tilde{R}_2(f) > 0$ such that for any $\delta > 0$, for any $\sigma \in (0, \tilde{\sigma}_2)$, for any point $x_0 \in M$ satisfying $R_\pm(f, B(x_0, 10\sigma)) > \tilde{R}_2$, there exists a partially hyperbolic diffeomorphism $g \in \mathscr{F}$ such that $d_{C^r}(f,g) < \delta$ and such that for some $\delta'' = \delta''(x_0, g) > 0$, it holds $\mathrm{Acc}_h(x_0) \supset B(x_0, \tilde{K}_2\sigma)$, for all $h \in \mathscr{F}$ with $d_{C^1}(g,h) < \delta''$.*

Proof. Fix some small $\delta > 0$. Let $\tilde{\sigma}_1 = \tilde{\sigma}_1(f) > 0$, $\tilde{K}_1 = \tilde{K}_1(f) \in (0,1)$ and $\tilde{R}_1 = \tilde{R}_1(f) > 0$ be the constants in Proposition 8.5. Let $x_0 \in M$ be a point satisfying $R_\pm(f, B(x_0, 10\sigma)) > \widetilde{R}$, for some $\widetilde{R} > \tilde{R}_1$ and $\sigma \in (0, \tilde{\sigma}_1)$, and take $\widetilde{K} \in (0, \tilde{K}_1)$. Then, by Proposition 8.5, there exists a partially hyperbolic diffeomorphism $f_1 \in \mathscr{F}$ such that $d_{C^r}(f, f_1) < \frac{\delta}{2}$ and such that for some $\delta' \in (0, \frac{\delta}{2})$, we have $x \notin \tilde{\Gamma}_g^0(\sigma)$, for all $x \in B(x_0, \widetilde{K}\sigma)$ and for all $g \in \mathscr{F}$ with $d_{C^1}(f_1, g) < \delta'$.

In the following, for any $x \in \Gamma_g^1 \cap \mathcal{W}_{f_1}^c(x_0, \widetilde{K}\sigma)$, we denote by $\Pi_x^c \colon M \to \mathbb{R}^2$ the map in Lemma 4.5 for the diffeomorphism f_1. By Proposition 4.7 and Proposition 4.4, if δ' is sufficiently small, then for any $g \in \mathscr{F}$ with $d_{C^1}(f_1, g) < \delta'$ and for any $x \in \Gamma_g^1 \cap \mathcal{W}_{f_1}^c(x_0, \widetilde{K}\sigma)$, it holds

$$\Pi_x^c \mathrm{C}_g(x, 10\sigma) \subset \mathscr{C}_1, \tag{8.2}$$

for some cone $\mathscr{C}_1$ centered at $0_{\mathbb{R}^2}$; as in Section 5, we let $\mathscr{C} := \big(\mathbb{R}^2 \setminus \mathscr{C}_1\big) \cup \{0_{\mathbb{R}^2}\}$, and let $\mathscr{C}^+$, $\mathscr{C}^-$ be the closures of the two connected components of $\mathscr{C} \setminus \{0_{\mathbb{R}^2}\}$. For any $x \in \mathcal{W}_{f_1}^c(x_0, \widetilde{K}\sigma)$, we let $\gamma_1^x = [x, \alpha_1^x, \ldots, \omega_1^x]$, resp. $\gamma_2^x = [x, \alpha_2^x, \ldots, \omega_2^x]$ be the non-degenerate closed 10 us-loop, resp. non-degenerate closed 10 su-loop at (f_1, x) given by Lemma 5.3 for f_1 in place of f, with

$$\big(\Pi_x^c \omega_1^x, \Pi_x^c \omega_2^x\big) \in \big(\mathscr{C}^+ \times \mathscr{C}^-\big) \cup \big(\mathscr{C}^- \times \mathscr{C}^+\big). \tag{8.3}$$

In the following, we will define a new deformation $\hat{f}$ obtained by considering infinitesimal deformations localized near the points α_1^x and α_2^x for $x \in \mathcal{W}_{f_1}^c(x_0, \widetilde{K}\sigma)$. Arguing as in Lemma 6.12, for $\star = 1, 2$, we can construct an infinitesimal C^r deformation at f_1 with 2-parameters $V_\star \colon \mathbb{R}^2 \times M \to TM$ such that $\mathrm{supp}(V_\star) \subset B(x_0, 10\sigma)$, and such that for some constants $\widetilde{C} > 0$, $\tilde{\kappa} > 0$, we have: for any $x \in \mathcal{W}_{f_1}^c(x_0, \widetilde{K}\sigma)$,

(1) $V_\star$ is adapted to $(\gamma_\star^x, \sigma, \widetilde{C}, \widetilde{R})$;
(2) for any corner $z \neq \alpha_\star^x$ of $\gamma_\star^x$, it holds

$$D_B(\pi_c V_\star(B, z)) = 0,$$

where $\pi_c \colon TM \to E_f^c$ denotes the canonical projection, and

$$\Big|\det\, D_B\big(\pi_c V_\star\big(B, x_1^x\big(\frac{k}{k_0}\big)\big)\big)\Big| > \tilde{\kappa}.$$

Indeed, for $\star = 1, 2$, as the map $\mathcal{W}_{f_1}^c(x_0, \widetilde{K}\sigma) \ni x \mapsto \gamma_\star^x$ is continuous, and by (5.2), we can construct the infinitesimal deformation $V_\star$ such that the $\mathrm{supp}(V)$ is localized around the set $\{\alpha_\star^x\}_x$ of the first corners of the loops $\gamma_\star^x$.

Let then $V \colon \mathbb{R}^4 \times M \to TM$ be the infinitesimal C^r deformation defined as

$$V(B, \cdot) := B^1 V_1(\cdot) + B^2 V_2(\cdot), \quad \forall\, B = (B^1, B^2) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

In particular, V satisfies $\mathrm{supp}(V) \subset B(x_0, 10\sigma)$, for any $x \in \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)$, V is adapted to $(\gamma_1^x, \sigma, \widetilde{C}, \widetilde{R})$ and $(\gamma_2^x, \sigma, \widetilde{C}, \widetilde{R})$, and for any corner $z \neq \alpha_1^x, \alpha_2^x$ of γ_1^x, γ_2^x,

$$D_B(\pi_c V(B, z)) = 0,$$

while for $E_1 := \mathbb{R}^2 \times \{0_{\mathbb{R}^2}\}$, $E_2 := \{0_{\mathbb{R}^2}\} \times \mathbb{R}^2$, we have

$$\left|\det \left(E_\star \ni B \mapsto D_B\big(\pi_c V\big(B, \alpha_\star^x\big)\big)\right)\right| > \tilde{\kappa}, \quad \star = 1, 2. \tag{8.4}$$

For some small $\delta_1 > 0$, let us consider the C^r deformation $\hat{f} \colon B(0_{\mathbb{R}^4}, \delta_1) \times M \to M$ at f_1 with 4-parameters generated by the infinitesimal C^r deformation V. As before, for any $b \in B(0_{\mathbb{R}^4}, \delta_1)$, we set $f_b := \hat{f}(b, \cdot)$. By (8.2), if δ_1 and σ are sufficiently small, then for all $b \in B(0_{\mathbb{R}^4}, \delta_1)$, and for all $x \in \Sigma_b(\sigma) := \Gamma^1_{f_b} \cap \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)$, it holds

$$\Pi_x^c \mathrm{C}_{f_b}(x, 10\sigma) \subset \mathscr{C}_1. \tag{8.5}$$

Let $T = T(\hat{f})$ be as in (6.1). We denote by $\hat{\gamma}_1^x$, $\hat{\gamma}_2^x$ the respective lifts of γ_1^x and γ_2^x for T according to Definition 6.5. By (8.4), thanks to Proposition 6.11, and arguing as in Corollary 6.13, we obtain:

Lemma 8.10. *The map*

$$\Phi \colon \begin{cases} B(0_{\mathbb{R}^4}, \delta_1) \times \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma) & \to \quad \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \\ (b, x) & \mapsto \quad \Big(\Pi_x^c H_{T, \hat{\gamma}_1^x}(b, x), \Pi_x^c H_{T, \hat{\gamma}_2^x}(b, x)\Big) \end{cases}$$

satisfies

$$\big|D_b|_{b=0}\Phi(\cdot, x) - D_b|_{b=0}\Phi(\cdot, y)\big| \leq \rho(\sigma), \qquad \forall\, x, y \in \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma), \tag{8.6}$$

for some function $\rho \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{\sigma \to 0} \rho(\sigma) = 0$, and there exists $\kappa > 0$ such that for any $x \in \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)$, it holds

$$|\det\, D_b|_{b=0}\big(\Phi(\cdot, x)\big)| > \kappa. \tag{8.7}$$

Indeed, for $\star = 1, 2$, since the map $\mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma) \ni x \mapsto \gamma_\star^x$ is continuous, it follows from Lemma 4.1 and Corollary 4.3 that the partial derivatives of the holonomies $H_{T, \hat{\gamma}_\star^x}$ with respect to b are uniformly close, for all $x \in \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)$. Hence, by the definition of Φ, and by Proposition 6.11, the maps $\{\Phi(\cdot, x)\}_{x \in \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)}$ are uniform submersions, which gives (8.6) and (8.7).

By (8.3), for each $x \in \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)$, we have

$$\Phi(0, x) \in \big(\mathscr{C}^+ \times \mathscr{C}^-\big) \cup \big(\mathscr{C}^- \times \mathscr{C}^+\big).$$

Let us denote by S^+, resp. S^- the set of all points $x \in \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)$ such that $\Phi(0, x) \in \mathscr{C}^+ \times \mathscr{C}^-$, resp. $\Phi(0, x) \in \mathscr{C}^- \times \mathscr{C}^+$, so that $S^+ \cup S^- = \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)$. By (8.6)-(8.7), there exists a perturbation parameter $b \in B(0_{\mathbb{R}^4}, \delta_1)$ such that

$$\Pi_x^c H_{T, \hat{\gamma}_1^x}(b, x) \in \mathscr{C}_*^+ = \mathscr{C}^+ \setminus \{0_{\mathbb{R}^2}\}, \qquad \text{for all } x \in S^+,$$
$$\Pi_x^c H_{T, \hat{\gamma}_2^x}(b, x) \in \mathscr{C}_*^- = \mathscr{C}^- \setminus \{0_{\mathbb{R}^2}\}, \qquad \text{for all } x \in S^-.$$

As $S^+ \cup S^- = \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)$, we deduce that for each $x \in \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)$,

$$\text{either} \quad \Pi_x^c H_{T, \hat{\gamma}_1^x}(b, x) \notin \mathscr{C}_1, \quad \text{or} \quad \Pi_x^c H_{T, \hat{\gamma}_2^x}(b, x) \notin \mathscr{C}_1.$$

By (8.5), we deduce that $\Sigma_b(\sigma) = \emptyset$, i.e., $\Gamma^1_{f_b} = \emptyset$. Therefore, by Theorem 2.10, the accessibility class $\mathrm{Acc}_{f_b}(x)$ of each point $x \in \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)$ is open. Moreover, if δ_1 is sufficiently small, then by construction, the diffeomorphism $g := f_b$ satisfies

$$d_{C^r}(f, g) \leq d_{C^r}(f, f_1) + d_{C^r}(f_1, f_b) < \delta,$$

which concludes the proof of Proposition 8.9. □

Proof of Proposition 8.1. Fix $\delta > 0$ arbitrarily small. Let $\tilde{\sigma}_2 = \tilde{\sigma}_2(f) > 0$, $\tilde{K}_2 = \tilde{K}_2(f) \in (0, 1)$ and $\tilde{R}_2 = \tilde{R}_2(f) > 0$ be the C^2-uniform constants given by Proposition 8.9. By Proposition 8.4, there exist C^1-uniform constants $\overline{N} = \overline{N}(f, \tilde{R}_2) > 0$, $\overline{\rho} = \overline{\rho}(f, \tilde{R}_2) \in (0, \tilde{R}_2^{-1})$ and $\overline{\sigma} = \overline{\sigma}(f, \tilde{R}_2) > 0$ such that for all diffeomorphism g sufficiently C^1-close to f, there exists a $(\frac{1}{40}, 4)$-spanning c-family $\mathcal{D}_g$ for g with at most $\overline{N}$ elements such that

(1) $\overline{\rho} < \underline{r}(\mathcal{D}_g) \leq \overline{r}(\mathcal{D}_g) < \tilde{R}_2^{-1}$;
(2) $\mathcal{D}_g$ is $\overline{\sigma}$-sparse;
(3) $R_\pm(g, (\mathcal{D}_g, \overline{\sigma})) > \tilde{R}_2$.

and such that the map $g \mapsto \mathcal{D}_g$ is continuous. Let $\sigma \in (0, \frac{1}{10} \min(\tilde{\sigma}_2, \overline{\sigma}))$. By compactness, we can take a finite collection of points $x_1, \ldots, x_m \in M$ such that

$$\frac{1}{20}\mathcal{D}_f \subset U := \bigcup_{i=1}^m B(x_i, \tilde{K}_2\sigma) \subset (\mathcal{D}_f, \overline{\sigma}).$$

Note that $x_i \in M$ satisfies $R_\pm(f, B(x_i, 10\sigma)) > \tilde{R}_2$, for each $i \in \{1, \ldots, m\}$. Therefore, we can apply Proposition 8.9 inductively to get a partially hyperbolic diffeomorphism $g \in \mathscr{F}$ such that $d_{C^r}(f, g) < \delta$ and such that $\mathrm{Acc}_g(x_i) \supset B(x_i, \tilde{K}_2\sigma)$, for all $i \in \{1, \ldots, m\}$. By connectedness of the disks in $\mathcal{D}_f$, each center disk in the family $\frac{1}{20}\mathcal{D}_f$ is contained in a single accessibility class for g. Moreover, if δ is sufficiently small, and by continuity of the map $h \mapsto \mathcal{D}_h$, each center disk in the family $\frac{1}{40}\mathcal{D}_g$ is contained in a single accessibility class for g. As $\mathcal{D}_g$ is a $(\frac{1}{40}, 4)$-spanning c-family for g, we deduce that g is accessible, as wanted. □

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